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Finding efficient and properly efficient solutions of MOLFP problems based on linear approaches

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Abstract

This study focuses on solving multi-objective linear fractional programming (MOLFP) problems, which include linear fractional objective functions and a feasible polyhedron set. Based on a linearization technique, two new approaches are presented to determine efficient (Pareto) and properly efficient solutions to MOLFP problems. In the first approach, the efficiency status of an arbitrary feasible point of MOLFP is examined. If this point is not efficient, an efficient projection onto the efficient space is obtained. This method is able to find an efficient solution in just one step in a linear manner; therefore, it needs much less time and computation compared to some famous previously presented methods. Two numerical examples are given to show the applicability and advantages of the new approaches.

Keywords: *multi-objective linear fractional programming, efficient solution, properly efficient solution, trade-offs*

1. Introduction and backgrounds

Multi-objective programming (MOP) is a well-known research field in optimization and operations research, which involves optimizing several objective functions over a given feasible set. Due to inconsistencies among some objectives, efficiency concepts are used in MOP instead of optimality (a feasible solution is efficient if there is no other feasible solution that improves at least one objective while causing no deterioration in the others). In general, there is no guarantee of the existence of a feasible solution that simultaneously optimizes all objective functions. However, many methods have been developed to find efficient solutions to MOP problems, which are based on iterative, scalarization, interactive, and other techniques.

One of the well-known methods for finding efficient solutions in MOP is the weighted sum method [9]. This method converts the MOP problem into a single-objective optimization problem. If the weights

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are non-negative, the optimal solutions are called weakly efficient, and the solutions are defined as efficient if the weights are positive. In addition to the weighted sum approach, the ε-constraint method is also a recognized technique for solving MOP problems. In some cases, when the objective functions are not aggregated, one of the original objectives is optimized while the others are converted into constraints. This approach was first introduced by Haimes and Lasdon [14] and is extensively discussed in [4]. Numerous other studies have been conducted to find efficient solutions in MOP, such as [27], [18], [9], [1], and [12].

On the other hand, several methods have been proposed to assess the efficiency status of a feasible point in MOP. Benson's method [3] is the most well-known of these methods. In this approach, if a feasible point is inefficient, an efficient point that dominates it is identified. It is worth noting that some efficient solutions may have undesirable properties (see [16]), as they can lead to arbitrarily large trade-offs between objectives [11], [16]. To address this issue, the concepts of properly efficient and improperly efficient solutions have been introduced. Efficient solutions with bounded trade-offs between objectives are referred to as properly efficient solutions. Identifying such solutions is crucial for both the theoretical development and practical application of MOP problems.

The concept of properly efficient solutions was first introduced by Kuhn and Tucker [16], and many authors have proposed methods for finding properly efficient solutions in MOP (see, for example, [11], [3], [13], [10], [1], [24] and [15]). It should be noted that one of the most important tools for obtaining an efficient solution with bounded trade-offs, or a properly efficient solution, in MOP is the weighted sum scalarization method. In this method, if the weights are positive and sum to one, the resulting optimal solutions are properly efficient [11].

One significant class of MOP problems is multi-objective linear fractional programming (MOLFP), which consists of multiple linear fractional objective functions and a feasible polyhedral set. In an MOLFP problem, the objective functions are fractional, with linear numerators and denominators. Determining the efficiency status of a feasible solution and identifying efficient and properly efficient solutions using traditional approaches requires solving a fractional programming model, which is not as straightforward as solving a linear problem. It is known that an MOLFP problem with a single objective function is considered a linear fractional programming (LFP) problem. Charnes and Cooper [5] showed how an LFP problem can be solved using linear programming techniques, provided that the sign of the objective function's denominator does not change across the feasible set. Additionally, some interesting studies have addressed solving LFP problems, such as [28] and [20].

Most of the techniques proposed for solving MOLFP problems are based on interactive, iterative, linearization, parametric, and decomposition methods. Dinkelbach [8] solved an MOLFP using a parametric technique. His approach was later extended by Skiscimi and Palocsay [25] and by Schaible and Shi [23]. Metev and Gueorguieva [17] identified a weakly efficient solution using a nonlinear programming problem. Additionally, some approaches for solving MOLFP problems rely on iterative techniques, such as [6], [7] and [29]. It is important to note that most of these techniques aim to identify an efficient solution to the MOLFP problem. In this regard, Mirdehghan and Rostamzadeh [19] introduced a linear approach to determine the efficiency status of feasible solutions in an MOLFP problem. Their method involves obtaining an efficient projection onto the efficient space for an inefficient solution. Perić et al. [21] solved the MOLFP problem using the goal programming method and analyzed the applicability of linearization techniques, including Taylor polynomial linearization approximation, the method of variable change, and a modification of the method of variable change. Since the objective functions in MOLFP are pseudo-convex and quasi-convex (see [2]), Rostamzadeh and Fakharzadeh [22] presented methods to find extreme efficient solutions to MOP problems with quasi-convex objective functions and a polyhedral feasible set; their method allows finding extreme efficient solutions of MOLFP problems.

Among the efficient solutions obtained for an MOLFP problem, some do not establish bounded trade-offs between the objective functions; therefore, such solutions should be filtered. For this reason, finding properly efficient solutions for the MOLFP problem is an important concept. In this study, we first propose an approach to determine the efficiency status of an arbitrary feasible solution in an MOLFP problem and obtain its efficient projection. Unlike the iterative methods mentioned above, which usually require several iterations to identify an efficient solution, our method determines it in a single step by solving a linear problem, simultaneously checking the efficiency status and obtaining an efficient solution. Moreover, we propose an approach to find a properly efficient solution for the MOLFP problem.

The proposed approaches are structured within the framework of linear programming. First, we show that when evaluating a feasible solution under assessment using linear programming, if the optimal value is zero, then the under-assessment solution is efficient. Otherwise, the solution under assessment is inefficient. Moreover, the optimal solution of the linear problem is efficient and if the optimal solution of the proposed model is unique, it is strictly efficient. Since the behavior of our proposed approaches is similar to the well-known weighted sum method, they have a wide range of applications. It is worth noting that our approaches employ linear techniques, whereas in the weighted sum approach, the objective functions are not linear. Additionally, in the proposed approaches, the weights of each criterion function in generating the solutions are also determined.

The rest of this paper is organized as follows. Section 2 presents important notions, definitions and necessary properties for the main discussion. Section 3 is devoted to the primary discussions and methods for finding efficient and properly efficient solutions for MOLFPs. Two numerical examples illustrating our approaches, along with some comparisons with other methods, are presented in Sections 4 and 5. The final section provides concluding remarks.

2. Preliminary concepts

In MOP, the objectives are often in conflict, which makes it generally impossible to find a feasible solution that simultaneously optimizes all of them. Therefore, instead of seeking a single optimal solution, MOP focuses on the concepts of strictly efficient, efficient, weakly efficient and properly efficient solutions. To illustrate these concepts, we first consider the following multi-objective programming model:

$$\begin{array}{ll} \min & f(x) = (f_1(x), f_2(x) \cdots, f_p(x))^t \\ s.t.: & x \in X, \end{array} \tag{1}$$

where for $k = 1, 2, \ldots, p$, $f_k(x)$, are the objective functions and $X \subseteq \mathbf{R}^n$ is the feasible set.

Definition 1. A point $y \in X$ is called an efficient solution or Pareto-optimal solution of (1) if there does not exist another point $x \in X$ such that $f_k(x) \leq f_k(y)$ for $k = 1, 2, \ldots, p$ and $f_j(x) < f_j(y)$ for at least one $j \in \{1, 2, \ldots, p\}$. This condition ensures that y is not dominated by any other feasible solution in X.

Definition 2. A point $y \in X$ is called a weakly efficient solution of (1) if there does not exist another point $x \in X$ such that $f_k(x) < f_k(y)$ for $k = 1, 2, \ldots, p$.

Definition 3. A point $y \in X$ is called a strictly efficient solution of problem (1) if there does not exist another point, $x \in X$, with $x \neq y$, such that $f_k(x) \leq f_k(y)$ for $k = 1, 2, \ldots, p$.

As mentioned earlier, one of the most well-known methods for solving MOP problems is the weighted sum method. This approach transforms the MOP into a single-objective optimization problem. If the weights are nonnegative, the resulting solutions are weakly efficient; in particular, if all weights are positive, the solutions are efficient. The following theorem formalizes this result.

Theorem 1. Suppose that x is an optimal solution of the weighted sum optimization problem $\min\limits_{x \in X} \sum_{k=1}^{p} \lambda_k f_k(x)$ with $\sum_{k=1}^{p} \lambda_k = 1$ where for $k = 1, 2, \ldots, p$, $\lambda_k \geqslant 0$. Then the following statements hold.

1. If for $k = 1, 2, ..., p$, $\lambda_k \geqslant 0$, then x is a weakly efficient solution of (1).
2. If for $k = 1, 2, ..., p$, $\lambda_k > 0$, then x is an efficient solution of (1).
3. If for $k = 1, 2, ..., p$, $\lambda_k \geqslant 0$ and x is the unique optimal solution, then x is a strictly efficient solution of (1).

Proof. Refer to [9]. □

Benson's method [3] is another approach for determining the efficiency status of a feasible point in MOP. In this method, if a given feasible point is not efficient, an efficient point that dominates it is identified. The following theorem formalizes this result.

Theorem 2. The feasible point $x_0 \in X$ is an efficient solution of (1) if and only if the optimal objective value of the following model is zero. Otherwise, the optimal solution of (2) is an efficient solution for (1).

$$\begin{array}{lll} \min & \sum_{k=1}^{p} l_k & \\ s.t.: & f_k(x_0) - l_k - f_k(x) = 0, & k = 1, 2, \ldots, p; \\ & \quad x \in X; & \\ & l_k \geqslant 0, & k = 1, 2, \ldots, p. \end{array} \tag{2}$$

Proof. Refer to [9]. □

By combining the weighted sum method and Benson's method (2), Ehrgott [9] proposed the following model to solve MOP problems. This model evaluates a feasible solution of the MOP and determines its efficiency status. If the given solution is inefficient, the model generates an efficient solution as its projection. Let x^0 be an arbitrary feasible solution of the MOP problem (1), and define $Z_k = f_k(x^0)$. Then, we have:

$$\begin{array}{lll} \min & \sum_{k=1}^{p} \lambda_k (f_k(x) - Z_k) & \\ s.t. & f_k(x) \leqslant Z_k, & k = 1, 2, \ldots, p; \\ & x \in X; & \\ & \sum_{k=1}^{p} \lambda_k = 1; & \\ & \lambda > 0. & \end{array} \tag{3}$$

Theorem 3. If the optimal value of the objective function in (3) is zero, then x^0 is an efficient solution of (1). Otherwise, the optimal solution of (3) is an efficient solution for (1), and if it is unique, then the optimal solution of (3) is strictly efficient.

Proof. Refer to [9]. □

Definition 4. [18] Let x^* be a feasible solution of (1) and for $j \in \{1, 2, \ldots, p\}$, $S_j(x^*) = \{x \in X \mid f_j(x) > f_j(x^*),\ f_i(x) \leqslant f_i(x^*),\ \text{for } i \in \{1, 2, \ldots, p\} \setminus \{j\}\}$. Then $\sup_{x \in S_j(x^*)} \frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)}$ is called the trade-off between objective functions f_i and f_j.

Some efficient solutions of the MOP problem (1) may exhibit undesirable properties, as they can lead to arbitrarily large trade-offs between objectives [11], [16]. To address these issues, the concepts of properly efficient and improperly efficient solutions have been introduced. Identifying such solutions is essential for both the theoretical development and practical applications of MOP problems [18], [27].

Definition 5. [11] A feasible point $y \in X$ is a properly efficient solution of problem 1 if it is efficient and if for some $i \in \{1, 2, \ldots, p\}$ and $x \in X$, $f_i(x) < f_i(y)$, and for $j \in \{1, 2, \ldots, p\} \setminus \{i\}$, $f_j(y) < f_j(x)$, there exists a real number $M > 0$ such that $\frac{f_i(y) - f_i(x)}{f_j(x) - f_j(y)} \leqslant M$.

In other words, properly efficient solutions are those efficient solutions that maintain bounded trade-offs between objectives. An efficient point that does not satisfy this condition is referred to as improperly efficient. In fact, a properly efficient solution with extremely high or low trade-offs may be indistinguishable from a weakly efficient solution from the perspective of a decision-maker.

One widely used method for identifying efficient solutions with bounded trade-offs is weighted sum scalarization; the corresponding theorem is presented below.

Theorem 4. Suppose that x^* is an optimal solution of the weighted sum optimization problem $\min_{x \in X} \sum_{k=1}^{p} \lambda_k f_k(x)$ with $\sum_{k=1}^{p} \lambda_k = 1$ and for $k = 1, 2, \ldots, p$, $\lambda_k > 0$. Then x^* is a properly efficient solution of (1).

Proof. Refer to [9]. □

Let X_{WE}, X_E, X_{SE} and X_{PE} denote the sets of all weakly efficient, efficient, strictly efficient, and properly efficient solutions of problem (1), respectively. Hence, we have $X_{PE} \subseteq X_E \subseteq X_{WE}$ and $X_{SE} \subseteq X_E \subseteq X_{WE}$.

3. MOLFP problem

MOLFP is a significant class of MOP problems that involves multiple linear fractional objective functions and a polyhedral feasible set. MOLFPs are formulated as follows:

$$\begin{array}{ll} Min & f(x) = (\frac{c_1x+\alpha_1}{d_1x+\beta_1}, \frac{c_2x+\alpha_2}{d_2x+\beta_2}, \cdots, \frac{c_px+\alpha_p}{d_px+\beta_p})^t \\ S.\,to: & x \in X = \{x \in R^n : Ax \leqslant b,\, x \geq 0\}, \end{array} \tag{4}$$

where X is a nonempty and bounded set, $A = [a_{ij}]_{m\times n}$, $b = [b_i]_{m\times 1}$, $c_k, d_k \in R^n$, and $\alpha_k, \beta_k \in R$ for $k = 1, 2, \ldots, p$.

Moreover, the feasible set X is a bounded polyhedron, and $d_kx+\beta_k$ is not zero in all feasible solutions of MOLFP problem (4) for $k = 1, 2, \ldots, p$. Here, without loss of generality, we suppose that $d_kx+\beta_k > 0$ for all $x \in X$ and $k = 1, 2, ..., p$.

3.1. Finding the efficient solution of MOLFP

To find efficient solutions to the MOLFP problem (4), we employ model (3). This model determines the efficiency status of a feasible point $x^0 \in X$ in (4). If x^0 is not efficient, model (3) generates an efficient solution as the projection of the under-assessment of the inefficient solution x^0. Accordingly, to solve problem (4), model (3) can be expressed as follows:

$$\begin{array}{ll} \min & \sum_{k=1}^{p} \lambda_k(\frac{c_kx+\alpha_k}{d_kx+\beta_k} - Z_k) \\ s.t. & x \in X = \{x \in R^n : Ax \leqslant b,\, x \geq 0\}; \\ & \frac{c_kx+\alpha_k}{d_kx+\beta_k} \leqslant Z_k, k = 1, 2, \ldots, p; \\ & \sum_{k=1}^{p} \lambda_k = 1; \\ & \lambda_k \geqslant 0, k = 1, 2, \ldots, p, \end{array} \tag{5}$$

where, for $k = 1, 2, \ldots, p$, $Z_k = \frac{c_kx^0+\alpha_k}{d_kx^0+\beta_k}$.

According to Theorems 3, if the optimal value of objective function of (5) is zero, then $x^0 \in X$ is efficient point for problem (4). Also, if $x^0 \in X$ is not efficient, then the optimal solution of (5) as a projection of the under-assessment inefficient solution is considered, and it is an efficient solution for (4). Moreover, if the optimal solution of (5) is unique, it is a strictly efficient solution for (4).

To solve model (5), for $k = 1, 2, \ldots, p$, we choose $\lambda_k = \frac{(d_kx+\beta_k)}{\sum_{k=1}^{p}(d_kx+\beta_k)}$, so that $x \in X$. It is clear that $\sum_{k=1}^{p} \lambda_k = 1$, and since $d_kx+\beta_k > 0$ for $k = 1, 2, \ldots, p$, we have $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) > 0$. By selecting the vector λ in this way, model (5) is transformed into the following fractional linear programming model:

$$\begin{array}{lll} \min & \sum_{k=1}^{p}(\frac{c_kx+\alpha_k}{\sum_{k=1}^{p} d_kx+\beta_k} - Z_k\lambda_k) & \\ s.t. & x \in X = \{x \in R^n : Ax \leqslant b,\, x \geq 0\}; & \\ & \frac{c_kx+\alpha_k}{d_kx+\beta_k} \leqslant Z_k, k = 1, 2, \ldots, p; & \\ & \lambda_k = \frac{d_kx+\beta_k}{\sum_{k=1}^{p} d_kx+\beta_k}, & k = 1, 2, \ldots, p; \\ & \sum_{k=1}^{p} \lambda_k = 1; & \\ & \lambda_k \geqslant 0, & k = 1, 2, \ldots, p. \end{array} \tag{6}$$

To solve the LFP problem (6), we employ the method of Charnes and Cooper [5] to transform this model into an equivalent linear formulation. By setting $t = \frac{1}{\sum_{k=1}^{p} d_k x + \beta_k}$ and $y = xt$, and noting that $d_k x + \beta_k > 0$ for $k = 1, 2, \ldots, p$, therefore, model (6) can be transformed into the following linear programming model:

$$
\begin{array}{lll}
\min & \sum_{k=1}^{p} (c_k y + \alpha_k t - Z_k \lambda_k) & \\
s.t. & Ay \leqslant bt; & \\
& \sum_{k=1}^{p} d_k y + \beta_k t = 1; & \\
& c_k y + \alpha_k t \leqslant (d_k y + \beta_k t) Z_k, & k = 1, 2, \ldots, p; \\
& \lambda_k = d_k y + \beta_k t, & k = 1, 2, \ldots, p; \\
& \sum_{k=1}^{p} \lambda_k = 1; & \\
& \lambda_k \geqslant 0, & k = 1, 2, \ldots, p; \\
& y \geqslant 0; & \\
& t \geqslant 0. &
\end{array}
\tag{7}
$$

Since $\lambda_k = d_k y + \beta_k t$ for $k = 1, 2, \ldots, p$ and $\sum_{k=1}^{p} \lambda_k = 1$, constraint $\sum_{k=1}^{p} d_k y + \beta_k t = 1$ is redundant and can be removed from (7).

Theorem 5. Let (y^*, t^*, λ^*) be an optimal solution of model (7), then $(x^* = \dfrac{y^*}{t^*}, \lambda^*)$ is an optimal solution of model (5).

Proof. First, we claim that $t^* \neq 0$. By contradiction, suppose that $(y, t) = (\overline{y}, 0)$ is a feasible solution of (7). Then, $A\overline{y} \leqslant 0$ and $\overline{y} \geqslant 0$ which implies that $\overline{y}$ is a recession direction of X (the feasible set of (4)), contradicting the assumption that X is bounded. Therefore, $t > 0$ for all feasible solutions, and in particular, $t^* > 0$ for all optimal solutions of (7).
Let (y^*, t^*, λ^*) be an optimal solution of (7) with $t^* > 0$ and define $x^* = \frac{y^*}{t^*}$. We now prove that (x^*, λ^*) is an optimal solution of (5). Since $Ax^* t^* = Ay^* \leqslant bt^*$, it follows that $Ax^* \leqslant b$. Moreover, for $k = 1, 2, \ldots, p$, $c_k y^* + \alpha_k t^* \leqslant (d_k y^* + \beta_k t^*) Z_k$, $\lambda^* \geqslant 0$ and $\sum_{k=1}^{p} \lambda_k^* = 1$; which together imply that (x^*, λ^*) is a feasible solution of (5). We now show that (x^*, λ^*) is indeed an optimal solution of (5). By contradiction, suppose that (x^*, λ^*) is not optimal. Then, there exists a feasible solution $(\overline{x}, \overline{\lambda})$ for (5) such that

$$\sum_{k=1}^{p} \overline{\lambda}_k \left(\frac{c_k \overline{x} + \alpha_k}{d_k \overline{x} + \beta_k} - Z_k\right) < \sum_{k=1}^{p} \lambda_k^* \left(\frac{c_k x^* + \alpha_k}{d_k x^* + \beta_k} - Z_k\right) \tag{8}$$

where $\overline{\lambda}_k = \frac{d_k\overline{x}+\beta_k}{\sum_{k=1}^p d_k\overline{x}+\beta_k}$. Let $\overline{t} = \frac{1}{\sum_{k=1}^p d_k\overline{x}+\beta_k} \geqslant 0$ and $\overline{y} = \overline{x}\overline{t} \geqslant 0$. Then, we have

$$\begin{aligned}
&A\overline{y} = A\overline{x}\overline{t} \leqslant b\overline{t};\\
&\textstyle\sum_{k=1}^p d_k\overline{y} + \beta_k\overline{t} = \overline{t}(\sum_{k=1}^p d_k\overline{x} + \beta_k) = \frac{1}{\sum_{k=1}^p d_k\overline{x}+\beta_k}(\sum_{k=1}^p d_k\overline{x} + \beta_k) = 1;\\
&\tfrac{c_k\overline{y}+\alpha_k\overline{t}}{d_k\overline{y}+\beta_k\overline{t}} = \tfrac{\overline{t}(c_k\overline{x}+\alpha_k)}{\overline{t}(d_k\overline{x}+\beta_k)} \leqslant Z_k, \quad k = 1, 2, ..., p;\\
&\overline{\lambda}_k = \tfrac{d_k\overline{x}+\beta_k}{\sum_{k=1}^p d_k\overline{x}+\beta_k} = \overline{t}(d_k\overline{x} + \beta_k) = d_k\overline{y} + \beta_k\overline{t}, \quad k = 1, 2, ..., p;\\
&\textstyle\sum_{k=1}^p \overline{\lambda}_k = \sum_{k=1}^p \frac{d_k\overline{x}+\beta_k}{\sum_{k=1}^p d_k\overline{x}+\beta_k} = \sum_{k=1}^p \overline{t}(d_k\overline{x} + \beta_k) = \sum_{k=1}^p d_k\overline{y} + \beta_k\overline{t} = 1;\\
&\overline{\lambda}_k = \tfrac{d_k\overline{x}+\beta_k}{\sum_{k=1}^p d_k\overline{x}+\beta_k} \geqslant 0, \quad k = 1, 2, ..., p;\\
&\overline{t} = \tfrac{1}{\sum_{k=1}^p d_k\overline{x}+\beta_k} \geqslant 0;\\
&\overline{y} = \overline{x}\overline{t} \geqslant 0.
\end{aligned} \tag{9}$$

Hence, $(\overline{y}, \overline{t}, \overline{\lambda})$ is a feasible solution of (7). Furthermore, from (8) we obtain $\sum_{k=1}^p(c_k\overline{y}+\alpha_k\overline{t}-Z_k\overline{\lambda}_k) < \sum_{k=1}^p(c_ky^* + \alpha_kt^* - Z_k\lambda_k^*)$ which contradicts the optimality of (y^*, t^*, λ^*). □

3.2. Finding properly efficient solution of MOLFP

As noted, identifying properly efficient solutions is crucial for both the theoretical development and practical applications of multi-objective optimization problems. To determine a properly efficient solution for the MOLFP problem (4), we apply Theorem 4 and compute this solution by solving the following problem:

$$\begin{aligned}
Min \quad & \textstyle\sum_{k=1}^p \lambda_k \frac{c_kx+\alpha_k}{d_kx+\beta_k}\\
S.\,to: \quad & x \in X = \{x \in R^n : \; Ax \leqslant b, \; x \geq 0\};\\
& \textstyle\sum_{k=1}^p \lambda_k = 1;\\
& \lambda_k \geqslant 0, k = 1, 2, \ldots, p.
\end{aligned} \tag{10}$$

Here, for $k = 1, 2, ..., p$; we define λ_k as $\frac{\theta_k(d_kx+\beta_k)}{\sum_{k=1}^p \theta_k(d_kx+\beta_k)}$, where $x \in X$ and θ_k is a positive real number. Since $d_kx + \beta_k > 0$ for $k = 1, 2, \ldots, p$, it follows that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) > 0$ and $\sum_{k=1}^p \lambda_k = 1$. Therefore, according to Theorem 4, any optimal solution of problem (10) corresponding to such λ is a properly efficient solution for the MOLFP problem (4).

It is worth noting that by selecting positive real values for θ_k for $k = 1, 2, \ldots, p$, different weight vectors λ_k can be generated. Consequently, this allows for the computation of multiple properly efficient solutions for problem (4).

By substituting $\frac{\theta_k(d_kx+\beta_k)}{\sum_{k=1}^p \theta_k(d_kx+\beta_k)}$, for $k = 1, 2, \ldots, p$ into model (10) and applying the Charnes and Cooper [5] transformation with $t = \frac{1}{\sum_{k=1}^p \theta_k(d_kx+\beta_k)}$ and $y = xt$, model (10) can be converted into the following linear programming model:

$$\begin{array}{lll} \min & \sum_{k=1}^{p} \theta_k(c_k y + \alpha_k t) & \\ s.t. & Ay \leqslant bt; & \\ & \sum_{k=1}^{p} \theta_k(d_k y + \beta_k t) = 1; & \\ & \lambda_k = \theta_k(d_k y + \beta_k t), & k = 1, 2, \ldots, p; \\ & \sum_{k=1}^{p} \lambda_k = 1; & \\ & \lambda_k \geqslant 0, & k = 1, 2, \ldots, p; \\ & y \geqslant 0; & \\ & t \geqslant 0, & \end{array} \tag{11}$$

where, the constraint $\sum_{k=1}^{p} \theta_k(d_k y + \beta_k t) = 1$ is redundant because for $k = 1, 2, \ldots, p$, $\lambda_k = \theta_k(d_k y + \beta_k t)$ and $\sum_{k=1}^{p} \lambda_k = 1$. Therefore, this constraint can be omitted.

Theorem 6. Let (y^*, t^*, λ^*) be an optimal solution of model (11), then $(x^* = \frac{y^*}{t^*}, \lambda^*)$ is an optimal solution of model (10).

Proof. The proof is the same as the proof of Theorem 5. □

It should be noted that in an MOLFP problem, efficient solutions are not necessarily properly efficient. To illustrate this, consider the following two-objective linear fractional programming problem:

$$\begin{array}{rl} & \min\ f_1(x) = \frac{1}{x} \\ & \min\ f_2(x) = x \\ S.\ to: & \{x \in \Re | 0 < x \leqslant 1\}. \end{array} \tag{12}$$

The point $x^* = 1$ is an efficient solution for (12), because no $x \in X \setminus \{1\}$ dominates $x^* = 1$. However, $x^* = 1$ is not properly efficient. According to Definition 5, consider any $x \in X$ with $x < 1$. Here, $f_2(x) < f_2(1)$, so index $i = 2$ is improving and $j = 1$ is deteriorating. Then we have:

$$\frac{f_1(x) - f_1(1)}{f_2(1) - f_2(x)} = \frac{1/x - 1}{1 - x} = \frac{1}{x}.$$

As $x \to 0^+$, this ratio tends to infinity. Therefore, no finite M can bound this ratio for all improving x, which shows that $x^* = 1$ is not properly efficient.

4. Example 1

Here, we consider a MOLFP problem with two objectives, adapted from Rostamzadeh and Fakharzadeh [22], as follows:

$$\begin{array}{rl} & \min\ f_1(x) = \frac{-x_1 - x_2}{x_1 + 2} \\ & \min\ f_2(x) = \frac{-x_1}{x_2 + 3} \\ s.t. & \frac{-3}{2} x_1 + x_2 \leqslant 4; \\ & x_1 + x_2 \leqslant 11; \\ & 2x_1 + x_2 \leqslant 16; \\ & x_1, x_2 \geqslant 0. \end{array} \tag{13}$$

The feasible region and the efficient solutions of (13) are illustrated in Figure 1. In Rostamzadeh and Fakharzadeh [22], only the extreme efficient points $(2.8, 8.2)$ and $(8, 0)$ were obtained for problem (13), while the other efficient points remained inaccessible. Using the proposed approach, we can identify not only the extreme efficient solutions but also other efficient solutions.

4.1. Finding efficient solutions

To find the efficient solutions of problem (13), we apply model (7) as follows:

$$
\begin{aligned}
& \min \ -2y_1 - y_2 - \lambda_1 Z_1 - \lambda_2 Z_2 \\
s.t. \quad & \tfrac{-3}{2} y_1 + y_2 - 4t \leqslant 0; \\
& y_1 + y_2 - 11t \leqslant 0; \\
& 2y_1 + y_2 - 16t \leqslant 0; \\
& -y_1 - y_2 - Z_1 y_1 - 2Z_1 t \leqslant 0; \\
& -y_1 - Z_2 y_2 + 3Z_2 t \leqslant 0; \\
& y_1 - \lambda_1 + 2t = 0; \\
& y_2 - \lambda_2 + 3t = 0; \\
& \lambda_1 + \lambda_2 = 1; \\
& y_1, y_2 \geqslant 0; \\
& \lambda_1 \geqslant 0; \\
& \lambda_2 \geqslant 0; \\
& t \geqslant 0,
\end{aligned}
\tag{14}
$$

where $Z_1 = f_1(x^0)$ and $Z_2 = f_2(x^0)$, and x^0 is a feasible solution of problem (13). For each arbitrary feasible point, referred to as an under-assessment feasible solution, we solve model (14) to obtain its projection onto the efficient space of (13). The optimal solutions of problem (14) which correspond to the efficient solutions of (13) for various under-assessment feasible points along with their efficiency statuses, are presented in Table 1. The weights of each objective function used in determining the efficient solutions are also reported in column 5 of Table 1. Note that if (y^*, t^*, λ^*) is the unique optimal solution of model (14), then $x^* = \frac{y^*}{t^*}$ is a strictly efficient solution for the MOLFP problem (13). By solving model (14) for multiple feasible points of problem (13), we identify the extreme points B, C, D, and all points on the bold segment $ED \setminus \{E\}$ in Figure 1, in which all these points are efficient solutions of (13). The results for 10 examined feasible solutions of (13) are summarized in Table 1.

The feasible points listed in rows 1 to 4 coincide with their corresponding projection points, confirming their efficiency. In contrast, the feasible solutions in rows 5 to 10 are inefficient; their projections correspond to the extreme points B, C, D, and points on segment $ED \setminus \{E\}$.

4.1.1. Some comparisons

Here, the better performance of the presented technique in Subsection 3.1 relative to some presented techniques such as [6], [7], [29], [19] and [22] are discussed.

In the method proposed by Costa [6], efficient solutions to the MOLFP problem corresponding to a given vector of weights for the objective functions are obtained. The main idea of these methods is to iteratively divide the feasible region and consequently the criterion space into two parts and analyze

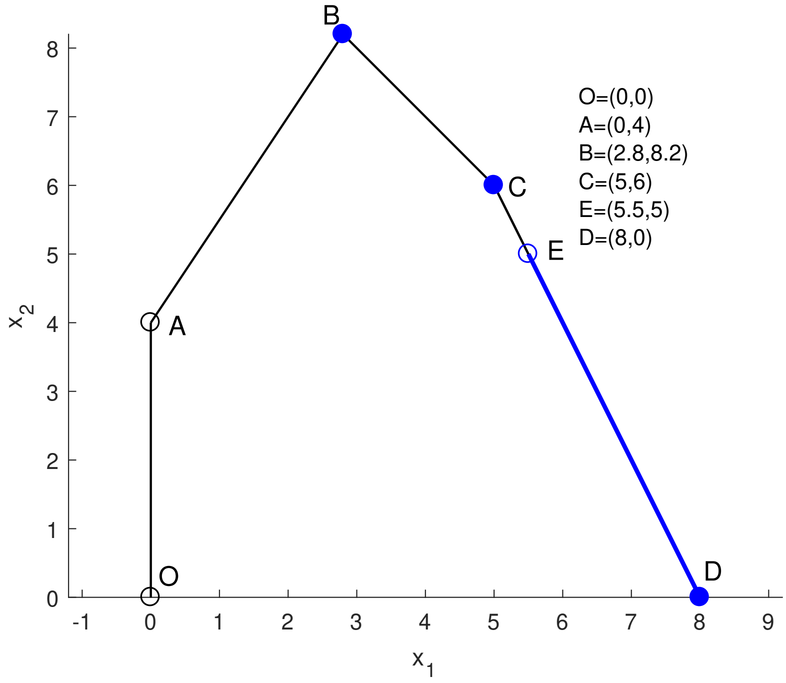

Figure 1. Feasible and efficient regions of problem (13).

Table 1. Finding some efficient solutions of problem (13) using several feasible solutions

Row	Feasible solution (x^0)	Projection (efficient solution)	Efficiency status of the projection	Optimal weight λ^*
1	$(8.0, 0.0)$	$(8.0, 0.0)$	Efficient	$(0.77, 0.23)$
2	$(2.8, 8.2)$	$(2.8, 8.2)$	Efficient	$(0.30, 0.70)$
3	$(5.6, 4.8)$	$(5.6, 4.8)$	Efficient	$(0.49, 0.51)$
4	$(7.1, 1.8)$	$(7.1, 1.8)$	Efficient	$(0.65, 0.35)$
5	$(0.0, 0.0)$	$(8.0, 0.0)$	Efficient	$(0.77, 0.23)$
6	$(0.0, 4.0)$	$(2.8, 8.2)$	Efficient	$(0.30, 0.70)$
7	$(2.9, 2.3)$	$(6.7, 2.6)$	Efficient	$(0.61, 0.39)$
8	$(2.0, 7.0)$	$(2.8, 8.2)$	Efficient	$(0.30, 0.70)$
9	$(5.3, 5.4)$	$(5.0, 6.0)$	Efficient	$(0.44, 0.56)$
10	$(5.5, 5.0)$	$(5.0, 6.0)$	Efficient	$(0.44, 0.56)$

each of them. Specifically, if the maximum weighted-sum model lies within one part, the other part is discarded, and this process is repeated for the remaining region. The division of the feasible region is performed approximately at the midpoint, based on the range of the objective functions derived from the pay-off table. However, in the pay-off table of multi-objective maximization problems with more than two objectives, the minimum value of each objective function is not always well defined. As a result, the boundaries of the criterion space cannot be determined precisely from the pay-off table. The algorithm terminates when the remaining regions become sufficiently small such that the difference between their non-dominated solutions, or equivalently, the boundaries of their criterion spaces on each side, is less than a predefined tolerance.

In 2009, Costa and Alves [7] proposed another method for determining the efficient solutions of the MOLFP problem. This approach is conceptually similar to Costa's earlier method, but it differs in the way the objective functions are aggregated. Instead of employing a traditional weighted-sum model, Costa and Alves introduced a formulation based on the maximum relative deviation between the objective function values and a reference point in the objective space. In this method, rather than assigning fixed weights to each objective, a measure is defined to minimize the deviation of each objective value from its corresponding value at a chosen reference point, while maintaining a balance among all objectives. A small positive parameter is included in the model to control precision and to avoid solution multiplicity. Consequently, this method provides an alternative way to generate efficient solutions that are located near selected reference points in the objective space, thus offering a more comprehensive representation of the Pareto front.

In [29], Valipour et al. another method introduced for finding the efficient solutions of the MOLFP problem. This approach is based on the parametric Dinkelbach method [8], which is an extension of Dinkelbach's algorithm for solving single-objective fractional programming problems. The idea is to reformulate the MOLFP as a sequence of weighted linear optimization problems, where in each iteration, a parameter vector is updated to gradually approach the efficient frontier. Specifically, for a given parameter vector, a corresponding weighted-sum problem is solved. When the optimal value of this problem approaches zero, the associated solution is an efficient solution of the original MOLFP. Accordingly, the algorithm iteratively updates the parameter vector so that the optimal value of the weighted problem converges toward zero. In each iteration, one linear programming problem needs to be solved, and the feasible region of each subsequent iteration becomes a subset of the previous one.

Mirdahghan et al.[19] proposed two linear programming-based approaches for solving MOLFP problems. In the first approach, they presented a method to determine the efficiency status of an arbitrary feasible solution. In the second approach, they extended this idea to not only identify the efficiency status but also to obtain an efficient projection of a given feasible solution. In both approaches, corresponding linear programming problems are constructed with respect to the original MOLFP problem. They showed that, in the first approach, if the optimal value of the proposed model is zero, the assessed feasible solution is efficient. In the second approach, the optimal solution of the constructed LP model is weakly efficient; moreover, if this optimal solution is unique, it becomes an efficient solution.

Rostamzadeh and Fakharzadeh [22] proposed two methods for finding strictly efficient and weakly efficient solutions of MOP problems, in which the objective functions are pseudo-convex and the feasible region is polyhedral. Their approaches are based on the gradient information of the objective functions and the simplex-like table, which are used to identify the efficient extreme points of the feasible region. Since pseudo-convex functions are also quasi-convex, one can use these gradient–simplex-based methods to solve MOFP problems.

Some of the advantages of the proposed model in Subsection 3.1 over the mentioned models are summarized below.

1. In [6], [7], [29] and [22], iterative algorithms are employed, in which an efficient solution of the MOLFP is usually obtained after several iterations and by solving multiple subproblems, while in the proposed technique, the efficient solution is obtained in a single stage by solving only one linear programming problem.

2. In [6] and [7], the proposed iterative algorithms require the computation of two pay-off tables in each iteration, which involves solving p (the number of objective functions) linear programming

problems per iteration. Similarly, in [29], one linear programming problem must be solved in each iteration until an efficient solution is obtained. In contrast, in our proposed technique, only a single linear programming problem is required to be solved.

3. In [19], an efficient or strictly efficient solution of the MOLFP is obtained by solving two linear programming problems. However, in the technique presented in Subsection 3.1, an efficient or strictly efficient solution of the MOLFP is achieved by solving only one linear programming problem.

4. In [22], only the extreme efficient solutions of the MOLFP are obtained, whereas in the proposed technique, all efficient solutions can be determined.

Based on the above points, the proposed approach for determining efficient solutions of MOLFPs is significantly less time-consuming and more computationally efficient than the previously mentioned methods. Moreover, it can identify multiple efficient solutions if they exist. From a computational complexity perspective, since the proposed method is formulated as a linear programming problem, it can be solved in polynomial time using well-established algorithms such as the simplex method or interior-point methods. Therefore, the overall computational cost is predictable and relatively low compared to iterative methods that require solving multiple linear or nonlinear problems per iteration. In particular, the Costa [6] and Costa and Alves [6] algorithms require, in each iteration, the decomposition of the criterion space and the computation of pay-off tables, which increases computational complexity. In [29], a linear programming problem similar in size to our model (in terms of constraints and variables) is solved at each iteration; however, multiple iterations are necessary, resulting in a higher overall computational effort. Similarly, in [19], each efficient solution requires solving two linear programming problems, both have a size similar to our model, which also increases the total computational cost. In [22], each iteration involves a simplex-like table combined with gradient calculations of the objective functions, further increasing computational complexity.

4.2. Finding properly efficient solutions

In the following, by using model (11), we find the properly efficient solutions of (13):

$$
\begin{aligned}
& \min \ \theta_1(-y_1 - y_2) - \theta_2 y_2 \\
s.t. \quad & \tfrac{-3}{2} y_1 + y_2 - 4t \leqslant 0; \\
& y_1 + y_2 - 11t \leqslant 0; \\
& 2y_1 + y_2 - 16t \leqslant 0; \\
& \theta_1(y_1 + 2t) - \lambda_1 = 0; \\
& \theta_2(y_2 + 3t) - \lambda_2 = 0; \\
& \lambda_1 + \lambda_2 = 1; \\
& y_1, y_2 \geqslant 0; \\
& \lambda_1 \geqslant 0; \\
& \lambda_2 \geqslant 0; \\
& t \geqslant 0,
\end{aligned}
\tag{15}
$$

where θ_1 and θ_2 are positive real numbers, and by varying their values, different weights are assigned to the objective functions. In this regard, we apply the presented algorithm by Smith and Tromble [26] (see

Appendix A). By assigning various uniformly random values to θ_1 and θ_2 and solving (15), two properly efficient solutions and their corresponding objective function weights for problem (13) are obtained. Some of these results are presented in Table 2. It is worth noting that although many uniformly random values were considered for θ_1 and θ_2, only two properly efficient solutions, $D = (8.0, 0.0)$ and $B = (2.8, 8.2)$, were found. Specifically, when $\theta_1 > \theta_2$, the solution $B = (2.8, 8.2)$ is obtained, and when $\theta_1 < \theta_2$, the solution $D = (8.0, 0.0)$ is obtained. As illustrated in Figure 1, these points correspond to the extreme points.

Table 2. Finding properly efficient solutions of (13) using different values for θ_1 and θ_2.

Row	θ_1 and θ_2	Properly efficient solutions	Optimal weight (λ^*)
1	$\theta_1 = 0.2127$, $\theta_2 = 0.5961$	$(8.0, 0.0)$	$(0.77, 0.23)$
2	$\theta_1 = 0.5961$, $\theta_2 = 0.2127$	$(2.8, 8.2)$	$(0.46, 0.54)$
3	$\theta_1 = 0.2188$, $\theta_2 = 0.7369$	$(8.0, 0.0)$	$(0.63, 0.37)$
4	$\theta_1 = 0.7369$, $\theta_2 = 0.2188$	$(2.8, 8.2)$	$(0.89, 0.11)$

5. Example 2

To demonstrate the applicability of the proposed methods to problems with more than two objectives, we now consider a multi-objective linear fractional programming problem with four objective functions. This example illustrates that the proposed approach can be directly extended to higher-dimensional cases without any modification in the main procedure. Consider the following model.

$$\begin{aligned}
& \min\ f_1(x) = \tfrac{2x_1+3x_2-x_3+4}{x_1+x_2+2x_3+3} \\
& \min\ f_2(x) = \tfrac{4x_1-x_2+5x_3+2}{2x_1+3x_2+x_3+5} \\
& \min\ f_3(x) = \tfrac{-x_1+6x_2+2x_3+1}{3x_1+x_2+2x_3+2} \\
& \min\ f_4(x) = \tfrac{3x_1+2x_2+4x_3+3}{x_1+4x_2+3x_3+4} \\
s.t.\ & x_1 + 2x_2 + x_3 \leqslant 15; \\
& 3x_1 + x_2 + 2x_3 \leqslant 20; \\
& x_1 + x_2 + x_3 \geqslant 5; \\
& 2x_1 - x_2 + x_3 \leqslant 10; \\
& x_1,\ x_2,\ x_3 \geqslant 0.
\end{aligned} \tag{16}$$

To find the efficient solutions of problem (16), we apply model (7). Then the transformed linear program becomes

$$\begin{aligned}
&\min\ 8y_1 + 10y_2 + 10y_3 + 10t - \lambda_1 Z_1 - \lambda_2 Z_2 - \lambda_3 Z_3 - \lambda_4 Z_4 \\
s.t.\ \ & y_1 + 2y_2 + y_3 - 15t \leqslant 0; \\
& 3y_1 + y_2 + 2y_3 - 20t \leqslant 0; \\
& y_1 + y_2 + y_3 - 5t \geqslant 0; \\
& 2y_1 - y_2 + y_3 - 10t \leqslant 0; \\
& 2y_1 + 3y_2 - y_3 + 4t - (y_1 + y_2 + 2y_3 + 3t)Z_1 \leqslant 0; \\
& 4y_1 - y_2 + 5y_3 + 2t - (2y_1 + 3y_2 + y_3 + 5t)Z_2 \leqslant 0; \\
& -y_1 + 6y_2 + 2y_3 + t - (y_1 + y_2 + 2y_3 + 2t)Z_3 \leqslant 0; \\
& 3y_1 + 2y_2 + 4y_3 + 3t - (y_1 + 4y_2 + 3y_3 + 4t)Z_4 \leqslant 0; \\
& y_1 + y_2 + 2y_3 + 3t - \lambda_1 = 0; \\
& 2y_1 + 3y_2 + y_3 + 5t - \lambda_2 = 0; \\
& y_1 + y_2 + 2y_3 + 2t - \lambda_3 = 0; \\
& y_1 + 4y_2 + 3y_3 + 4t - \lambda_4 = 0; \\
& \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1; \\
& y_1,\ y_2,\ y_3 \geqslant 0; \\
& \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geqslant 0; \\
& t \geqslant 0,
\end{aligned} \tag{17}$$

where $Z_1 = f_1(x^0)$, $Z_2 = f_2(x^0)$, $Z_3 = f_3(x^0)$ and $Z_4 = f_4(x^0)$ and x^0 is a feasible solution of problem (16).

For each arbitrary feasible point of model (16), we solve model (17) to obtain its projection onto the efficient space of (16). The resulting efficient solutions of model (16), corresponding to various under-assessment feasible points, are summarized in Table 3. Additionally, model (17) provides the weights associated with each objective function in determining the efficient solutions, which are reported in column 5 of Table 3.

Table 3. Finding some efficient solutions of problem (16) using several feasible solutions.

Row	Feasible solution (x^0)	Projection (efficient solution)	Efficiency status of the projection	Optimal weight λ^*
1	$(5, 0, 0)$	$(4.62, 0.30, 0.06)$	Efficient	$(0.20, 0.38, 0.17, 0.25)$
2	$(0, 5, 0)$	$(0, 5, 0)$	Efficient	$(0.13, 0.34, 0.12, 0.41)$
3	$(0, 0, 5)$	$(0, 0, 5)$	Efficient	$(0.24, 0.19, 0.22, 0.35)$
4	$(0, 5, 2)$	$(0, 5, 2)$	Efficient	$(0.16, 0.29, 0.15, 0.4)$
5	$(4.4, 0.4, 1)$	$(3.90, 0.27, 0.85)$	Efficient	$(0.21, 0.34, 0.18, 0.27)$
6	$(0, 7, 0.5)$	$(0, 4.8, 0.19)$	Efficient	$(0.14, 0.33, 0.12, 0.40)$
7	$(0, 7.5, 0)$	$(0, 7.5, 0)$	Efficient	$(0.13, 0.34, 0.12, 0.42)$

Now, using model (11), we determine the properly efficient solutions of problem (16):

$$
\begin{aligned}
& \min\ \theta_1(2y_1 + 3y_2 - y_3 + 4t) + \theta_2(4y_1 - y_2 + 5y_3 + 2t) + \\
& \theta_3(-y_1 + 6y_2 + 2y_3 + t) + \theta_4(3y_1 + 2y_2 + 4y_3 + 3t) \\
s.t.\ & y_1 + 2y_2 + y_3 - 15t \leqslant 0; \\
& 3y_1 + y_2 + 2y_3 - 20t \leqslant 0; \\
& y_1 + y_2 + y_3 - 5t \geqslant 0; \\
& 2y_1 - y_2 + y_3 - 10t \leqslant 0; \\
& \theta_1(y_1 + y_2 + 2y_3 + 3t) - \lambda_1 = 0; \\
& \theta_2(2y_1 + 3y_2 + y_3 + 5t) - \lambda_2 = 0; \\
& \theta_3(y_1 + y_2 + 2y_3 + 2t) - \lambda_3 = 0; \\
& \theta_4(y_1 + 4y_2 + 3y_3 + 4t) - \lambda_4 = 0; \\
& \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1; \\
& y_1,\ y_2,\ y_3 \geqslant 0; \\
& \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geqslant 0; \\
& t \geqslant 0,
\end{aligned}
\tag{18}
$$

where, θ_1, θ_2, θ_3, and θ_4 are positive real parameters whose variations determine the weights of the objective functions. Using the proposed algorithm by Smith and Tromble [26] in Appendix A, we assign multiple uniformly random values to these parameters and solve problem (18). The resulting properly efficient solution for problem (16), along with the associated weights, is summarized in Table 4. Despite testing numerous random values, only one properly efficient solution, $(0, 7.5, 0)$, was identified. This point is the only properly efficient point for problem (16).

Table 4. Finding properly efficient solutions of (16) using different values for θ_1, θ_2, θ_3 and θ_4.

Row	Vector $(\theta_1, \theta_2, \theta_3, \theta_4)$	Properly efficient solutions	Optimal weight (λ^*)
1	$(0.1241, 0.6723, 0.089, 0.0921)$	$(0, 7.5, 0)$	$(0.13, 0.33, 0.12, 0.42)$
2	$(0.0839, 0.1556, 0.3043, 0.3162)$	$(0, 7.5, 0)$	$(0.05, 0.23, 0.15, 0.57)$
3	$(0.1564, 0.7939, 0.0073, 0.0349)$	$(0, 7.5, 0)$	$(0.07, 0.88, 0.002, 0.04)$

6. Conclusion

MOLFP problems play a vital role in a variety of fields, including resource allocation, transportation, production planning, performance evaluation and finance. They serve as important planning tools within the framework of operations research. MOLFPs have a distinctive structure compared to other multi-objective programming problems: their feasible sets are polyhedral, and their objective functions are fractional, with both numerators and denominators being affine functions. Like other multi-objective problems, identifying efficient and properly efficient solutions in MOLFPs is crucial. In this paper, we propose two linear programming models to obtain such solutions. The first model employs a linear programming technique to determine the efficiency status of an arbitrary feasible solution and to generate an efficient solution. If the initial feasible solution is not efficient, the method produces an efficient or strictly efficient solution that dominates the under-assessment point. Compared to previously proposed methods such as [6], [7], [29], and [19], this approach requires significantly less computational effort and time. The second linear programming model is designed to identify properly efficient solutions for MOLFP problems. Both proposed models utilize the weighted sum technique from multi-objective programming, allowing not only the determination of efficient and properly efficient solutions but also the computation of the weights associated with each objective function in generating these solutions.

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A. Appendix

The MATLAB code for generating uniform random numbers, presented by Smith and Tromble [26], is as follows:

```
function result = simplexUnit(n)
    %Determine n-1 random numbers (uniform) and sort
    x = sort(rand(1, n-1));
    % Add 0 and 1
    x = [0, x, 1];
    % Determine difference vector y
    x1 = x(1, 2:n+1);
    x2 = x(1, 1:n);
    y = x1 - x2;
    % Determine random value z and take n-th root
    z = rand^(1/n);
    % Multiply component-wise
    result = y .* z;
end
```