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Managing Critical Rank Reversals in TOPSIS: A Mathematical Framework for Ensuring Stable Ideal Solutions

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Abstract

This research examines the impact of non-dominated alternatives on rankings and explores rank reversal (RR) in the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS), a widely used distance-based MCDM method. A theoretical analysis reveals how the mathematical operations of TOPSIS contribute to RR through the relative closeness and separation measures. Four scenarios are outlined to identify conditions where RR becomes unavoidable. The study provides new insights into the mathematical foundations of RR and its implications for decision makers. To address this issue, three strategies are proposed: identifying non-dominated alternatives, recognizing conditions leading to close performance margins, and normalizing ideal solutions to fixed reference values. These findings offer practical guidance for developing distance-based MCDM methods that minimize rank reversal.

Keywords: *Rank reversal, Dominance, Relative closeness, Linear normalization, Extreme alternative, TOPSIS*

1. Introduction

Rank reversal (RR) in multi-criteria decision making (MCDM) refers specifically to a change in the relative ordering of existing alternatives caused by the addition or removal of other alternatives in the decision matrix [34]. Among various types of RR phenomena [5], this research specifically addresses the most problematic case: changes in ranking order resulting from the addition or deletion of alternatives, while holding all other parameters (including criterion weights) constant. Rather than developing a comprehensive taxonomy of all RR types, this research provides a rigorous mathematical framework for understanding when and why RR caused by the addition or deletion of alternatives occurs in Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS). This phenomenon, also known as rank anomaly or rank irregularity, raises significant concerns over the consistency and robustness of MCDM techniques. The occurrence of RR has been widely debated in both MCDM and traditional decision

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theories, as it challenges the reliability of decision-making methods [44]. Given the importance of ranking stability in decision making, this research investigates the underlying causes of RR in MCDM with a specific focus on TOPSIS, which has demonstrated relative stability in maintaining rank consistency [42].

It is obvious that RR violates independence of irrelevant alternatives and independence of irrelevant alternatives (IIA) [1] goes against our intuitions as well. Belton and Gear [2] initially noted RR in the Analytic Hierarchy Process (AHP), while Salo and Hämäläinen [29] ascribed the finding to the AHP's relative measurement mode. The issue of having RR when the TOPSIS method is applied was first reported by Triantaphyllou [34]. Many debates have arisen in some major operations research journals, such as Saaty [26–28], Saaty and Vargas [28], Finan and Hurley [7], Triantaphyllou [34], and Kułakowski et al. [15]. Zanakis et al. [42] compared eight MCDM approaches using simulation data, including ELimination Et Choice Translating REality (ELECTRE), Simple Additive Weighting (SAW), Multiplicative Exponential Weighting (MEW), TOPSIS, and four variations of AHP. Their results showed that SAW and MEW are the best two methods, while TOPSIS, AHPs, and ELECTRE are the next best. Regardless of how well TOPSIS performs, it is still influenced by the RR phenomenon. Shekhovtsov et al. [31] noted that even the interval TOPSIS extension is not robust to rank reversal, highlighting that the problem persists in more advanced variants of the method. De Farias Aires and Ferreira [5] categorized RR (ranking reversal) phenomena into five types. The change in the final ranking order of alternatives that results from the addition or deletion of irrelevant alternatives is widely regarded as the most problematic. We note that Arrow's IIA [1] addresses preference independence from irrelevant alternatives in social choice theory, whereas our focus is on the specific mathematical mechanisms in TOPSIS that cause ranking changes when alternatives are added or removed. This study provides a comprehensive mathematical analysis of RR in TOPSIS, identifies four scenarios determining when RR occurs, introduces a gap function δ for predicting RR, and proposes three practical management strategies based on our theoretical findings.

Numerous experts have attempted to provide various examples to show the occurrence of RR, but we find that these examples may not accurately describe RR due to the existence of non-dominated alternatives [25, 36]. In MCDM, non-dominated alternatives are those that are clearly superior on at least one criterion, and choosing among them depends on decision makers' preferences or weights. Under different weight settings, any of these non-dominated alternatives can be ranked first, thus affecting the rankings of other alternatives [12]. Therefore, failing to initially identify possible non-dominated alternatives in these examples makes the proposed approaches questionable.

Most papers on RR have presented two main areas: the normalization process using linear methods and the determination of PIS and NIS through the addition of extreme fictitious alternatives [8, 14]. While these heuristic approaches only offer partial solutions, limited attention has been paid to theoretical insights examining how TOPSIS's mathematical operations, specifically in calculating the relative closeness - contribute to RR. Understanding the relationship between these mathematical components and RR is fundamental to comprehending the phenomenon in TOPSIS. Our research addresses this critical knowledge gap by investigating the mathematical properties of TOPSIS for an extensive understanding.

The rest of the paper runs as follows. Section 2 reviews the resolutions for RR in TOPSIS. Section 3 demonstrates the effect of non-domination on ranking alternatives by TOPSIS. Section 4 examines the rankings of TOPSIS and presents a mathematical view of RR in TOPSIS. Section 5 introduces strategies for preventing RR. Section 6 and 7 present the discussion and conclusions, respectively.

2. Literature Review

2.1. The TOPSIS Method

TOPSIS is a widely used distance-based MCDM method referred to Zavadskas et al. [43]. To better understand the following discussions, we first list the traditional algorithm of TOPSIS for ranking and selection in the following seven steps [10].

Step 1: Create a decision or evaluation matrix $D = \{x_{ij}\}$. The matrix refers to m alternatives and n criteria with its performance element $x_{ij}, i = 1, \ldots, m$, and $j = 1, \ldots, n$.

Step 2: Construct the vector normalized decision matrix $R = \{r_{ij}\}$. Matrix D is normalized to matrix R with:

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \tag{1}$$

In this study, to simplify the derivation of the mathematical model, a common linear normalization is adopted. For a benefit-type criterion, where a larger value is preferred, the normalized value r_{ij} is defined as

$$r_{ij} = \frac{x_{ij}}{\max_i x_{ij}}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \tag{2}$$

For a cost-type criterion, where a smaller value is preferred, the normalized value r_{ij} is defined as [9]

$$r_{ij} = 1 - \frac{x_{ij}}{\max_i x_{ij}}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \tag{3}$$

Step 3: Construct the weighted normalized decision matrix V. The matrix $V = \{w_j r_{ij}\}$ is calculated by multiplying the elements at each column of the matrix R by their associated weights w_j, where $j = 1, \ldots, n$, and $i = 1, \ldots, m$.

Step 4: Determine the positive-ideal and negative-ideal solutions V^+ (PIS) and V^- (NIS), respectively.

$$V^+ = \{v_1^+, \ldots, v_n^+\} = \left\{\max_i v_{ij} \mid j \in J, \min_i v_{ij} \mid j \in J^*\right\} \tag{4}$$

$$V^- = \{v_1^-, \ldots, v_n^-\} = \left\{\min_i v_{ij} \mid j \in J, \max_i v_{ij} \mid j \in J^*\right\} \tag{5}$$

J is associated with the benefit criteria (those properties whose higher values are desirable), and J^* is associated with the cost criteria.

Step 5: For each alternative $i = 1, \ldots, m$, calculate the separation measures (or distance measures) S_i^+ and S_i^-. We measure them by the n-dimensional Euclidean distance as:

$$S_i^+ = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^+)^2}, \quad S_i^- = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2} \tag{6}$$

Step 6: For each alternative $i = 1, \ldots, m$, calculate its relative closeness (or ranking index) C_i^*:

$$C_i^* = \frac{S_i^-}{S_i^+ + S_i^-}, \quad 0 \leq C_i^* \leq 1 \tag{7}$$

The larger the index value is, the better is the performance of the alternative.

Step 7: Rank the preference order of all alternatives in descending order of C_i^*'s value.

2.2. Rank Reversal

In addressing RR in TOPSIS, Ren et al. [25] conducted one of the earliest studies attempting to resolve this issue. They proposed a modified synthetic evaluation method that transforms two separation measures into a two-dimensional plane, with relative closeness based on the distance to PIS minus the minimum PIS distance across all alternatives. However, their approach lacked a systematic framework for preventing RR, and their validation relied solely on two illustrative examples from public health and student evaluation contexts. Wang and Luo [36] subsequently documented RR in TOPSIS through a single example, but their analysis overlooked the crucial role of non-dominated alternatives in causing RR. Kong [14] advanced the field by proposing two modifications: replacing vector normalization with linear normalization and introducing extreme fictitious alternatives to stabilize PIS and NIS values. However, their validation was limited to a simple case of five alternatives with two criteria under non-domination.

Building upon Kong's work [14], García-Cascales and Lamata [8] expanded the analysis to multiple scenarios, though their investigation was similarly constrained to a four-alternative, two-criteria case with non-domination. İç [11] explored integrating design of experiments (DoE) with TOPSIS for financial performance evaluation. That study demonstrated the approach using a five-alternative, five-criteria case with non-domination to show rank preservation. However, the proposed DoE-TOPSIS method made no substantive improvements to the core TOPSIS algorithm, despite claims about RR avoidance.

Cables et al. [3] introduced a reference ideal method that redefines the distance to the reference ideal as an interval, allowing performance data to be linearly normalized into a single value. Their method also modified separation measures by using the weight vector as the reference ideal. While they demonstrated their approach using a personnel selection case (five candidates, six criteria with non-domination) and claimed data-type independence, the validation's reliability is questionable due to its reliance on assumed interval values in a single example.

Senouci et al. [30] addressed RR in mobile network interface selection by implementing linear normalization with four variants and dynamic max/min value settings. Their simulation analysis, involving seven alternatives and five criteria with non-domination, demonstrated that two normalization types could achieve zero RR ratio. However, the broader applicability of their findings remains to be established.

Mufazzal and Muzakkir [18] proposed a proximity index to address RR, arguing that traditional predefined extreme values are unrealistic. Their approach employed Manhattan (linear) distance for separation measures and utilized a linear index based on the sum of distances for ranking. Although they analyzed seven historical cases with non-dominated alternatives to demonstrate their method's effectiveness, their study had two significant limitations: they failed to explain the discrepancies between their rankings and those of other methods, and they did not thoroughly examine the mathematical properties of their proposed index.

De Farias Aires and Ferreira [5] employed two forms of linear normalization to modify traditional TOPSIS. They showed the approach to be robust through a student selection case, twenty alternatives with three criteria, and their subsets, but the illustration is a bit strange due to plenty of non-dominated candidates and the fact that the ranking is not the focus of the problem. Yang [38] modified de Farias

Aires and Ferreira's approach [5] with linear normalization of historical extreme values and only giving weights to PIS. That study illustrated Wang and Luo's dubious example [36] to demonstrate its rank stability.

Yang et al. [39] applied a similar concept of Kong's linear normalization and absolute maximum and minimum values [14], to rank five alternatives with non-domination for material selection. Through deleting an alternative, they demonstrated the proposed method is better than traditional TOPSIS in rank preservation. Yang et al. [37] considered a similar idea of Kong [14] and a relative closeness proposed by Kuo [16] for a robust method in TOPSIS. Yu et al. [41] utilized S-shape utility functions for normalization to reduce the effect of extreme values on the upper and lower bounds of index data. Their proposed approach, demonstrated through a case study involving four criteria and eight non-dominated alternatives, was found to mitigate RR in TOPSIS. Although the above works claimed their suggested methods via a heuristic could avoid RR by different examples, the actual situation of RR occurring is still unknown. This is the prime motivation of our study.

Researchers have systematically investigated the mathematical underpinnings of TOPSIS beyond its heuristic origins with particular focus on the RR phenomenon. Opricovic and Tzeng [21] demonstrated how RR could occur under certain irregular conditions, from the non-linearity mentioned by Hwang and Yoon [10], by examining TOPSIS's relative closeness. Building on this work, Kuo [16] investigated the effect under different TOPSIS ranking indices and analyzed the theoretical decidability of TOPSIS by studying the relationships among four separation measures from two alternatives, finding that 8 of 24 possible cases are mathematically undecidable and could result in RR. He also demonstrated that the linear distance combined index shows better ranking consistency compared to traditional TOPSIS.

Further advancing this line of research, Shih and Olson [32] developed a definitive method for detecting RR by examining the deviation between two product terms of the separation measure for any two alternatives, where negative deviation values indicate RR occurrence. Shyur and Shih [33] explored how different distance metrics and normalization methods affect RR in TOPSIS, identifying three out of six conditions that could trigger RR. Following these established research directions, we shall conduct a detailed investigation of the RR phenomenon.

As a similar method to TOPSIS, Opricovic [19, 20] originally proposed VIKOR (VlseKriterijumska Optimizacija I Kompromisno Resenje [from Serbian]; All/Multiple Criteria Optimization and Compromise Solution), which was later organized by Opricovic and Tzeng [21]. Both paper's approaches are categorized as distance-based MCDM. VIKOR is designed with linear normalization and considers two distances, Mahattan and Chebyshev, instead of the Euclidean distance. It generates less RR phenomena than traditional TOPSIS, because of the two linear distances versus non-linearity of the Euclidean distance.

A few works had dealt with RR phenomena in VIKOR. Mousavi-Nasab and Sotoudeh-Anvari [17] suggested an extra process averting RR in VIKOR, TOPSIS, and COPRAS (Complex Proportional Assessment). Based on the original rankings of TOPSIS and COPRAS, or possibly VIKOR, the process is able to obtain a final ranking for sustainable material selection problems by modifying the inconsistent ranks from the result by a simple additive weighting method, which was thought not to generate any RR. Ceballos et al. [4] proposed an improvement by setting the extreme values for the performances on each criterion so that the virtual PIS would be fixed as Kong [14] did for TOPSIS. Similarity results with three alternatives and five criteria show that its RR occurs much less frequently.

Following the same stream, Yang and Wu [40] considered selecting historical extreme values of the performance so that the values of the group utility and individual regret would be within the interval of [0, 1]. They also examined several examples and random experiments to validate and verify credibility of the proposed method. Papathanasiou [22] provided an example showing that RR also occurs in fuzzy VIKOR. Among the above four studies, only Ceballos et al. [4] and Yang and Wu [40] presented a method to eliminate the RR phenomenon, but no deep insights were found by reason of its early development stage.

From the preceding discussions, we summarize that while other works proposed new ways to avert the RR phenomenon, few of them used TOPSIS properties. By way of simulation and the Spearman ranking test, they are still unable to guarantee that the approaches will be immune to RR. The first reason is the choice behavior of TOPSIS [10, 32], which causes the decision to slightly distort away from the central line of the relative closeness; i.e., relative closeness equal to 0.5. The second reason is that the non-dominated alternative disturbs the ranking of alternatives under different weight settings [12]. This is the gap in the literature that we look to fill. In the following parts we first demonstrate the non-dominated effect on TOPSIS ranking and later derive the TOPSIS formulation to find properties of the TOPSIS decision.

Beyond methods derived from TOPSIS and VIKOR, several RR-free MCDA methods have been proposed. The SPOTIS method (Stable Preference Ordering Towards Ideal Solution) [6] eliminates RR by relying on fixed, user-defined ideal and anti-ideal reference points. Although this ensures full rank stability, its weakness is that results may depend heavily on how decision makers choose these reference values, which may introduce subjectivity or require domain expertise that is not always available. The COMET (Characteristic Objects Method) [23] builds a rule-based preference model using characteristic objects and fuzzy rules, making evaluations independent of the alternative set and robust to RR. Its main drawback is the computational effort and expert involvement required to construct the rule base, particularly when dealing with high-dimensional criteria spaces, where the number of characteristic objects can grow rapidly.

These approaches demonstrate the broader effort within MCDA to design rank-reversal-free methods. However, their reliance on predefined reference points or complex preference modeling highlights important trade-offs: while they ensure stability, they may require additional domain knowledge, parameter selection, or computational effort. This contrasts with TOPSIS, where RR originates not from preference modeling choices but from the mathematical properties of its separation measures and the dynamic nature of PIS/NIS, motivating the need for a deeper mathematical understanding such as the one developed in this paper.

3. Non-dominated Alternatives

We first give the definitions of dominance and non-dominance by Ramesh and Zionts [24] and Vincke [35].

Definition 1. Given two alternatives a and b in the alternative set A, a dominates b iff:

$$g_j(a) \geq g_j(b), \quad \text{for } j \in \text{benefit criteria} \tag{8}$$

$$g_j(a) \leq g_j(b), \quad \text{for } j \in \text{cost criteria} \tag{9}$$

where at least one of the inequalities is strict. In addition, g_j represents the jth criterion in the evaluation. In the above definition, if all the inequalities hold as strict inequalities, then the dominance is said to be strong; otherwise, it is called weak. The following concept is a logical extension of the dominance concept.

Definition 2. An alternative a is said to be "non-dominated" or "efficient" in A if there is no other alternative in A that dominates it, even weakly. As Section 3 describes, many studies have demonstrated the RR occurrence. However, we discover that these demonstrations may not accurately represent RR. The outcomes of non-dominated alternatives are the main problem. When Kaliszewski [12] examined the rankings of 50 Polish universities, he found that the best ranking went to an underrepresented institution. The highest ranking varied depending on the assigned weight set. This finding raises a crucial question: Can different ranks characterize the RR phenomenon? The answer is probably no. Here, we use a simple example to show how the ranking changes through the existing non-dominated alternatives.

Example 1: An example of four alternatives with four criteria. To clearly understand the effect, we use the example from Wang and Luo [36], which are four alternatives evaluated by four criteria, as shown in Table 1.

Table 1. Performance of alternatives by four criteria

Alternative	C_1	C_2	C_3	C_4
A_1	36	42	43	70
A_2	25	50	45	80
A_3	28	45	50	75
A_4	24	40	47	100

We see that the above four alternatives are non-dominated or Pareto optimal, as alternative A_1 excels in criterion C_1, alternative A_2 demonstrates superiority in criterion C_2, alternative A_3 dominates in criterion C_3, and alternative A_4 ranks highest in criterion C_4. It is obvious that alternative A_1 would be ranked first if the weight set is $W = (w_1, w_2, w_3, w_4) = (1, 0, 0, 0)$, regardless of the MCDM method used. Similarly, alternative A_2 would be ranked first when setting w_2 = 1, or $W = (w_1, w_2, w_3, w_4) = (0, 1, 0, 0)$, and the same applies for alternatives A_3 and A_4. Consequently, each alternative could emerge as the top choice under different weight combinations. Different weight sets can generate various ranking orders after the first alternative is determined. For instance, there are six ranking orders if alternative A_1 is ranked first (i.e., $A_1 \succ A_2 \succ A_3 \succ A_4; A_1 \succ A_2 \succ A_4 \succ A_3; A_1 \succ A_3 \succ A_2 \succ A_4; A_1 \succ A_3 \succ A_4 \succ A_2; A_1 \succ A_4 \succ A_2 \succ A_3$; and $A_1 \succ A_4 \succ A_3 \succ A_2$), resulting in 6 × 4 = 24 ranking orders in total in this example. Furthermore, when considering the weight within the interval [0, 1], the possible combinations of ranking orders and the weight sets become innumerous. This makes it difficult to comprehend the complete picture of the decision space.

Using TOPSIS, the ranking of the example would be $A_1 \succ A_4 \succ A_3 \succ A_2$ when assigning equal weights to all criteria, $W = (w_1, w_2, w_3, w_4) = (0.25, 0.25, 0.25, 0.25)$. To determine the impact of criterion weights on the ranking, we fix alternative A_1 in the first rank and employ a non-linear program to find the minimum and maximum possible weights for its first criterion (i.e., w_1), while adhering to the weight constraints $w_1 + w_2 + w_3 + w_4 = 1$. To prevent any criterion from being disregarded (i.e., avoiding wash criteria), we enforce a lower bound of 0.01 empirically for the weight of each criterion. This program allows us to systematically investigate the influence of criterion weights on the overall ranking.

The resulting weights of criteria and the corresponding rankings appear in Table 2. Here, we observe that alternative A_1 can be ranked first when its weight (w_1) falls within the range of 0.1841-0.97, or when w_1 is above the value of 0.97. By repeating this process for the minimum and maximum possible weights of w_2, w_3, and w_4, we obtain a range of possible weight sets and corresponding alternative rankings. This analysis demonstrates that the ranking of alternatives is significantly influenced by the assigned weights to the criteria. Consequently, the example presented by Wang and Luo [36] may have limited practical significance, because the final ranking is highly sensitive to the chosen criterion weights.

Table 2. Representative weight sets on the rankings of alternatives for A_1 being the first

	Rank			
Alternative	equal weight	min weight for w_1	max weight for w_1	Note
A_1	1	1	1	Equal weights on four criteria: (0.25, 0.25, 0.25, 0.25)
A_2	4	1	3	Min weight for w_1: (0.0794, 0.3292, 0.3780, 0.2134)
A_3	3	1	2	Max weight for w_1: (0.97, 0.01, 0.01, 0.01) under constraints $w_1, w_2, w_3, w_4 \geq 0.01$
A_4	2	1	4	Enormous weight sets under the specific ranking could be obtained

In the provided example, non-dominated alternatives significantly influence TOPSIS's ranking process, as any non-dominated alternative can potentially achieve the top rank, thus reshaping the order of all other alternatives. We note that different criteria weight settings can cause RR to occur. However, even when the criteria weights remain unchanged, the addition or removal of an alternative can still result in RR. This highlights the complexity of the RR phenomenon and the limitations of existing heuristic approaches, such as Yang [38], in effectively addressing it. To tackle this issue, we leverage the mathematical formulation of TOPSIS to identify the specific conditions under which RR occurs. This analytical approach with providing management strategies constitute a novel contribution of our work.

4. Mathematical View of TOPSIS's RR

To analyze the terms of relative closeness in the TOPSIS methodology, we compare two alternatives, A_p and A_q, which are not dominated by each other. If A_p dominates A_q or vice versa, then the preference between them remains unchanged under any condition, making further analysis unnecessary. Their respective relative closeness values are calculated as:

$$C_p^* = \frac{S_p^-}{S_p^+ + S_p^-}, \quad C_q^* = \frac{S_q^-}{S_q^+ + S_q^-} \tag{10}$$

where S_p^+ and S_p^- are the separation measures of A_p from the positive-ideal solution (PIS) and negative-ideal solution (NIS), respectively. Similarly, S_q^+ and S_q^- represent the corresponding measures for A_q. For A_p to be preferred over A_q in TOPSIS, the relative closeness of A_p must be greater than that of A_q:

$$C_p^* > C_q^* \tag{11}$$

This can be expressed as:

$$\frac{S_p^-}{S_p^+ + S_p^-} > \frac{S_q^-}{S_q^+ + S_q^-}. \tag{12}$$

By rearranging terms, this inequality can be rewritten as:

$$S_p^- S_q^+ - S_p^+ S_q^- > 0. \tag{13}$$

The expression mentioned above is indeed the fundamental formula used to determine whether alternative A_p is preferable over alternative A_q in the TOPSIS methodology. By evaluating the result of $S_p^- S_q^+ - S_p^+ S_q^-$, we can ascertain the preference relationship between the two alternatives. If the value is greater than 0, then A_p is preferred over A_q.

To gain a deeper understanding of how the calculation method of separation measures affects the values of relative closeness, we express the normalized evaluation value of alternative i with respect to criterion j using v_{ij} for ease of derivation in subsequent calculations. In this context we assume that criterion j is a benefit criterion. However, for convenience, it is feasible to convert cost criteria into benefit criteria and vice versa, as proposed by Houska et al. [9]. For instance, the normalized evaluation matrix can employ linear normalization for cost criteria using the formula $r_{ij} = 1 - x_{ij}/\max_i x_{ij}$. Therefore, according to Eq. (6), the separation measures are further expanded as follows:

$$S_p^- = \sqrt{\sum_{j=1}^{n}(v_{pj} - v_j^-)^2}, \quad S_p^+ = \sqrt{\sum_{j=1}^{n}(v_{pj} - v_j^+)^2} \tag{14}$$

$$S_q^+ = \sqrt{\sum_{j=1}^{n}(v_{qj} - v_j^+)^2}, \quad S_q^- = \sqrt{\sum_{j=1}^{n}(v_{qj} - v_j^-)^2} \tag{15}$$

where $j = 1, \ldots, n$ for criteria; and v_j^+ and v_j^- are taken to be the PIS and NIS values of the jth criterion, respectively. It is important to note that these separation measures are dependent on v_j^+ and v_j^-.

We next explore the impact on separation measures' calculation if a new alternative is added to or removed from the candidate list. The new corresponding separation measures and relative closeness for alternatives A_p and A_q are denoted as $\bar{S}_p^+, \bar{S}_p^-, \bar{C}_p^*$, and $\bar{S}_q^+, \bar{S}_q^-, \bar{C}_q^*$, respectively. To simplify the mathematical derivation, we make the assumption that only the achievement of the jth criterion changes and that linear normalization is applied, resulting in a modification of its PIS from v_j^+ to $\bar{v}_j^+$. In this context, x_j^+ denotes the maximum value of the jth benefit criterion in the original set of achievements. After the inclusion of the new alternative, the extreme values of the jth criterion are replaced with $\bar{x}_j^+$. To quantify the change on PIS, we introduce the notation Δ_j^+, which represents the ratio of this change.

$$\Delta_j^+ = \bar{x}_j^+ / x_j^+, \tag{16}$$

where $\Delta_j^+ \geq 1$ (here, $0 \leq \Delta_j^+ \leq 1$ for cost criteria) if a new alternative is added, and $0 \leq \Delta_j^+ \leq 1$ (here, $\Delta_j^+ \geq 1$ for cost criteria) if an alternative is removed.

By assuming $S_p^- S_q^+ - S_p^+ S_q^- > 0$, to maintain the preference order of A_p and A_q after introducing a new alternative one must ensure $\bar{S_p^-} \bar{S_q^+} - \bar{S_p^+} \bar{S_q^-} > 0$. For the sake of convenience in subsequent explanations, we set $\delta = \bar{S_p^-} \bar{S_q^+} - \bar{S_p^+} \bar{S_q^-}$. If $\delta > 0$, then regardless of variations in the relevant parameters, alternative A_p remains preferred over alternative A_q, and there is no need to address the RR phenomenon between them. Indeed, $\delta = 0$ indicates that A_p and A_q have the same preference. However, in cases where $\delta < 0$, RR occurs. The negative value of δ aids in comprehending the onset of RR, offering insight into the connection between the alteration in PIS and the occurrence of RR. Example 2 illustrates a scenario demonstrating how the alteration of Δ_j^+ impacts the value of δ.

Example 2: A simplified fighter selection problem. Hwang and Yoon [10] proposed a fighter selection problem. Shih and Olson [32] adapted it by focusing solely on the first three benefit criteria. To illustrate our study, we keep the same example, and its decision matrix appears in Table 3.

Table 3. Decision matrix of the four alternatives evaluated by three criteria

Alternative	C_1	C_2	C_3	S^+	S^-	C^*
A_1	2.0	1,500	20,000	0.4897	0.1244	0.2025
A_2	2.5	2,700	18,000	0.1429	0.5253	0.7862
A_3	1.8	2,000	21,000	0.3816	0.2339	0.3800
A_4	2.2	1,800	20,000	0.3575	0.2168	0.3776

We employ the TOPSIS method for ranking. The decision matrix is linearly normalized using Eq. (2), and equal weights are assigned to all criteria. The initial ranking result shows $A_2 \succ A_3 \succ A_4 \succ A_1$. To simulate a situation where RR may occur, alternative A_5 with three criteria is assumed to be added to evaluate the achievement; i.e., $x_{51} = 2$, $x_{52} = 2,700$, and $x_{53} = 21,000$. PIS does not change due to the newly added A_5, and the new ranks are $A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$. The ranking order of A_1, A_2, A_3, and A_4 remains unchanged; i.e., no RR occurs. We try to gradually increase the achievement rating of alternative A_5 on criterion C_2, which leads to a change in PIS.

In the current case, when the achievement rating of alternative A_5 on criterion C_2 continues to increase, the ratio of change $\Delta_{C_2}^+$ will be greater than 1 and increase synchronously. Based on the given data, A_3 and A_4 are non-dominated alternatives, and alternative A_3 is preferred over alternative A_4 originally. Figure 1 illustrates the effect of increasing $\Delta_{C_2}^+$ on alternatives A_3 and A_4.

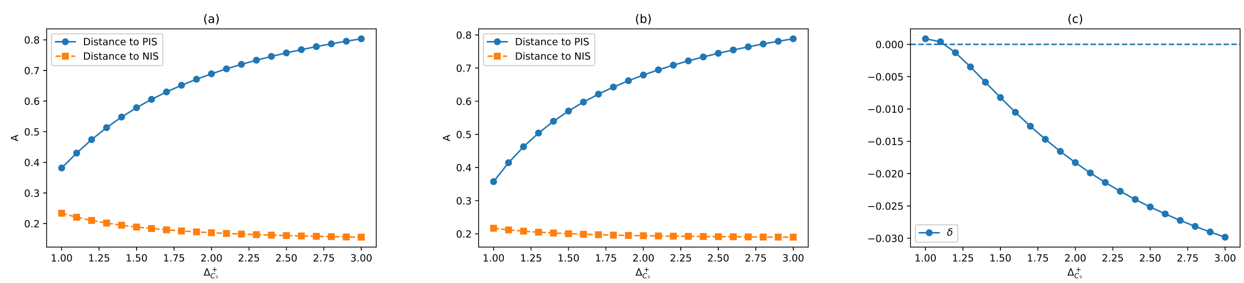

Figure 1. Effect of the change ratio $\Delta_{C_2}^+$ on alternative $A_3(a)$, alternative $A_4(b)$, and gap $\delta(c)$. The gap δ shown in panel (c) is calculated for alternatives A_3 and A_4

When the change ratio $\Delta_{C_2}^+$ increases, several effects are observed. First, the distance between NIS and alternatives $A_3(\bar{S_{A_3}^-})$ and $A_4(\bar{S_{A_4}^-})$ decreases. Second, the distance between PIS and alternatives

$A_3(S(\bar{A_3})^+)$ and $A_4(S(\bar{A_4})^+)$ increases (as shown in Figs. 1a and 1b). However, directly using these four measures may not provide a clear indication of the occurrence of RR. To address this, introducing gap δ offers an indicator for assessing the presence of the perceived RR phenomenon. Initially, when we raise the achievement value of alternative A_5 on criterion C_2 with $\Delta^+_{C_2} = 1$, the value of δ is 0.000862, indicating no RR ($A_3 \succ A_4$) based on Eq. (13). Figure 1c illustrates when the change ratio exceeds 1.2 that RR occurs ($A_4 \succ A_3$), resulting in a negative δ value. As the change turns larger, the magnitude of δ becomes more negative, and the phenomenon does not reverse with a larger value.

To gain deeper theoretical insights into the RR occurrence, we examine the function $\bar{S_p^-}\bar{S_q^+} - \bar{S_p^+}\bar{S_q^-}$ in more detail. For simplicity, we assume that all criteria are benefit criteria. The normalization process adheres to a linear normalization method. In our scenario, the maximum value of criterion l changes, and we use Δ^+_l to represent the change ratio.

To reflect the change in criterion l, we introduce a new weight-adjusted normalized value for all alternatives i on criterion l, denoted as $\bar{v_{il}}$. This is calculated as:

$$\bar{v_{il}} = v_{il}/\Delta^+_l. \tag{17}$$

When comparing alternatives to PIS, all weight-adjusted normalized values for each criterion are equal to their respective weights. Conversely, for NIS, the normalized values for criterion l are adjusted by dividing the original NIS value v^-_l by Δ^+_l. These relationships are expressed as:

$$[(v^+_1), (v^+_2), \ldots, (v^+_l), \ldots, (v^+_n)] = [w_1, w_2, \ldots, w_l, \ldots, w_n], \tag{18}$$

and

$$[(v^-_1), (v^-_2), \ldots, (v^-_l), \ldots, (v^-_n)] = [v^-_1, v^-_2, \ldots, v^-_l/\Delta^+_l, \ldots, v^-_n] \tag{19}$$

For the change in criterion l, we present the four new corresponding separation measures as shown in Eqs. 20-23. In these expressions the information related to the lth criterion is extracted from the original summation sign, while the remaining criteria are kept in their original form as the last term. The new separation measures are as follows.

$$\bar{S_p^-} = \sqrt{([(1/(\Delta^+_l))^2 - 1](v_{pl} - v^-_l)^2 + (S^-_p)^2} \tag{20}$$

$$\bar{S_q^+} = \sqrt{((w_l - v_{ql}/(\Delta^+_l))^2 - (w_l - v_{ql})^2 + (S^+_q)^2} \tag{21}$$

$$\bar{S_p^+} = \sqrt{((w_l - v_{pl}/(\Delta^+_l))^2 - (w_l - v_{pl})^2 + (S^+_p)^2} \tag{22}$$

$$\bar{S_q^-} = \sqrt{([(1/(\Delta^+_l))^2 - 1](v_{ql} - v^-_l)^2 + (S^-_q)^2} \tag{23}$$

We observe here that Δ^+_l plays a significant role, often leading to logical conflicts in the TOPSIS method, particularly when alternatives A_p and A_q exhibit similar preferences. This occurs when $S^-_p S^+_q \cong S^-_q S^+_p$ and $v_{pl} \cong v_{ql}$. To streamline the description of Δ^+_l and to facilitate mathematical inferences, we introduce the following variables:

$$\begin{aligned} \alpha_i &= ((1/(\Delta_l^+))^2 - 1)(v_{il} - v_l^-)^2 \\ \beta_i &= (w_l - v_{il}/(\Delta_l^+))^2, \\ \gamma_i &= (w_l - v_{il})^2. \end{aligned} \tag{24}$$

If we consider only the case where a new alternative is added ($\Delta_l^+ \geq 1$), with $0 \leq v_l^- \leq v_{il} \leq v_l^+ \leq w_l$, then it readily appears that $-1 \leq \alpha_i \leq 0$ and $0 \leq \gamma_i \leq \beta_i \leq w_l$. Based on the above variable setting, $\bar{S}_p^- \bar{S}_q^+ - \bar{S}_p^+ \bar{S}_q^-$ is reformulated as:

$$\sqrt{\alpha_p + (S_p^-)^2)}\sqrt{(\beta_q - \gamma_q + (S_q^+)^2)} - \sqrt{(\alpha_q + (S_q^-)^2}\sqrt{(\beta_p - \gamma_p + (S_p^+)^2}, \tag{25}$$

where $(S_i^-)^2 > |\alpha_i|$, and $(S_i^+)^2 > \beta_i - \gamma_i$.

Under the given assumptions and definitions, we now explore four different scenarios to understand why the TOPSIS method might result in RR occurring under logical decision making. We investigate when this problem arises and when it does not and explore the possibility of avoiding the RR issue. These scenarios encompass the following situations.

1. The new alternative's introduction has no impact on the PIS value for criterion j.
2. The new alternative contributes to an increased PIS value for criterion j (the new alternative excels in criterion j), with A_q exhibiting higher performance in criterion j compared to A_p.
3. The new alternative results in a greater PIS value for criterion j, and A_p demonstrates higher performance in criterion j compared to A_q.
4. The new alternative contributes to a greater PIS value for criterion j, and both A_p and A_q share identical performance values in criterion j.

Scenario 1: $\Delta_l^+ = 1$, indicating that $v_l^+ = \bar{v}_l^+$. The PIS value in the lth criterion remains unchanged. Based on Eq. (24), we determine that:

$$\alpha_p = \alpha_q = 0, \quad \beta_q - \gamma_q = \beta_p - \gamma_p = 0. \tag{26}$$

As a result, we have:

$$\bar{S}_p^- \bar{S}_q^+ - \bar{S}_p^+ \bar{S}_q^- = S_p^- S_q^+ - S_q^- S_p^+ > 0. \tag{27}$$

Therefore, $\delta > 0$, indicating no occurrence of RR. Maintaining $\Delta_l^+ = 1$ effectively prevents the RR phenomenon in the modified TOPSIS method, as Kong [14] and others did.

Scenario 2: $\Delta_l^+ > 1$ and $v_{pl} < v_{ql}$. The PIS value in the lth criterion changes, and the weighted achievement of alternative A_p on criterion l is lower than alternative A_q on criterion $l(v_{pl} < v_{ql})$. We derive the following condition using Eq. (24):

$$-1 \leq \alpha_q \leq \alpha_p \leq 0, \text{ and } 0 \leq \beta_p - \gamma_p \leq \beta_q - \gamma_q. \tag{28}$$

These findings give evidence that $\bar{S}_p^- \bar{S}_q^+ - \bar{S}_p^+ \bar{S}_q^-$ is always greater than 0 when $\Delta_l^+ > 1$ and $v_{pl} < v_{ql}$. Therefore, $\delta > 0$, indicating no occurrence of RR in this scenario.

Scenario 3: $\Delta_l^+ > 1$ and $v_{pl} > v_{ql}$. When the PIS value in the lth criterion changes, and the weighted achievement of alternative A_p on criterion l exceeds that of alternative A_q ($v_{pl} > v_{ql}$), this implies that:

$$-1 \leq \alpha_p \leq \alpha_q \leq 0. \tag{29}$$

Furthermore, we can guarantee that:

$$\begin{cases} \frac{w_l - v_{pl}}{\Delta_l^+}^2 - (w_l - v_{pl})^2 > \frac{w_l - v_{ql}}{\Delta_l^+}^2 - (w_l - v_{ql})^2 & \text{if } v_{pl} + v_{ql} < \frac{2w_l\Delta_l^+}{\Delta_l^+ + 1} \\ \frac{w_l - v_{ql}}{\Delta_l^+}^2 - (w_l - v_{ql})^2 > (w_l - v_{pl}\Delta_l^+)^2 - (w_l - v_{pl})^2 & \text{otherwise} \end{cases} \tag{30}$$

The term w_l directly scales the normalized values and affects the magnitude of both S_p^+ and S_q^+. A larger w_l increases the sensitivity of separation measures to changes in v_{pl} and v_{ql}. Notably, we observe that $v_{pl} + v_{ql} < w_l$ and $w_l \leq 2w_l\Delta_l^+/(\Delta_l^+ + 1) \leq 2w_l$. Hence, $v_{pl} + v_{ql} < 2w_l\Delta_l^+/(\Delta_l^+ + 1)$, implying that $\beta_p - \gamma_p$ consistently exceeds $\beta_q - \gamma_q$. In essence, $(S_p^-)^2$ and $(S_q^-)^2$ notably surpass α_p and α_q, while $(S_p^+)^2$ and $(S_q^+)^2$ are significantly larger than $\beta_p - \gamma_p$ and $\beta_q - \gamma_q$.

In most cases, the condition $\bar{S}_p^- \bar{S}_q^+ > \bar{S}_p^+ \bar{S}_q^-$ remains valid upon introducing a new alternative. However, if $S_p^- S_q^+ \cong S_q^- S_p^+$, then the final ranking between alternatives p and q can vary, especially when $\alpha_p \ll \alpha_q$ and $\beta_p - \gamma_p \gg \beta_q - \gamma_q$. Clearly, Δ_l^+amplifies the difference between α_p and α_q, leading to an expanded gap. Similarly, the disparity between $\beta_p - \gamma_p$ and $\beta_q - \gamma_q$ also widens.

Scenario 4: $\Delta_l^+ > 1$ and $v_{pl} = v_{ql}$. The PIS value in the lth criterion changes, and alternative A_p is initially preferred to alternative A_q while having the same weighted achievement on criterion $l(v_{pl} = v_{ql})$. We deduce the following conditions using Eq. (24):

$$-1 \leq \alpha_p = \alpha_q \leq 0, 0 \leq \beta_p = \beta_q \leq w_l^2, \text{and } 0 \leq \gamma_p = \gamma_q \leq w_l^2. \tag{31}$$

These conditions ensure that $\bar{S}_p^- \bar{S}_q^+ - \bar{S}_p^+ \bar{S}_q^- > 0$ is established. Therefore, $\delta > 0$, and no RR phenomenon occurs.

Table 4 summarizes the key characteristics of the four scenarios, highlighting the critical roles of w_l, v_{pl}, v_{ql}, and Δ_l^+ in influencing the occurrence of RR. Scenarios 1, 2, and 4 demonstrate stability under their respective conditions, effectively preventing RR. In contrast, Scenario 3 introduces the possibility of RR due to the interplay between Δ_l^+, w_l, and the relative magnitudes of v_{pl} and v_{ql}. Among these, the RR phenomenon may occur only in Scenario 3, where $\Delta_l^+ > 1$ leads to an outward shift of PIS from its original position. This shift becomes critical when alternative A_p is preferred over A_q due to a higher weighted achievement on criterion $l(v_{pl} > v_{ql})$.

The potential for RR increases in cases where alternatives A_p and A_q have similar preferences, indicated by $S_p^- S_q^+ \cong S_q^- S_p^+$, and v_{pl} is only slightly larger than v_{ql}. Under such conditions, the separation measures closely balance, amplifying the likelihood of RR. For instance, in Example 2, $S_{A_3}^- S_{A_4}^+ = 0.08361 \cong S_{A_4}^- S_{A_3}^+ = 0.08271$, and $v_{A_3C_2} = 0.7407 > v_{A_4C_2} = 0.6667$. This provides evidence that if the ratio of change $\Delta_{C_2}^+$ increases, then the RR phenomenon may occur.

It is evident that RR can conversely be prevented if a modified TOPSIS method finds a way to maintain the ratio of change at $\Delta_l^+ = 1$. Keeping Δ_l^+ at 1 acts as a preventive measure against RR occurrences. This can be accomplished in a number of ways, including by presenting hypothetical extreme options, applying extreme values from the past, or deciding on values based on the expertise and experience of decision makers [14][3]. This observation underscores the importance of carefully analyzing the impact of Δ_l^+ and weight assignments in decision-making processes to mitigate the risk of RR.

Table 4. Key characteristics of the four scenarios

Scenario	Condition	Key Expressions	Outcome	RR Occurrence
1	$\Delta_l^+ = 1$	$\alpha_p = \alpha_q = 0$, $\beta_q - \gamma_q = \beta_p - \gamma_p = 0$	$S_p^- S_q^+ - S_p^+ S_q^- > 0$	No RR occurs
2	$\Delta_l^+ > 1$, $v_{pl} < v_{ql}$	$-1 \leq \alpha_q \leq \alpha_p \leq 0$, $0 \leq \beta_p - \gamma_p \leq \beta_q - \gamma_q$	$S_p^- S_q^+ - S_p^+ S_q^- > 0$	No RR occurs
3	$\Delta_l^+ > 1$, $v_{pl} > v_{ql}$	Complex inequality expressions involving w_l, v_{pl}, v_{ql}, Δ_l^+	RR may occur if $S_p^- S_q^+ \approx S_q^- S_p^+$	Possible RR under specific conditions
4	$\Delta_l^+ > 1$, $v_{pl} = v_{ql}$	$-1 \leq \alpha_p = \alpha_q \leq 0$, $0 \leq \beta_p = \beta_q \leq w_l^2$, $0 \leq \gamma_p = \gamma_q \leq w_l^2$	$S_p^- S_q^+ - S_p^+ S_q^- > 0$	No RR occurs

5. RR Management Strategies

After presenting the analysis in Sections 3 and 4, we propose three RR prevention strategies for TOPSIS as follows:

- Identify non-dominated alternatives

  Non-dominated alternatives in the decision matrix represent candidates that could potentially achieve the top rank in the final decision. Therefore, these alternatives must be carefully examined. If they rank first later, no concern arises. However, if a non-dominated alternative fails to achieve the top rank despite its non-dominated status, the evaluation procedure warrants re-examination.

  In practice, after identifying multiple non-dominated alternatives, the decision maker could: (i) retain the current weight set if the ranking aligns with their preferences, (ii) select an alternative weight set through sensitivity analysis (a posteriori analysis) to explore different preference structures, or (iii) provide additional preference information to refine the weight values for further processing.

- Check the condition of Scenario 3 for close performed alternatives in TOPSIS evaluation In the TOPSIS evaluation process, ranking decisions rely on relative closeness values. Typically, the gap between closely performing alternatives, especially between the second-ranked and top-ranked alternatives, is less than 0.001, which warrants verification.

  In Scenario 3, the condition of $\Delta_l^+ > 1$ and $v_{pl} > v_{ql}$ indicates that RR may occur, particularly with $S_p^- S_q^+ \cong S_q^- S_p^+$. This implies that RR is likely only among alternatives with small performance margins. If such alternatives are not ranked among the top positions, the issue is less critical; however, if they appear in the top ranks, remedial action should be considered.

- Take a remedial action

  The analytical results demonstrate that RR can be completely prevented only when the change ratios associated with the positive and negative ideal solutions satisfy $\Delta_l^+ = \Delta_l^- = 1$.

As shown in Scenario 1, this condition guarantees that the ranking-determining expression remains invariant under the addition or removal of alternatives, thereby preserving the relative ordering of all existing alternatives. In contrast, any increase in Δ_l^+ or decrease in Δ_l^- alters the separation measures and increases the likelihood of RR occurrence. Consequently, maintaining fixed ideal solutions constitutes a necessary and sufficient condition for eliminating RR in TOPSIS.

To entirely prevent RR, the PIS and NIS should therefore be fixed at predefined reference values. Specifically, the PIS is set to $[w_1, w_2, \ldots, w_n]$ (or [1, 1,. . . , 1] if without consider criteria weight) and the NIS to [0, 0,. . . ,0], corresponding to the natural bounds of normalized criteria. This ensures $\Delta_l^+ = \Delta_l^- = 1$ for all criteria, independent of the set of evaluated alternatives.

Operationally, this is accomplished by augmenting the decision matrix with a reference alternative that exhibits maximum values for benefit criteria and minimum values for cost criteria. Consequently, the PIS and NIS values become $[w_1, w_2, \ldots, w_n]$ and [0, 0,. . . , 0], respectively. TOPSIS is then applied to this augmented matrix, with the reference alternative excluded from the final ranking.

6. Discussions

Despite some occurrences of RR in TOPSIS, the literature suggests two approaches to mitigate the problem: (i) using linear normalization on the achievements under each criterion, and (ii) maintaining PIS and NIS at their maximum values, even when incorporating fictitious alternatives. Additionally, using the Manhattan distance to linearize the choice behavior has been proposed. However, these initiatives only partially alleviate the RR phenomenon, as demonstrated by a counterexample [18]. Linear normalization cannot completely eliminate the phenomenon. From the mathematical analysis, we know that to use fixed PIS and NIS values at 1 and 0, respectively, is a better choice to effectively reduce the likelihood of RR though extra step is needed.

Weight assignments play a critical role in ranking stability. Larger weights increase the sensitivity of separation measures to changes in alternative performance. In Scenario 3, when $v_{pl} > v_{ql}$, the rankings between alternatives may become unstable, especially when the separation measures are nearly balanced (e.g., $S_p^- S_q^+ \cong S_q^- S_p^+$). In contrast, when $v_{pl} \leq v_{ql}$ (Scenarios 2 and 4), the rankings remain stable even when PIS changes.

For a simplified procedure, we assume all criteria are benefit criteria and use a linear normalization method. We make this assumption because converting cost criteria to benefit criteria is a straightforward process [9]. Additionally, the use of linear normalization is a good approximation to traditional TOPSIS with vector normalization, especially in the middle of their contour plot [13].

7. Conclusions

This research provides a comprehensive mathematical analysis of RR in TOPSIS, moving beyond previous heuristic approaches to establish a rigorous theoretical framework for understanding when and why RR occurs. The study identifies four distinct scenarios that determine RR occurrence, revealing that rank reversal only happens in Scenario 3 when the change ratio $\Delta_l^+ > 1$ and the superior alternative has higher performance on the changed criterion l. The research introduces an innovative gap function δ that serves as a clear mathematical indicator for predicting RR, with $\delta < 0$ signifying when RR will occur.

A critical finding is that many previous examples claiming to demonstrate RR were actually showing the natural behavior of non-dominated alternatives under different weight settings, rather than true rank reversal phenomena. The study's primary recommendation is to maintain the NIS at 0 and PIS at 1 before weight consideration, which effectively prevents RR by keeping the change ratio $\Delta_l^+ = 1$. This can be implemented through various practical methods including extreme fictitious alternatives, historical extreme values, or expert-determined bounds. Additionally, RR management strategies have been presented to demonstrate how to avoid RR in practice.

The research reveals that criterion weights directly influence RR probability, with larger weights increasing sensitivity to changes in ideal solutions, particularly when comparing non-dominated alternatives with similar preferences. These findings provide immediate practical value for decision makers while establishing a theoretical foundation for developing more robust distance-based MCDM methods that minimize RR phenomena.

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