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A new and simple method to get the priority vector for reciprocal matrices

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Abstract

A new optimization method is proposed to find the priority vector of a non-consistent reciprocal matrix, which leads to the solution of a constrained least squares problem, specially suited for large matrices. Two different approaches are used to solve this problem, the first approach by using the QR decomposition and a second approach by performing the singular value decomposition leading to simple solutions. In addition, a comparison of the proposed method with other methods, used in the consulted bibliography, is made to obtain the priority vector and finally an illustrative example is shown.

Keywords: *AHP, Comparison matrices, Priority Vector, Least Squares Method.*

1. Introduction

Analytic hierarchy process (AHP) is one of the most important methods in leading multi-attribute decision-aiding model. This process is designed to help make better choices when an expert (a decision maker) deals with complex decisions involving several choices, and it was developed by T. Saaty [12]. AHP has been applied in several areas, such as business, manufacturing, industry, government, education, ... [7].

The expert evaluates the possible choices by using pairwise comparisons between the alternatives forming a pairwise comparison matrix $A = (a_{ij})_{i,j=1}^n$. When the expert makes the comparisons, he can use concrete data about the choices and his judgments about the elements employing his knowledge about the matter. Therefore, human judgments is used in performing the evaluations. Each entry $a_{ij} > 0$ of a pairwise comparison matrix measures the preference of alternative i over alternative j, and if we use a multiplicative criterion, $a_{ij}a_{ji} = 1$ holds. Evidently, from this we deduce $a_{ii} = 1$, which says that an alternative i is equal to itself.

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In the final step of the pairwise comparison process, numerical weights (the vector of priorities) are computed for each of the decision alternatives. These weights represent the relative importance of each alternative: The largest (smallest) coordinate corresponds to the best (worst) alternative.

Unfortunately, due to the human condition, the expert does not express his opinions in a completely consistent manner. This means that for a pairwise comparison matrix, there can exist three alternatives i, j, k such that $a_{ij}a_{jk} \neq a_{ik}$. And this is a major handicap in AHP since the final result of the process (the priority vector) may not be entirely optimal. It is therefore of paramount importance to know how the inconsistency of the expert opinion can be reduced. If the consistency of judgements is unacceptable, it should be improved. Several alternatives, mostly based on various optimization techniques, have been previously proposed to improve consistency (see, for example, [1, 4, 10, 11, 13, 14]).

In this paper we give a new optimization method based on the eigenvector method proposed by Saaty [11], which only requires to solve a linear system.

2. A mathematical review of AHP and the eigenvector method

In this section we review the mathematical foundations of AHP paying attention to the eigenvector method to extract the priority vector.

The set of $n \times m$ real matrices and its subset composed of positive matrices will be denoted, respectively, by $M_{n,m}$ and $M^+_{n,m}$. We simplify the notation for square matrices: $M_n = M_{n,n}$ and $M^+_n = M^+_{n,n}$. Any vector of $\mathbb{R}^n$ will be considered as a column vector, i.e., an element of $M_{n,1}$. If A is a matrix, then A^T will denote the transpose of A. The vector $[1, \ldots, 1]^T \in \mathbb{R}^n$ will be denoted by $\mathbf{e}$.

A **reciprocal matrix** $A = (a_{ij})$ is a square in M^+_n that satisfies $a_{ij}a_{ji} = 1$ for all $1 \leq i, j \leq n$. The element a_{ij} expresses the multiplicative importance of the ith alternative over the jth (many times Saaty's scale is used, see, e.g., [12]). Evidently, by setting $i = j$ in $a_{ij}a_{ji} = 1$, one has that $a_{ii} = 1$. A **consistent matrix** $A = (a_{i,j}) \in M^+_n$ satisfies $a_{ij}a_{jk} = a_{ik}$ for all $1 \leq i, j, k \leq n$. It is trivial to prove that any consistent matrix is reciprocal (set $k = i$ in the previous equality).

It can be easily proved that the rank of any consistent matrix is 1 [2, Theorem 1]. In fact for a consistent matrix $A = (a_{ij}) \in M^+_n$, if we define $\mathbf{v} = [a_{11}, a_{21}, \ldots, a_{n1}]^T$ (the first column of A) one has $v_i/v_j = a_{i1}/a_{j1} = a_{i1}a_{1j} = a_{ij}$ in view of the consistency of A. Also, $\mathbf{v}$ is an eigenvalue of A associated to the eigenvalue n (i.e., $A\mathbf{v} = n\mathbf{v}$).

As it was said, usually the decision maker builds a comparison matrix $A \in M^+_n$ which is reciprocal, but is not consistent. In such case, it is important to determinate the priority vector $\mathbf{v} \in \mathbb{R}^n$, which is not explicitly known to the decision maker.

By using the Perron theorem (see, e.g., [9]), we get that n is the Perron root of any consistent $n \times n$ matrix and the Perron vector is the priority vector. However, if A is a reciprocal matrix which is not consistent, Saaty [12] recommends to use the Perron vector of A as the priority vector under the assumption that A is close to be consistent.

If the inconsistency of a reciprocal matrix A is larger than desired, then we cannot use the Perron vector of A as the priority vector, and some extra computation is needed. Most of the methods to find an appropriate priority vector $\mathbf{v} = [v_1, \ldots, v_n]^T$ of A use $a_{ij} \approx v_i/v_j$ (see [4]). In this paper, we propose a method to find the priority vector which agrees with the idea of the eigenvector method proposed by Saaty and has a very simple solution unlike other ways to find the priority vector because only a linear system is required.

3. A new optimization method to find the priority vector

Let $A \in M_n^+$ be a reciprocal matrix which will be assumed throughout this section non consistent. Also, we will assume that $A - nI_n$ is a non singular matrix (which is the most common of the cases). Hence the linear system $A\mathbf{x} = n\mathbf{x}$ has the unique trivial solution $\mathbf{x} = \mathbf{0}$. Therefore, the following system

$$(A - nI_n)\mathbf{x} = \mathbf{0}, \qquad x_1 + \cdots + x_n = 1$$

has no solution. Observe that the normalisation condition $x_1 + \cdots + x_n = 1$ can be rewritten as $\mathbf{e}^T\mathbf{x} = 1$.

The new method that we propose to find the priority vector $\mathbf{x}$ of A is given by the solution of the following problem.

$$\text{Minimize } \|(A - nI_n)\mathbf{x}\| \text{ restricted to } \mathbf{e}^T\mathbf{x} = 1. \tag{1}$$

Here, $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$, i.e., $\|\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{x}$. Observe that when A is consistent, the solution of the problem formulated in (1) is the priority vector of A. If $A - nI_n$ is singular, the solution of (1) is trivial: solve $(A - nI_n)\mathbf{x} = 0$ constrained to $\mathbf{e}^T\mathbf{x} = 1$.

We have not added the condition $x_i > 0$ in (1) in view of the following reasons:

(i) If A is near to a consistent matrix M, then the solution of (1) is near to the normalized priority vector of M whose components are positive (by Perron's theorem), therefore, if the solution $\mathbf{x}$ has a negative component, then the matrix A is far to be consistent, hence the expert should be asked to modify the comparison matrix to improve its inconsistency. We have given references at the end of section 1 to several articles where various methods are proposed to reduce the inconsistency.

(ii) If only the order of preferences is required, then it is not necessary to ensure the condition $x_i > 0$. However, the expert should be advised that the comparison matrix is very inconsistent, and some judgment should be changed to improve its inconsistency.

We will use the following theorem whose proof will be given in an appendix. We give a purely algebraic proof that only uses standard linear algebra (inner product spaces) avoiding the use of the method of the multipliers' Lagrange. Notice that this last method only establishes the critical points of the objective function, and not the points minimising the objective function.

Theorem 1. Let M and C be matrices $n \times m$ and $m \times p$, respectively; $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{d} \in \mathbb{R}^p$. If there exist $\mathbf{z} \in \mathbb{R}^p$ and $\mathbf{x}_0 \in \mathbb{R}^m$ such that

$$\begin{bmatrix} M^TM & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} M^T\mathbf{b} \\ \mathbf{d} \end{bmatrix}, \tag{2}$$

then

(i) $C^T\mathbf{x}_0 = \mathbf{d}$.

(ii) $\|M\mathbf{x}_0 - \mathbf{b}\| \leq \|M\mathbf{x} - \mathbf{b}\|$ for any $\mathbf{x} \in \mathbb{R}^m$ which satisfies $C^T\mathbf{x} = \mathbf{d}$.

(iii) If the columns of $\begin{bmatrix} M \\ C^T \end{bmatrix}$ are independent and the rows of C^T are independent, then the solution of (2) is unique.

In order to employ Theorem 1 to solve the problem given in (1), we set $M = A - nI_n$, $C = \mathbf{e}$, $\mathbf{b} = \mathbf{0}$, and $\mathbf{d} = 1$. Next, we will prove that condition (iii) of Theorem 2 holds. Since $C^T = \mathbf{e}^T$ is a nonzero row, then the rows of C^T are independent. Furthermore, $M = A - nI_n$ is nonsingular. So, the n columns of M are independent, hence the columns of $\begin{bmatrix} M \\ C^T \end{bmatrix}$ are independent. We have proved the following result.

Theorem 2. The problem stated in (1) has a unique solution.

In the following result we obtain that if we reorder the judgements, then the order of the components of the priority vector changes according to the order of the judgements.

Let us recall that a permutation matrix of order n is a square matrix of the form $P = [\mathbf{e}_{\pi(1)}, \cdots, \mathbf{e}_{\pi(n)}]$, where $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a bijection and $\{\mathbf{e}_1, \ldots \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$. If $\mathbf{v} = [v_1, \ldots, v_n]^T \in \mathbb{R}^n$, then $P\mathbf{v} = [v_{\pi(1)}, \ldots v_{\pi(n)}]^T$. If $A \in M_n$, then PAP^T is obtained from A by performing the permutation π to the rows and columns of A. Finally, if P is any permutation matrix, then P is orthogonal (i.e., $P^{-1} = P^T$).

Theorem 3. Let $A \in M_n^+$ be a reciprocal non consistent matrix and $P \in M_n^+$ be a permutation matrix. Let $\mathbf{x}$ be the unique solution of (1). Then $P\mathbf{x}$ is the unique solution of

$$\text{Minimize } \|(PAP^T - nI_n)\mathbf{y}\| \text{ restricted to } \mathbf{e}^T\mathbf{y} = 1. \tag{3}$$

Proof. Let $\mathbf{y} \in \mathbb{R}^n$. Observe that $(PAP^T - nI_n)\mathbf{y} = P(A - nI_n)P^T\mathbf{y}$. Taking into account that the orthogonal matrices preserve the Euclidean norm and $P\mathbf{e} = \mathbf{e}$, then the problem (3) is equivalent to

$$\text{Minimize } \|(A - nI_n)P^T\mathbf{y}\| \text{ restricted to } \mathbf{e}^T P^T\mathbf{y} = 1.$$

In view of the uniqueness of the solution of (1), the problem (3) has a unique solution $\mathbf{y}$, and $\mathbf{y}$ satisfies $P^T\mathbf{y} = \mathbf{x}$. By using that P is orthogonal, we get $\mathbf{y} = P\mathbf{x}$. □

To solve the constrained least squares problem posed in (1), we will use two different approaches.

3.1. Using the QR decomposition

According to Theorem 1, the unique solution $\mathbf{x} \in \mathbb{R}^n$ of the problem given in (1) must satisfy

$$\begin{bmatrix} (A - nI_n)^T(A - nI_n) & \mathbf{e} \\ \mathbf{e}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}.$$

Hence

$$(A - nI_n)^T(A - nI_n)\mathbf{x} = -z\mathbf{e}, \qquad \mathbf{e}^T\mathbf{x} = 1.$$

Obviously, $z \neq 0$. Define $B = (A - nI_n)^T(A - nI_n)$, a nonsingular matrix. If $\mathbf{x}_0$ is the unique solution of $B\mathbf{x}_0 = \mathbf{e}$, then $\mathbf{x}_0 = B^{-1}\mathbf{e} = -z^{-1}\mathbf{x}$, so $\mathbf{e}^T\mathbf{x}_0 = -z^{-1}\mathbf{e}^T\mathbf{x} = -z^{-1} \neq 0$ and

$$\frac{1}{\mathbf{e}^T\mathbf{x}_0}\mathbf{x}_0 = \mathbf{x} \text{ is the unique solution of problem (1).}$$

Solving $B\mathbf{x}_0 = \mathbf{e}$ directly is not adequate when n is large because the cross matrix $B = (A - nI_n)^T(A - nI_n)$ is ill-conditioned. The QR factorization is suitable to avoid numerical errors. Let $A - nI_n = QR$ be the QR factorization of $A - nI_n$. We have $(A - nI_n)^T(A - nI_n) = (QR)^T(QR) = R^TQ^TQR = R^TR$. Therefore, the system $M\mathbf{x}_0 = \mathbf{e}$ $B\mathbf{x}_0 = \mathbf{e}$ is equivalent to $R^TR\mathbf{x}_0 = \mathbf{e}$. Therefore, the solution $\mathbf{x}$ of problem (1) can be got by solving

$$R^T\mathbf{y} = \mathbf{e}, \quad R\mathbf{x}_0 = \mathbf{y}, \quad \mathbf{x} = \frac{1}{\mathbf{e}^T\mathbf{x}_0}\mathbf{x}_0. \tag{4}$$

Theorem 4. Let $A \in M_{n,n}$ be a reciprocal, non consistent matrix such that $A - nI_n$ is nonsingular. Let QR be the QR factorization of $A - nI_n$, Then the solution $\mathbf{x}$ of Problem (1) is given by (4).

3.2. Using the singular value decomposition

One of the most useful decompositions is the singular value decomposition (SVD), see e.g. [6, Chapter 2]. If $A - nI_n = U\Sigma V^T$ is the SVD of $A - nI_n$, then since the Euclidean norm is preserved under orthogonal premultiplication, problem (2) is equivalent to

$$\text{Minimize } \|\Sigma V^T\mathbf{x}\| \text{ restricted to } \mathbf{e}^T\mathbf{x} = 1. \tag{5}$$

Setting $\mathbf{y} = V^T\mathbf{x}$, then (5) is equivalent to

$$\text{Minimize } \|\Sigma\mathbf{y}\| \text{ restricted to } \mathbf{e}^TV\mathbf{y} = 1 \tag{6}$$

because V is orthogonal. Let $\mathbf{c} = [c_1, \ldots, c_n]^T = V^T\mathbf{e}$ and $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are the singular values of $A - nI_n$. Observe that $\mathbf{c} \neq \mathbf{0}$ (because V is non singular) and $\sigma_n > 0$ (because $A - nI_n$ is non singular). Problem (6) is equivalent to

$$\text{Minimize } \sigma_1^2y_1^2 + \cdots + \sigma_n^2y_n^2 \text{ restricted to } c_1y_1 + \cdots + c_ny_n = 1.$$

By using the method of Lagrange multipliers, exists $\alpha \in \mathbb{R}$ such that $2\sigma_i^2y_i = \alpha c_i$ for $i = 1, \ldots, n$. Therefore,

$$\mathbf{x} = V\mathbf{y} = \frac{\alpha}{2}V\left[\frac{c_1}{\sigma_1^2}, \ldots, \frac{c_n}{\sigma_n^2}\right]^T.$$

If we define $\mathbf{x}_0 = V\left[c_1/\sigma_1^2, \ldots, c_n/\sigma_n^2\right]^T = 2\alpha^{-1}\mathbf{x}$ (observe that $\alpha \neq 0$ since $\mathbf{x} \neq \mathbf{0}$), then

$$\frac{1}{\mathbf{e}^T\mathbf{x}_0}\mathbf{x}_0 = \frac{1}{2\alpha^{-1}\mathbf{e}^T\mathbf{x}}2\alpha^{-1}\mathbf{x} = \mathbf{x} \tag{7}$$

because $\mathbf{e}^T\mathbf{x} = 1$. Therefore, the solution of the problem stated in (1) is the following:

$$\mathbf{x} = \frac{1}{\mathbf{e}^T\mathbf{x}_0}\mathbf{x}_0, \qquad \mathbf{x}_0 = V \begin{bmatrix} c_1/\sigma_1^2 \\ \vdots \\ c_n/\sigma_n^2 \end{bmatrix} = V\Sigma^{-2}\mathbf{c} = V\Sigma^{-2}V^T\mathbf{e}. \tag{8}$$

Theorem 5. Let $A \in M_{n,n}$ be a reciprocal, non consistent matrix such that $A - nI_n$ is nonsingular. Let $U\Sigma V^T$ be the SVD decomposition of $A - nI_n$, Then the solution $\mathbf{x}$ of Problem (1) is given by (7) and (8).

Observe that if we use the SVD approach to solve (1), then matrix U is not necessary to be computed.

4. Comparison with other methods and illustrative examples

There are many methods in the literature to get a priority vector ([8, Chapter 3]), we can cite the Eigenvector Method (EM), the Geometric Mean Method (GMM) and a family of optimizaton methods such as the Least Square Method (LSM), Weighted Least Square Method (WLSM), Least Worst Squares Method (LWSM), and many others. All of these (including the proposed in this article) are based on certain heuristic or on an optimization problem. Hence the obtained rankings depend on the optimization function. Therefore we cannot claim that our method is better than the others because our method is better if we consider the minimization problem (1), but if we change the optimization function, we do not get the optimal result.

However, we think that the minimization problem (1) considered here is a natural consequence of the fact that the priority vector of a consistent matrix $n \times n$ is an eigenvector associated to the eigenvalue n.

It is important to note that many of the methods to get the priority vector require to solve non linear equations. For example, the EM requires to solve the equation $\det(A - \lambda I_n) = 0$; the LSM problem is difficult to solve as the objective function is non linear, usually non convex, and no unique solution exists [3]. The LWSM has the same handicaps as the LSM. Other methods such as Weighted Least Absolute Error Method and Weighted Least Worst Method require to solve a linear program. The unique two methods with closed solutions are the Weighted Least Square Method (WLSM) and the Logarithm Least Square Method, whose solution is the Geometric Mean Method (GMM) [8, Chapter 3]. The method proposed in this paper only requires to solve a square linear system, and we give two robust approaches: the QR and the SVD decomposition. In the appendix, we will give an Octave (Matlab) code showing the easiness of the computation of the priority vector using our approach.

As a final remark, we will say that the solution of the problem (1) is always unique provided that $A - nI_n$ is nonsingular.

Let us see two illustrative examples. Consider the following two 4×4 reciprocal matrices.

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1/2 & 1 & 4 & 2 \\ 1/4 & 1/4 & 1 & 1 \\ 1/3 & 1/2 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 4 & 1/5 \\ 1/2 & 1 & 4 & 2 \\ 1/4 & 1/4 & 1 & 1 \\ 5 & 1/2 & 1 & 1 \end{bmatrix}.$$

In Table 1, we list three priority vectors using three methods: the Eigenvalue Method (EM); the Geometric Mean Method (GMM) and the given by our approach.

Table 1. Comparison of the priority vectors

Matrices	A	B
EM	$[0.4622, 0.2992, 0.1054, 0.1332]^T$	$[0.2513, 0.2904, 0.1109, 0.3474]^T$
GMM	$[0.4644, 0.2967, 0.1049, 0.1340]^T$	$[0.2618, 0.3292, 0.1164, 0.2927]^T$
Our approach	$[0.4665, 0.2998, 0.1022, 0.1316]^T$	$[0.2295, 0.3248, 0.0434, 0.4022]^T$

We get closed results and most important: the ranking of the alternatives of both matrices are not changed.

The first matrix A has, according to Saaty a consistency acceptable because the consistency ratio of A (0.02) is less than 0.1. See [12] for a revisitation of the consistency index and the consistency ratio of a reciprocal, non consistent matrix and its criterion of acceptable consistency.

Observe that to get B, we have modified the (1,4) and the (4,1) entry of A making more inconsistent this matrix B. In fact, the consistency ratio of B is 0.51, which is greater than 0.1, and according to Saaty's criterion, the inconsistency of B is not acceptable.

Bana e Costa and Vansnick [5] formulated the following condition of order preservation. If $A = (a_{ij}) \in M_n^+$ is a reciprocal matrix and $\mathbf{x} = [x_1, \ldots, x_n]^T$ is a priority vector, then A is said to preserve the order of intensity of preference condition (POIP) if there are four alternatives such that i dominates j more that p dominates q (i.e., $a_{ij} > a_{pq} > 1$), then this relationship transfer to the ranking:

$$\frac{x_i}{x_j} > \frac{x_p}{x_q}.$$

The following example was given in [5]. Let us consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 & 9 \\ 1/2 & 1 & 2 & 4 & 9 \\ 1/3 & 1/2 & 1 & 2 & 8 \\ 1/5 & 1/4 & 1/2 & 1 & 7 \\ 1/9 & 1/9 & 1/8 & 1/7 & 1 \end{bmatrix}.$$

The consistency index of this matrix is 0.051, and according to Saaty's criterion, A must be considered acceptable. However, the priority vector using the EM vector is

$$\mathbf{x}_{\text{EM}} = [0.426, 0.281, 0.165, 0.101, 0.0270]^T,$$

which satisfies

$$\frac{x_1}{x_4} = 4.23 > 3.75 = \frac{x_4}{x_5}.$$

Since $a_{14} = 5 < 7 = a_{45}$, the POIP condition is not satisfied by A and the priority vector obtained by the EM.

If we find the priority vector using the GM method we get

$$\mathbf{x}_{\mathrm{GM}} = [0.424, 0.284, 0.169, 0.0977, 0.0257]^T,$$

which violates the POIP condition because $x_1/x_4 = 4.32 > 3.80 = x_4/x_5$.

However, the priority vector solution of (1) is

$$\mathbf{x} = [0.435, 0.282, 0.164, 0.0971, 0.0222]^T.$$

We see that the three priority vectors are close: $\|\mathbf{x} - \mathbf{x}_{\mathrm{EM}}\| = 0.0106$, $\|\mathbf{x} - \mathbf{x}_{\mathrm{GM}}\| = 0.0121$, and $\|\mathbf{x}_{\mathrm{EM}} - \mathbf{x}_{\mathrm{GM}}\| = 5.65 \cdot 10^{-3}$, the ranking of the alternatives obtained by using the these three vectors is the same. But, the priority vector solution of (1) is closer to satisfy the POIP condition because $x_1/x_4 = 4.47 < 4.38 = x_4/x_5$.

Of course, one cannot draw any conclusion from one example. A comparison of several methods would require extensive numerical simulations, possibly via Monte Carlo method. Only then there could be a conclusion which method performs better concerning the POIP condition.

5. Conclusions

We give a new and simple method to find the priority vector of a $n \times n$ non consistent matrix. This method leads to a constrained least square problem, which is solved in an efficient manner by two standard matrix decompositions: the QR and the SVD of a non singular $n \times n$ matrix. We also compare our method with other methods to get the priority vector showing that, although the minimisation problem changes, we get close results. As a future work, it is needed extensive numerical simulations to compare the proposed method with others concerning the POIP condition.

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A. Octave Codes

This simple Octave function computes the priority vector of a given matrix using our approach.

```
function pv(A)
clc
[n,m] = size(A);
%%%%% Minimizing ||A-nI|| via SVD
[U S V] = svd(A-n*eye(n));
x0 = V*S^(-2)*V'*ones(n,1);
disp('Minimizing ||A-nI|| via SVD'); x = x0'/sum(x0)
%%%%% Minimizing ||A-nI|| via QR
[Q R] = qr(A-n*eye(n));
y = R'\ones(n,1);
x0 = R\y;
disp('Minimizing ||A-nI|| via QR'); x = x0'/sum(x0)
```

B. Proof of Theorem 1

The statement (i) is evident from (2). To prove (ii), it is useful to observe that

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}^T\mathbf{v} \tag{9}$$

for all $\mathbf{u}, \mathbf{v}$. Let $\mathbf{x} \in \mathbb{R}^m$ be such that $C^T\mathbf{x} = \mathbf{d}$. By using (9) for $\mathbf{u} = M(\mathbf{x}-\mathbf{x}_0)$ and $\mathbf{v} = M\mathbf{x}_0 - \mathbf{b}$,

$$\begin{aligned}
\|M\mathbf{x}-\mathbf{b}\|^2 &= \|M(\mathbf{x}-\mathbf{x}_0) + M\mathbf{x}_0 - \mathbf{b}\|^2 \\
&= \|M(\mathbf{x}-\mathbf{x}_0)\|^2 + \|M\mathbf{x}_0-\mathbf{b}\|^2 + 2\left(M(\mathbf{x}-\mathbf{x}_0)\right)^T(M\mathbf{x}-\mathbf{b}) \\
&= \|M(\mathbf{x}-\mathbf{x}_0)\|^2 + \|M\mathbf{x}_0-\mathbf{b}\|^2 + 2(\mathbf{x}-\mathbf{x}_0)^T M^T(M\mathbf{x}-\mathbf{b}).
\end{aligned}$$

We get from (2) $M^TM\mathbf{x}_0 + C\mathbf{z} = M^T\mathbf{b}$ hence $M^T(M\mathbf{x}_0-\mathbf{b}) = -C\mathbf{z}$. Therefore,

$$\|M\mathbf{x}-\mathbf{b}\|^2 = \|M(\mathbf{x}-\mathbf{x}_0)\|^2 + \|M\mathbf{x}_0-\mathbf{b}\|^2 - 2(\mathbf{x}-\mathbf{x}_0)^T C\mathbf{z}.$$

Since $C^T\mathbf{x}_0 = \mathbf{d} = C^T\mathbf{x}$, then $C^T(\mathbf{x}-\mathbf{x}_0) = \mathbf{0}$. So,

$$\|M\mathbf{x}-\mathbf{b}\|^2 = \|M(\mathbf{x}-\mathbf{x}_0)\|^2 + \|M\mathbf{x}_0-\mathbf{b}\|^2 \geq \|M\mathbf{x}_0-\mathbf{b}\|^2.$$

To prove (iii), it is enough to prove that the solution of the following linear system

$$\begin{bmatrix} M^TM & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{10}$$

is the trivial: $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$ (if we prove this, since the coefficient matrix of this system is square, then this matrix will be nonsingular). From (10) we get $M^TM\mathbf{x} + C^T\mathbf{y} = \mathbf{0}$ and $C\mathbf{x} = \mathbf{0}$. Hence,

$$0 = \mathbf{x}^T(M^TM\mathbf{x} + C^T\mathbf{y}) = \mathbf{x}^TM^TM\mathbf{x} + \mathbf{x}^TC^T\mathbf{y} = (M\mathbf{x})^T(M\mathbf{x}) + (C\mathbf{x})^T\mathbf{y} = \|M\mathbf{x}\|^2.$$

Therefore, $M\mathbf{x} = \mathbf{0}$. Since the columns of $\left[\begin{smallmatrix} M \\ C \end{smallmatrix}\right]$ are linearly independent, $M\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$.

From $M^TM\mathbf{x} + C^T\mathbf{y} = \mathbf{0}$ we get $C^T\mathbf{y} = \mathbf{0}$. Since the rows of C are linearly independent we get $\mathbf{y} = \mathbf{0}$. The theorem is proved. $\square$