Seat Allocation and Seat Bias under the Jefferson–D'Hondt Method

Daria Boratyn[1]   Wojciech Słomczyński[1,2]   Dariusz Stolicki[1,3]

[1]*Center for Quantitative Political Science, Jagiellonian University, Krakow, Poland*

[2]*Faculty of Mathematics and Computer Science, Jagiellonian University, Krakow, Poland*

[3]*Faculty of International and Political Studies, Jagiellonian University, Krakow, Poland*

*Corresponding author, email address: dariusz.stolicki@uj.edu.pl

Abstract

We prove that under the Jefferson–D'Hondt method of apportionment, given certain distributional assumptions regarding mean rounding residuals, as well as absence of correlations between party vote shares, district sizes (in votes), and multipliers, the seat share of each relevant party is an affine function of the aggregate vote share, the number of relevant parties, and the mean district magnitude. We further show that the first of those assumptions follows approximately from more general ones regarding smoothness, vanishing at the extremes, and total variation of the density of the distribution of vote shares. We also discuss how our main result differs from the simple generalization of the single-district asymptotic seat bias formulae, and how it can be used to derive an estimate of the natural threshold and certain properties thereof.

Keywords: *apportionment, Jefferson–D'Hondt method, seat bias, rounding*

1. Introduction

The **Jefferson–D'Hondt method of apportionment**[1] was originally devised in 1792 by Thomas Jefferson to apportion seats in the U.S. House of Representatives among the states [24], and later it was proposed by a Belgian mathematician and lawyer Victor D'Hondt [8, 10][2] for use in parliamentary elections[3]. It calls for finding such a divisor δ that if each party $i = 1, \ldots, n$, where $n \in \mathbb{N}$ is the number of

---

[1]The Jefferson–D'Hondt method is also known as the Hagenbach-Bischoff method, the method of greatest divisors, the method of highest averages, and the method of rejected fractions. In Israel the method is called the Bader-Ofer method after two members of the Knesset who proposed it in 1975: Yohanan Bader and Avraham Ofer.

[2]The method was also rediscovered by several authors in various contexts between 1860 and 1874, see [28, p. 6] for details.

[3]It is unclear whether D'Hondt knew of Jefferson's work on the subject. James [21, p. 36] has probably been the first to notice that the Jefferson method is equivalent to the D'Hondt method, but it appears that this finding has escaped the attention

parties, were to be allocated as many seats s_i as its number of votes v_i divided by δ, rounding down to the nearest integer, i.e., if $s_i = \lfloor v_i/\delta \rfloor$, no seats would remain unallocated, i.e., $\sum_{i=1}^{n} s_i = s$, where s is the overall number of seats. It is easy to demonstrate that – unless an electoral tie occurs – there are always uncountably many such divisors and they always yield the same apportionment of seats.

Whether because of its relative simplicity or a bias in favor of the largest parties (who are usually setting the rules), the Jefferson–D'Hondt method has attained wide popularity. It is currently employed to allocate all or some parliamentary seats in Albania, Argentina, Aruba, Austria, Belgium, Cape Verde, Chile, Croatia, the Czech Republic, Denmark, the Dominican Republic, East Timor, Faroe Islands, Fiji, Finland, Greenland, Iceland, Israel, Japan, Luxembourg, Macedonia, Montenegro, the Netherlands, Paraguay, Peru, Poland, Portugal, Săo Tome and Príncipe, Serbia, Spain, Suriname, Switzerland, and Turkey [13], and to allocate European Parliamentary seats in a majority of the EU member states, making it one of the most popular proportional representation formulae.

Drawing on earlier works by Janson [23] and Pukelsheim [35] (see Sec. 3), Flis et al. [13] have proposed a **seat allocation formula** describing, under assumptions discussed in Sec. 4, the relationship between the seat share of the i-th party q_i and the vector of aggregate electoral results $(v_1, \ldots, v_n)$, given mean district magnitude m and assuming that parties are sorted degressively by the number of votes

$$q_i = \begin{cases} \hat{p}_i + \hat{p}_i \dfrac{\hat{n}}{2m} - \dfrac{1}{2m} & \text{for } i \leq \hat{n}, \\ 0 & \text{for } i > \hat{n}, \end{cases} \tag{1.1}$$

where $\hat{n}$ is the number of "relevant" parties

$$\hat{n} := \max \left\{ l = 1, \ldots, n : \frac{v_l}{\sum_{j=1}^{l} v_j} > \frac{1}{2m + l} \right\}, \tag{1.2}$$

and $\hat{p}_i$ is the renormalized vote share of the i-th party

$$\hat{p}_i := \frac{v_i}{\sum_{j=1}^{\hat{n}} v_j}. \tag{1.3}$$

However, while they have established that (1.1) is (approximately) accurate as an empirical regularity, they did not explain why it works. In this article, we fill that gap by proving that the seat allocation formula (1.1) holds under its assumptions, and demonstrating how some of the latter follow from certain more fundamental probabilistic assumptions about vote distribution in district-based elections.

In Sec. 2 we discuss the Jefferson–D'Hondt method and some of its mathematical properties. In Sec. 3 we summarize the prior work on seat bias. In Sec. 4 we formalize our main result as Theorem 1 and prove it. In Sec. 5 we present a probabilistic model under which some of the technical assumptions appearing in Theorem 1 are asymptotically justified. Finally, in Sec. 6 we return to the concept of party relevance, using it to define the natural threshold of representation and derive certain properties thereof.

---

of the subsequent generations of scholars. As far as we are aware, Balinski and Young [1, p. 703] have been the first modern authors to credit Jefferson with the original authorship of the method. It should be noted that the Jefferson method was enacted into law (Act of Apr. 14, 1792, c. 23, 1 Stat. 253) and had remained in use for apportioning representatives among the states until 1842.

Throughout the remainder of this article, we use lowercase letters for numbers and density functions, uppercase letters for sets, random variables, and cumulative distribution functions, and bold font for vectors. For variables defined by aggregating over parties, the hat symbol denotes aggregation over relevant parties only, while its absence denotes aggregation over all parties.

Notation. Let:

- $n \in \mathbb{N}_+$ be the *number of parties*;

- $s \in \mathbb{N}_+$ be the *number of seats* to be allocated;

- $c \in \mathbb{N}_+$ be the *number of districts*;

- m_k be the *magnitude of the k-th district*, $k = 1, \ldots, c$, i.e., the number of seats to be allocated in that district; note that $\sum_{k=1}^{c} m_k = s$;

- $m := s/c$ be the *mean district magnitude*;

- v_i be the *aggregate number of votes for the i-th party*, $i = 1, \ldots, n$; we assume that parties are sorted degressively by the number of votes;

- v_i^k be the *number of votes for the i-th party in the k-th district*; obviously, $v_i = \sum_{k=1}^{c} v_i^k$;

- $\hat{n}$ be the *number of relevant parties* given by

$$\hat{n} := \max \left\{ l = 1, \ldots, n : \frac{v_l}{\sum_{j=1}^{l} v_j} > \frac{1}{2m + l} \right\}; \tag{1.4}$$

- $w_k := \sum_{i=1}^{n} v_i^k$ be the *number of votes cast for all parties in the k-th district*;

- $\hat{w}_k := \sum_{i=1}^{\hat{n}} v_i^k$ be the *number of votes cast for relevant parties in the k-th district*;

- $v := \sum_{i=1}^{n} v_i = \sum_{k=1}^{c} w_k$ be the *aggregate number of votes cast for all parties*;

- $\hat{v} := \sum_{i=1}^{\hat{n}} v_i = \sum_{k=1}^{c} \hat{w}_k$ be the *aggregate number of votes cast for relevant parties*;

- $p_i := v_i/v$ be the *(non-renormalized) aggregate vote share of the i-th party*;

- $\hat{p}_i := v_i/\hat{v}$ be the *renormalized aggregate vote share of the i-th party*;

- $p_i^k := v_i^k/w_k$ be the *(non-renormalized) vote share of the i-th party in the k-th district*;

- $\hat{p}_i^k := v_i^k/\hat{w}_k$ be the *renormalized vote share of the i-th party in the k-th district*;

- s_i^k be the *number of seats of the i-th party in the k-th district*; note that $\sum_{i=1}^{n} s_i^k = m_k$;

- $s_i := \sum_{k=1}^{c} s_i^k$ be the *total number of seats of the i-th party*; note that $\sum_{i=1}^{n} s_i = s$;

- $q_i^k := s_i^k/m_k$ be the *seat share of the i-th party in the k-th district*;

- $q_i := s_i/s$ be the *aggregate seat share of the i-th party*.

Notation. Moreover, let:

- Δ_k, $k \in \mathbb{N}_+$, be a $(k-1)$-dimensional *unit simplex*, i.e.,

$$\Delta_k := \left\{ \mathbf{x} \in \mathbb{R}^k_{\geq 0} : \sum_{i=1}^k x_i = 1 \right\};$$ (1.5)

- $\mathcal{G}^l_k$, $k, l \in \mathbb{N}_+$, be a $(k-1)$-dimensional *l-grid simplex*, i.e.,

$$\mathcal{G}^l_k := \left\{ \mathbf{x} \in \Delta_k : l\,\mathbf{x} \in \mathbb{N}^k \right\};$$ (1.6)

- $\langle x_i \rangle^b_{i=a}$ denote the *average* of x_i over $i = a, \ldots, b$, $a < b \in \mathbb{N}$, i.e.,

$$\langle x_i \rangle^b_{i=a} := \frac{1}{b - a + 1} \sum_{i=a}^b x_i;$$ (1.7)

- $X_i^\downarrow$ denote the *i-th largest element* of a sequence (X_j).

2. The Jefferson–D'Hondt Method

2.1. Divisor Methods

Let $i = 1, \ldots, n$. As in this section we focus solely on apportionment as applied to an individual district, aggregate and district-level variables are indistinguishable, wherefore index k will be omitted.

Definition 1 (Apportionment method). An *apportionment method* is a partial function that maps a vote share vector to a seat share vector, $\mathbf{q} : \Delta_n \to \mathcal{G}^m_n$.

Definition 2 (Rounding function). A *rounding function* is a non-decreasing function $\rho : \mathbb{R} \to \mathbb{Z}$ such that $|\rho(x) - x| < 1$ for every $x \in \mathbb{R}$.

Definition 3 (Divisor method of apportionment). An apportionment method $\mathbf{q}$ is called a *divisor method* if and only if there exists a rounding function ρ such that for every $i = 1, \ldots, n$ the seat share of the i-th party is given by

$$q_i = \rho(p_i/\delta)/m,$$ (2.1)

where the *divisor* $\delta \in \mathbb{R}_+$ is such that

$$\sum_{i=1}^n \rho(p_i/\delta) = m.$$ (2.2)

Remark 1. It is often more convenient to use an equivalent form of (2.1):

$$q_i = \rho(p_i\mu)/m,$$ (2.3)

where $\mu := 1/\delta$ is called the *multiplier*.

Proposition 1. If the rounding function ρ is right-continuous, the solution set of (2.2) is either a half-open interval $(\delta_{\inf}, \delta_{\sup}]$ or an empty set. Furthermore, q_i does not depend on the choice of $\delta \in (\delta_{\inf}, \delta_{\sup}]$.

Proof. If ρ is right-continuous, $\delta \mapsto \rho(p_i/\delta)$ is left-continuous and non-increasing. Thus, the sum of such functions over $i \in \{1, \ldots, n\}$, i.e., $\delta \mapsto \sum_{i=1}^{n} \rho(p_i/\delta)$, is also left-continuous and non-increasing. Accordingly, the preimage of every element of its codomain is necessarily either empty or an interval (due to monotonicity) that is right-half-closed (due to left-continuity). We will denote the set of such intervals by $\mathcal{D}$. Since $\mathcal{D}$ is a partition of $\mathbb{R}_+$, each $I \in \mathcal{D}$ must also be left-half-open, as desired.

Uniqueness of q_i for every divisor interval $I \in \mathcal{D}$ follows from $\rho(p_i/\delta)$ being non-increasing for every $i = 1, \ldots, n$ and every $\delta \in I$, and $\sum_{i=1}^{n} \rho(p_i/\delta)$ being constant over I. A sum of weakly monotonic functions is constant if and only if all of them are constant. Thus, q_i does not depend on the choice of $\delta \in I$ for every $i = 1, \ldots, n$. $\qquad\square$

Corollary 1. If the rounding function is right-continuous, $\mu \in [\mu_{\inf}, \mu_{\sup})$, where $\mu_{\inf} := 1/\delta_{\sup}$ and $\mu_{\sup} := 1/\delta_{\inf}$.

Definition 4 (Rounding thresholds). Let $\rho : \mathbb{R} \to \mathbb{Z}$ be a right-continuous rounding function. For every $k \in \mathbb{Z}$, the preimage of k under ρ is a left-half-closed interval. The sequence of *rounding thresholds* of ρ is a sequence $(\rho_k)_{k \in \mathbb{Z}}$ whose elements are given by $\rho_k := \min \rho^{-1}(k)$.

Definition 5 (Electoral quotients). Let ρ be a right-continuous rounding function, and let $(\rho_k)_{k \in \mathbb{Z}}$ be the sequence of its rounding thresholds. The j-th *electoral quotient* of the i-th party, where $j \in \mathbb{N}_+$ and $i = 1, \ldots, n$, is given by $\pi_{i,j} := p_i/\rho_j$.

Note that for every $i = 1, \ldots, n$ the electoral quotients form a decreasing sequence $(\pi_{i,j})_{j \in \mathbb{N}_+}$ such that for every $x \in \mathbb{R}_+$ there exist only finitely many terms satisfying $\pi_{i,j} > x$. Let $(Q_k)_{k \in \mathbb{N}_+}$ be a "merged" sequence given by $Q_{(j-1)n+i} := \pi_{i,j}$ for every $i = 1, \ldots, n$ and every $j \in \mathbb{N}_+$. Since the number of parties is finite, for every $x \in \mathbb{R}_+$ there exist only finitely many terms of $(Q_k)_{k \in \mathbb{N}_+}$ satisfying $Q_k > x$. Accordingly, for every $l \in \mathbb{N}_+$ there exists a (not necessarily unique) l-th largest element of $(Q_k)_{k \in \mathbb{N}_+}$.

Proposition 2. If ρ is a right-continuous rounding function, then $\delta_{\inf} = Q_{m+1}^{\downarrow}$ and $\delta_{\sup} = Q_m^{\downarrow}$.

Proof. By Proposition 1, it is sufficient to establish that for every $k \in \mathbb{N}_+$ and every $\delta \in \mathbb{R}_+$ we have $\sum_{i=1}^{n} \rho(p_i/\delta) < k$ if and only if $\delta > Q_k^{\downarrow}$.

We begin by showing that for every $i = 1, \ldots, n$ and every $\delta \in \mathbb{R}_+$ the following equality holds:

$$\rho(p_i/\delta) = |\{\pi_{i,j} : j \in \mathbb{N} \text{ and } \pi_{i,j} \geq \delta\}|. \tag{2.4}$$

Let $k_i := \rho(p_i/\delta)$. Then $p_i/\delta \in \rho^{-1}(k_i)$. It follows that $p_i/\delta \geq \min \rho^{-1}(k_i) = \rho_{k_i}$, wherefore $p_i/\rho_{k_i} \geq \delta$. By the same reasoning, $p_i/\rho_{k_i+1} < \delta$. Since $\pi_{i,j}$ is decreasing in j, it follows that $\pi_{i,j} \geq \delta$ if and only if $j \in \{1, \ldots, k_i\}$.

Then from (2.4) and the definition of $(Q_k)_{k \in \mathbb{N}_+}$ for every $\delta \in \mathbb{R}_+$ we obtain

$$\sum_{i=1}^{n} \rho(p_i/\delta) = \sum_{i=1}^{n} |\{\pi_{i,j} : j \in \mathbb{N} \text{ and } \pi_{i,j} \geq \delta\}| = |\{l \in \mathbb{N}_+ : Q_l \geq \delta\}|. \tag{2.5}$$

Fix any $k \in \mathbb{N}_+$ and any $\delta \in \mathbb{R}_+$. If $\delta > Q_k^{\downarrow}$, then from (2.5) it follows that

$$\sum_{i=1}^{n} \rho(p_i/\delta) = |\{l \in \mathbb{N}_+ : Q_l \geq \delta\}| \leq |\{l \in \mathbb{N}_+ : Q_l > Q_k^{\downarrow}\}| < k. \tag{2.6}$$

In the other direction, fix any $k \in \mathbb{N}_+$ and any $\delta \in \mathbb{R}_+$ such that $\sum_{i=1}^{n} \rho(p_i/\delta) < k$. Assume on the contrary that $\delta \leq Q_k^{\downarrow}$. From (2.5) it then follows that

$$\sum_{i=1}^{n} \rho(p_i/\delta) = |\{l \in \mathbb{N}_+ : Q_l \leq \delta\}| \geq |\{l \in \mathbb{N}_+ : Q_l \leq Q_k^{\downarrow}\}| \geq k, \tag{2.7}$$

contradicting the assumption that $\sum_{i=1}^{n} \rho(p_i/\delta) < k$, and thereby concluding the proof. $\square$

Remark 2 (Electoral ties). We refer to cases where the set of solutions of (2.2) is empty as *electoral ties*, although they do not necessarily involve two parties having the same number of votes.

Example 2.1. Fix $n = 2$, $p_1 = 2/3$, $p_2 = 1/3$, and $m = 2$, and let $\rho(x) := \lfloor x \rfloor$. Then for any $\delta \in (1/3, 2/3]$ we have $s_1 = 1$, $s_2 = 0$, wherefore $\sum_{i=1}^{n} s_i < m$. However, for $\delta \in (2/9, 1/3]$ we have $s_1 = 2$, $s_2 = 1$, and thus $\sum_{i=1}^{n} s_i > m$. Since the left side of (2.2) is weakly decreasing in δ, we need not check any other intervals to establish that there exists no $\delta \in \mathbb{R}_+$ such that $\sum_{i=1}^{n} s_i = m$. Thus, the election is tied.

Corollary 2. It follows from Proposition 2 that a necessary and sufficient condition for an electoral tie to occur is $Q_{m+1}^{\downarrow} = Q_m^{\downarrow}$. Accordingly, electoral ties are not possible if vote shares are incommensurable. If $m < w$ and vote counts are coprime, electoral ties are also not possible.

Remark 3 (Probability of an electoral tie – continuous models). Assume that vote shares are drawn at random from an absolutely continuous probability distribution on Δ_n. Then the set of points for which electoral ties occur is of Lebesgue measure zero. Thus, we can treat such ties are theoretically negligible.

Remark 4 (Probability of an electoral tie – a discrete toy model). Assume that $m < w$ and that vote counts are drawn independently from $\mathrm{Unif}\{1, \ldots, k\}$, where $k \in \mathbb{N}_+$. The probability of an electoral tie occurring for some value of m is bounded from the above by the probability of vote counts of all parties being coprime, which approaches $1/\zeta(n)$ as $k \to \infty$, where ζ is the Riemann zeta function [30].

Remark 5 (Probability of an electoral tie – empirical data). In a set of $61,416$ Polish local, regional, and national elections held under divisor rules since 1991 (with each electoral district regarded as an individual election), only 11 electoral ties occurred. For a more general discussion of the frequency of electoral ties, see [29]. For the reasons stated in this and the above two remarks, we do not concern ourselves with ties in the present article.

Remark 6 (Alternative formulations). Janson [23] distinguishes three alternative but equivalent formulations of divisor methods besides (2.1) and (2.3):

- **iterative formulation**: seats are distributed iteratively, with k-th seat being awarded to the party with the highest *comparative vote share*, defined for the i-th party as v_i/ρ_{z_i+1}, where z_i is the number of seats already allocated to the i-th party;

- **highest-quotients formulation**: the number of seats of the i-th party is given by

$$s_i = |\{\pi_{i,j} : j \in \mathbb{N}_+\} \cap \{Q_1^{\downarrow}, \ldots, Q_m^{\downarrow}\}|; \tag{2.8}$$

- **quotient-separation formulation**: the number of seats of the i-th party is such that the L_1 norm of the seat count vector, $\|\mathbf{s}\|_1$, equals m and $\max_{i=1,\ldots,n} \pi_{i,s_i+1} < \min_{i=1,\ldots,n} \pi_{i,s_i}$.

Definition 6 (Linear divisor methods). A divisor method is *linear* if and only if it is induced by a right-continuous rounding function $\rho : \mathbb{R} \to \mathbb{Z}$ such that the rounding thresholds are equidistant, i.e., there exists some $\beta \in [0, 1)$ such that $\rho_k = k + \beta$ for every $k \in \mathbb{Z}$.

2.2. Jefferson–D'Hondt

The Jefferson–D'Hondt method is the most popular linear divisor method in use in political elections. Accordingly, we will focus on it throughout the remainder of this paper. Nevertheless, the results presented in the following three sections can be easily generalized to other linear divisor methods.

Definition 7 (Jefferson–D'Hondt method). The *Jefferson–D'Hondt method* is a divisor method of apportionment induced by the floor function $\lfloor \cdot \rfloor$, i.e., a function $\mathbf{q} : \Delta_n \to \mathcal{G}_n^m$ such that

$$q_i = \lfloor p_i/\delta \rfloor /m, \tag{2.9}$$

for every $i = 1, \ldots, n$, where $\delta \in \mathbb{R}_+$ (a *Jefferson–D'Hondt divisor*) is such that

$$\sum_{i=1}^n \lfloor p_i/\delta \rfloor = m. \tag{2.10}$$

Remark 7. It is easy to see that the Jefferson–D'Hondt method is a linear divisor method, with the k-th rounding threshold given by $\rho_k = k$.

Remark 8. Under the original Jefferson proposal, the divisor has been fixed, while the number of seats has been allowed to vary [2]. The mathematical properties of the method are otherwise unaltered.

Remark 9. The iterative formulation of the Jefferson–D'Hondt method was first proposed in 1888 by Hagenbach-Bischoff [15]. It is used in legislative elections in Luxembourg, and it has been used in the United Kingdom for the European Parliament elections before 2019.

Remark 10. The highest-quotients formulation of the Jefferson–D'Hondt method was first introduced by D'Hondt himself [10] in 1885, and is by far the most popular among legislators and political scientists. For instance, all EU countries employing the D'Hondt method for legislative elections – except Luxembourg – employ it in their electoral legislation. Note that this formulation closely resembles an earlier proposal by Burnitz and Varrentrapp [5], who called for a modified version of the Borda count, with each elector ranking no more than m candidates, ranks being translated to scores harmonically, and seats being

awarded to m candidates with the highest scores. Should the whole electorate be divided into perfectly disciplined partisan voting blocks, and should each block unanimously vote for the same candidates in the same order, the Burnitz–Varrentrapp method would be equivalent to the Jefferson–D'Hondt method.

It is well known that the highest quotient formulation and the "standard" formulation (Definition 7) of the Jefferson–D'Hondt method are equivalent, i.e., they always generate an identical allocation of seats. For an early proof, see [12]. To check it quickly, assume κ_i $(i = 1, \ldots, n)$ is the number of seats awarded to the i-th party under the highest quotient method. Clearly, $\sum_{i=1}^{n} \kappa_i = m$. Recall that $\delta \in (Q_{m+1}^{\downarrow}, Q_m^{\downarrow}]$. It follows that $p_i/\kappa_i \geq Q_m^{\downarrow} \geq \delta > Q_{m+1}^{\downarrow} \geq p_i/(\kappa_i + 1)$, and, in consequence, $\kappa_i = \lfloor p_i/\delta \rfloor$ for every $i = 1, \ldots, n$, as desired.

Pukelsheim [35] defines a simple algorithm for finding a divisor given some fixed *divisor initialization* $\delta \in \mathbb{R}_+$:

Algorithm 1 (Jump-and-step). While $\sum_{i=1}^{n} \lfloor p_i/\delta \rfloor \neq m$ do:

- if $\sum_{i=1}^{n} \lfloor p_i/\delta \rfloor < m$, set $\delta \leftarrow \max\{p_i/(\lfloor p_i/\delta \rfloor + 1)\}$;

- if $\sum_{i=1}^{n} \lfloor p_i/\delta \rfloor > m$, set $\delta \leftarrow \min\{p_i/\lfloor p_i/\delta \rfloor\} + \varepsilon$, where $\varepsilon \in (0, 1/m)$.

Remark 11. Popular divisor initializations for the Jefferson–D'Hondt method include:

- the *simple quota*, $\delta_m := 1/m$ [8];

- the *Hagenbach-Bischoff quota*, $\delta_m^{HB} := (\lfloor w/(m+1) \rfloor + 1)/w$ [15];

- the *Gfeller-Joachim-Pukelsheim quota*, $\delta_m^{GP} := (m + n/2)^{-1}$ [14, 25].

For an in-depth discussion of the origins and attributions of the most popular electoral quotas, see [7].

Remark 12. Happacher and Pukelsheim [17, 18] have established the Gfeller-Joachim-Pukelsheim quota to have the unique property of being asymptotically unbiased under the Jefferson–D'Hondt method as the district magnitude approaches infinity.

Remark 13. The highest-quotients formulation of the D'Hondt method is informally equivalent to the jump-and-step algorithm with divisor initialization $\delta = \infty$. This is always the least optimal choice, as for every possible vote share vector it requires exactly m iterations to arrive at a correct divisor.

Remark 14. Let $\lambda_\mu := \sum_{i=1}^{n} \lfloor p_i\mu \rfloor - m$ be the *discrepancy* of the seat allocation under multiplier μ. The distribution of the discrepancies for the Jefferson–D'Hondt method has attracted much scholarly interest. Happacher [16] has provided an analytical formula for the probability distribution of the discrepancy under the assumption that vote shares are drawn from a uniform distribution on the unit simplex. Janson [22, Thm. 7.5] has established that as the number of seats approaches infinity, the discrepancy distribution approaches the Euler-Frobenius distribution. Finally, Heinrich et al. [19] have found that the discrepancy distribution can be approximated by applying standard rounding to a sum of uniformly distributed random variables. By the way of illustration, we plot the discrepancy distribution on the 2-dimensional standard unit simplex in Fig. 1.
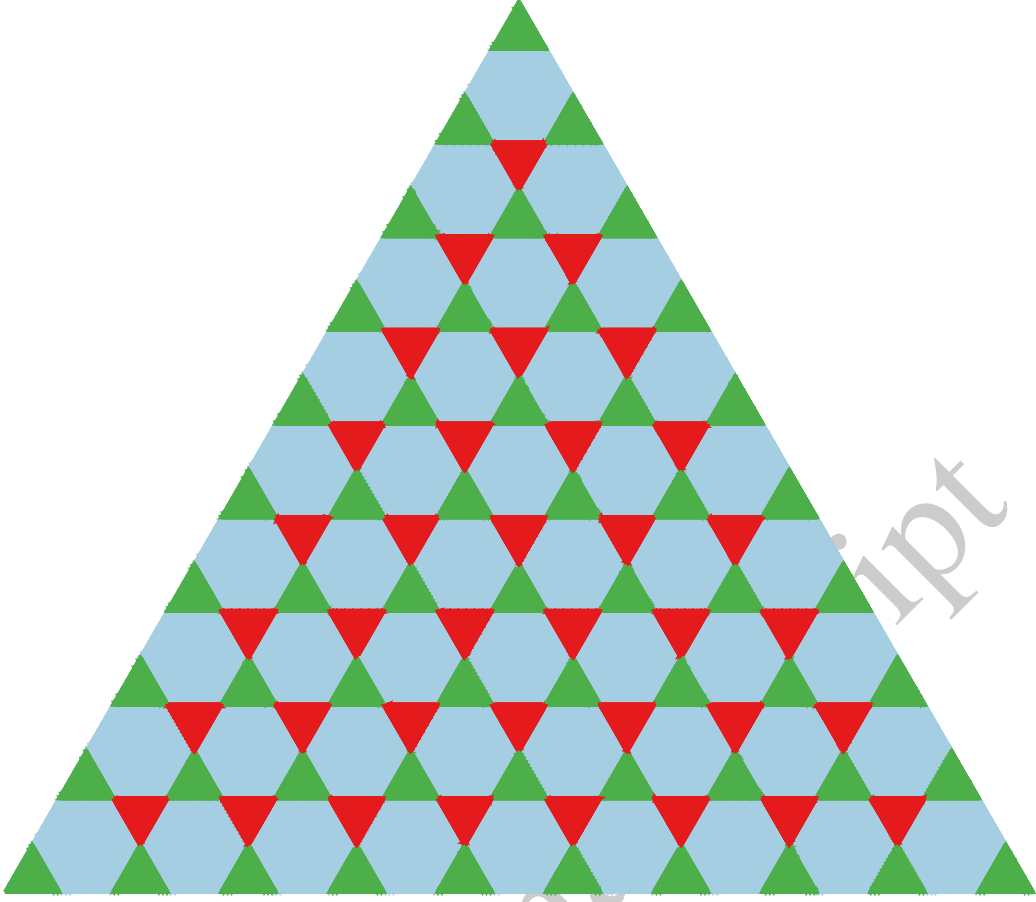
Figure 1. Disrepancy values on the unit simplex Δ_n for $n = 3$, $m = 8$, and $\mu = m + n/2 = 9.5$. Blue, green, and red regions represent, respectively, $\lambda_\mu = 0$, $\lambda_\mu = 1$, and $\lambda_\mu = -1$.

Definition 8. The *rounding residual* of the i-th party under multiplier $\mu \in [\mu_{\inf}, \mu_{\sup})$ is

$$r_i(\mu) := \{p_i\mu\} = p_i\mu - \lfloor p_i\mu \rfloor. \tag{2.11}$$

We can extend this definition for $\mu_{\sup}$:

$$r_i(\mu_{\sup}) := \lim_{\mu \nearrow \mu_{\sup}} r_i(\mu). \tag{2.12}$$

Lemma 1. For every $\mu \in [\mu_{\inf}, \mu_{\sup})$ and every $k = 1, \ldots, n$ the following equality holds:

$$\mu \sum_{i=1}^{k} p_i = \sum_{i=1}^{k} s_i + \sum_{i=1}^{k} r_i(\mu). \tag{2.13}$$

Proof. Recall that by (2.3) for every $\mu \in [\mu_{\inf}, \mu_{\sup})$ we have $s_i = \lfloor p_i\mu \rfloor$. Thus, summing (2.11) over $i = 1, \ldots, k$ we obtain $\sum_{i=1}^{k} r_i(\mu) = \mu \sum_{i=1}^{k} p_i - \sum_{i=1}^{k} s_i$, as desired. $\qquad\square$

Corollary 3. For every $\mu \in [\mu_{\inf}, \mu_{\sup})$ the following equality holds:

$$\mu = m + \sum_{i=1}^{n} r_i(\mu). \tag{2.14}$$

Lemma 2. For every party $i = 1, \ldots, n$, every $k = 1, \ldots, n$, and every $\mu \in [\mu_{\inf}, \mu_{\sup})$ the following equality holds:

$$s_i = \frac{p_i}{\sum_{j=1}^{k} p_j} \left(\sum_{j=1}^{k} s_j + \sum_{j=1}^{k} r_j(\mu) \right) - r_i(\mu). \tag{2.15}$$

Proof. Note that for every $x \in \mathbb{R}_+$, seat allocations and rounding residuals are invariant under simultaneous multiplication of the vote shares by x and division of the multiplier by x. Then, (2.15) follows immediately from Lemma 1 and (2.11). □

Recall that in (1.2) we have introduced the concept of *relevant parties*. Note that if the sum of seats over relevant parties, $\sum_{i=1}^{\hat{n}} s_i$, equals m, we can express the preceding results in terms of renormalized vote shares, renormalized multipliers, and renormalized rounding residuals:

Definition 9 (Renormalized multipliers). For every multiplier $\mu \in \mathbb{R}_+$ the corresponding *renormalized multiplier* is given by

$$\hat{\mu} := \mu \sum_{i=1}^{\hat{n}} p_i. \tag{2.16}$$

In particular,

$$\hat{\mu}_{\inf} := \mu_{\inf} \sum_{i=1}^{\hat{n}} p_i \quad \text{and} \quad \hat{\mu}_{\sup} := \mu_{\sup} \sum_{i=1}^{\hat{n}} p_i. \tag{2.17}$$

Definition 10 (Renormalized rounding residuals). For every party $i = 1, \ldots, \hat{n}$ its *renormalized rounding residual* under renormalized multiplier $\hat{\mu}$ is given by

$$\hat{r}_i(\hat{\mu}) := \{\hat{p}_i \hat{\mu}\}. \tag{2.18}$$

Observation 1. Note that for every $i = 1, \ldots, \hat{n}$ the following equality holds:

$$\hat{p}_i = \frac{v_i}{\hat{w}} = \frac{w p_i}{\sum_{j=1}^{\hat{n}} w p_j} = \frac{p_i}{\sum_{j=1}^{\hat{n}} p_j}. \tag{2.19}$$

Observation 2. Note that for every party $i = 1, \ldots, \hat{n}$ we have

$$\lfloor \hat{p}_i \hat{\mu} \rfloor = \lfloor p_i \mu \rfloor = s_i \tag{2.20}$$

and

$$\hat{r}_i(\hat{\mu}) = \{\hat{p}_i \hat{\mu}\} = \{p_i \mu\} = r_i(\mu). \tag{2.21}$$

Proposition 3. If $\sum_{j=1}^{\hat{n}} s_j = m$, (2.15) simplifies to:

$$s_i = \hat{p}_i \left(m + \sum_{j=1}^{\hat{n}} \hat{r}_j(\hat{\mu}) \right) - \hat{r}_i(\hat{\mu}). \tag{2.22}$$

3. Prior Research on Seat Bias

It is well known that the Jefferson–D'Hondt method is biased in favor of larger parties [see, e.g., 3, 20, 27, 31, 35, 39]. Sainte-Laguë [37] was the first to quantify this effect, finding that the expected seat bias equals $\log 2 - 1/2$ under the assumptions that $n = 2$ and p_2/p_1 is uniformly distributed over $(0, 1)$. Pólya [33, 34] has employed geometric approach to calculate expected seat biases for three-party elections, assuming instead a uniform distribution of the vector of party vote shares over the probability simplex. This line of research has been continued by Schuster et al. [38] and Drton and Schwingenschlögl [11] who have obtained analytical expression for the expected seat bias of the k-th largest party in an n-party election under the uniform distribution of the vote share vector[4]. However, one is frequently interested in estimating the expected seat bias of a specific party (characterized by a given vote share) rather than for an average k-th largest party. Moreover, the assumption about the uniform distribution is of uncertain empirical validity.

Analytical formulae for the expected seat bias of the i-th party in single-district elections conditioned on that party's vote share have been proposed by Bochsler [4], Janson [23], and Pukelsheim [35]. Prima facie, they appear identical to each other and very similar to our seat bias formula. However, despite those similarities, they address different problems and employ different assumptions. Under the assumption that the vote shares follow an arbitrary absolutely continuous distribution over the probability simplex, Pukelsheim [35, Sec. 6.10] has proven that the rounding residuals converge in distribution to a vector of n variables drawn from a uniform distribution and stochastically independent of each other and of the party vote shares as the district magnitude approaches infinity[5]. Hence he has deduced [35, Sec. 7.3] that the seat biases approach those given by (4.2) in this case. Janson [23, Thm. 3.4] has shown that for any choice of vote shares (under the mild assumption that they are linearly independent over rationals), the expected seat biases for the district magnitude randomly drawn from the uniform distribution on $\{1, \ldots, \eta\}$, where $\eta \in \mathbb{N}_+$, also converge to those given by (4.2) as η approaches infinity. Bochsler [4, p. 621] has obtained the single-district expected seat bias formula by assuming that the rounding residuals are independent of each party's vote share and always have an expected value of $1/2$.

While all three works discussed in the preceding paragraph significantly advanced our knowledge of seat biases, they share some common limitations. The single-district formulae described by the foregoing authors are only correct in the asymptotic sense – as the district magnitude approaches infinity[6], and little is known regarding their respective rates of convergence. Moreover, numerical simulations demonstrate that under realistic distributional assumptions, said rates are slow enough as to render them of limited usefulness if the district magnitude is of the order of 3 to 20, as is usually the case in real-life elections. It is primarily in this area that we seek to advance prior knowledge by demonstrating that when seat allocations and seat biases are summed over multiple districts, restrictive assumptions about the round-

---

[4]It should be noted that their results for the expected seat bias of the k-th largest party match exactly the results produced by our seat bias formula when $\hat{p}_i$ equals the expected vote share of the k-th largest party.

[5]The proof is a more general case of an earlier proof by Tukey [40], who has established that the rounding residuals of a scalar variable converge in distribution to a uniform distribution on $(0, 1)$.

[6]Technically, Janson treats the district magnitude as randomly drawn from a discrete uniform distribution on $(0, \eta)$, with η approaching infinity. But in such case the expected district magnitude also approaches infinity, while the value of its cumulative distribution function at x approaches 0 for every $x \in \mathbb{R}_+$.

ing residuals on which Bochsler, Janson, and Pukelsheim rely can be exchanged for more liberal ones, dealing only with interdistrict averages of residuals (which converge to $1/2$ far more rapidly).

While Theorem 1 is superficially similar to the the results obtained by Janson, Pukelsheim, and Bochsler, it is not merely a generalization of the latter three to the multi-district case. Indeed, the difference is one of kind. First, under assumptions **A1** to **A3** formulae (4.1) and (4.2) are deterministic rather than probabilistic. Second, if we relax assumption **A1a** and switch to the probabilistic model described in Sec. 5, we still avoid the problematic assumption that $m \to \infty$, instead providing an error bound for arbitrarily small values of $m \geq 1$.

4. The Seat Allocation Theorem

We have introduced all concepts necessary to state our main result regarding the relationship between the seat share and the renormalized vote share for each relevant party under three assumptions: that there exists such a selection of renormalized multipliers, one for every district, that for every relevant party the rounding residuals average to $1/2$ over all districts (**A1a**) and the renormalized district vote shares of that party are not correlated with the multipliers (**A1b**); that all seats are distributed only among relevant parties (**A2**); and that for every relevant party renormalized district vote shares are not correlated with the numbers of votes for relevant parties (**A3**). Note that even if these assumptions are not satisfied, the assertion of the theorem still holds approximately, as further discussed in the Appendix.

Throughout this section, we only deal with renormalized vote shares, renormalized multipliers, and renormalized rounding residuals. Since there is no risk of confusion, we omit the term "renormalized."

Theorem 1. Assume that $\hat{n}$ is the number of relevant parties given by (1.2). If:

(**A1**) there exists a sequence $(\hat{\mu}_k)_{k=1}^c \in \prod_{k=1}^c [\hat{\mu}_{\inf}^k, \hat{\mu}_{\sup}^k)$ such that for every party $i = 1, \ldots, \hat{n}$

  (**A1a**) $\left\langle \hat{r}_i^k (\hat{\mu}_k) \right\rangle_{k=1}^c = \frac{1}{2}$;

  (**A1b**) $\mathrm{Cov}\,(\hat{\mathbf{p}}_i, \hat{\boldsymbol{\mu}}) = 0$, where $\hat{\mathbf{p}}_i := (\hat{p}_i^1, \ldots, \hat{p}_i^c)$ and $\hat{\boldsymbol{\mu}} := (\hat{\mu}_1, \ldots, \hat{\mu}_c)$;

(**A2**) $\sum_{i=1}^{\hat{n}} s_i^k = m_k$ for every district $k = 1, \ldots, c$; and

(**A3**) $\mathrm{Cov}\,(\hat{\mathbf{p}}_i, \hat{\mathbf{w}}) = 0$, where $\hat{\mathbf{p}}_i = (\hat{p}_i^1, \ldots, \hat{p}_i^c)$ and $\hat{\mathbf{w}} := (\hat{w}_1, \ldots, \hat{w}_c)$, for every party $i = 1, \ldots, \hat{n}$;

then

$$q_i = \hat{p}_i + \hat{p}_i \frac{\hat{n}}{2m} - \frac{1}{2m}, \tag{4.1}$$

for every $i = 1, \ldots, \hat{n}$, where $\hat{p}_i$ is the renormalized vote share of the i-th party given by (1.3).

Corollary 4. Under assumptions **A1**-**A3** the *seat bias* of the i-th party, $\hat{\sigma}_i := q_i - \hat{p}_i$, i.e., the difference between its seat share and its vote share, is given by[7]

$$\hat{\sigma}_i = \frac{\hat{n}}{2m} \left(\hat{p}_i - \frac{1}{\hat{n}} \right). \tag{4.2}$$

For the proof of Theorem 1, we need the following lemma:

---

[7]For an alternative form of (4.2), see (6.8).

Lemma 3. Under assumptions **A1**-**A3** the mean multiplier $\langle \hat{\mu}_k \rangle_{k=1}^c = m + \hat{n}/2$, i.e., the inverse of the Gfeller-Joachim-Pukelsheim quota (Remark 11) with non-relevant parties excluded.

Proof. By **A2**, $\sum_{i=1}^{\hat{n}} s_i^k = m_k$. Hence, from Lemma 1 and (2.21) we obtain $\sum_{i=1}^{\hat{n}} \hat{r}_i^k(\hat{\mu}_k) = \hat{\mu}_k - m_k$. Summing this over all districts, we arrive at

$$\sum_{k=1}^c \sum_{i=1}^{\hat{n}} \hat{r}_i^k(\hat{\mu}_k) = \sum_{k=1}^c (\hat{\mu}_k - m_k) = c\langle \hat{\mu}_k \rangle_{k=1}^c - s. \tag{4.3}$$

On the other hand, from **A1a** it follows that $\sum_{k=1}^c \hat{r}_i^k(\hat{\mu}_k) = c/2$ for every $i = 1, \ldots, \hat{n}$. Therefore,

$$\sum_{i=1}^{\hat{n}} \sum_{k=1}^c \hat{r}_i^k(\hat{\mu}_k) = \sum_{i=1}^{\hat{n}} \frac{c}{2} = \frac{c\hat{n}}{2}. \tag{4.4}$$

Accordingly,

$$\langle \hat{\mu}_k \rangle_{k=1}^c = m + \frac{\hat{n}}{2}, \tag{4.5}$$

as desired. □

We can now proceed to the proof of Theorem 1.

Proof. Fix any $i = 1, \ldots, \hat{n}$. Note that by (2.18) and (2.20),

$$s_i^k = \hat{p}_i^k \hat{\mu}_k - \hat{r}_i^k(\hat{\mu}_k). \tag{4.6}$$

Taking a sum over all districts, we obtain

$$s_i = \sum_{k=1}^c \left(\hat{p}_i^k \hat{\mu}_k - \hat{r}_i^k(\hat{\mu}_k) \right) = \sum_{k=1}^c \hat{p}_i^k \hat{\mu}_k - \sum_{k=1}^c \hat{r}_i^k(\hat{\mu}_k). \tag{4.7}$$

From **A1b** and Lemma 3 it follows that

$$\sum_{k=1}^c \hat{p}_i^k \hat{\mu}_k = c\langle \hat{p}_i^k \rangle_{k=1}^c \langle \hat{\mu}_k \rangle_{k=1}^c = \langle \hat{p}_i^k \rangle_{k=1}^c \left(s + \frac{c\hat{n}}{2} \right). \tag{4.8}$$

However, by **A3**,

$$\langle \hat{p}_i^k \rangle_{k=1}^c = \frac{\sum_{k=1}^c v_i^k}{\sum_{k=1}^c \hat{w}_k} = \frac{v_i}{\hat{v}} = \hat{p}_i. \tag{4.9}$$

Thus,

$$s_i = \hat{p}_i \left(s + \frac{c\hat{n}}{2} \right) - \sum_{k=1}^c \hat{r}_i^k(\hat{\mu}_k). \tag{4.10}$$

Finally, by **A1a**, we have $\sum_{k=1}^c \hat{r}_i^k(\hat{\mu}_k) = c/2$. Substituting this into (4.10) and dividing both sides by s, we thus arrive at the seat allocation formula (4.1):

$$q_i = \hat{p}_i \left(1 + \frac{\hat{n}}{2m} \right) - \frac{1}{2m} = \hat{p}_i + \hat{p}_i \frac{\hat{n}}{2m} - \frac{1}{2m}. \tag{4.11}$$

$$\square$$

Observation 3. As an immediate consequence of **A2**, if $\hat{n} < n$, then

$$q_i = 0 \tag{4.12}$$

for every non-relevant party $i = \hat{n} + 1, \ldots, n$.

5. Discussion of Assumption A1

Assumptions **A2** and **A3** are essentially only for the convenience of application of Theorem 1, as it can be expressed in terms of the average renormalized district-level vote share of the i-th party, $\langle \hat{p}_i^k \rangle_{k=1}^c$, and the sum of seat shares of the non-relevant parties, $\sum_{j=\hat{n}+1}^n q_j$. Assumptions **A2** and **A3** would then be superfluous, as we would obtain the following corollary:

Corollary 5. If **A1**, then

$$q_i = \langle \hat{p}_i^k \rangle_{k=1}^c \left(1 - \sum_{j=\hat{n}+1}^n q_j + \frac{\hat{n}}{2m} \right) - \frac{1}{2m}, \tag{5.1}$$

for every $i = 1, \ldots, \hat{n}$.

The principal advantage of Theorem 1 over Corollary 5 is purely practical: in the context of most expected applications, $\hat{p}_i$ is known, while $\langle \hat{p}_i^k \rangle_{k=1}^c$ and $\sum_{j=\hat{n}+1}^n q_j$ are not. However, the above makes it clear that **A2** and **A3** do not give significant insight into the workings of the Jefferson–D'Hondt method.

On the other hand, assumption **A1** is of fundamental importance to Theorem 1 and cannot be avoided in a manner similar to the other ones. At the same time, it does not easily correspond to normative intuitions about electoral systems, instead requiring some additional justification. In this section, we seek to partially provide such a justification by demonstrating that, under a probabilistic model of elections involving certain reasonable distributional assumptions, we can expect at least **A1a** to be approximately satisfied if the number of districts is sufficiently large.

As a consequence of Lemma 2 and **A2**, non-relevant parties have no effect on seat allocation among relevant parties under Theorem 1. It follows that they can be disregarded at will. Accordingly, a probabilistic model of relevant parties only is sufficient for our present purposes. It follows that in this section we need not distinguish between renormalized and non-renormalized variables.

Let us assume that an election conforms to the following probabilistic model:

- let $c \in \mathbb{N}_+$ be the number of districts;

- let district magnitudes $M_1, \ldots, M_c$ be independent random variables identically distributed according to some discrete probability distribution $\mathcal{M}$ on $\mathbb{N}_+$ with an expectation $m \in [m_{\min}, \infty)$, where $m_{\min} \geq 1$;

- let $\mathbf{p} := (p_1, \ldots, p_n) \in \Delta_n$ be a vector of aggregate vote shares such that $p_i > (2m + n)^{-1}$ for every $i = 1, \ldots, n$, i.e., that all parties are relevant;

- let $\mathcal{V}$ be an absolutely continuous probability distribution on Δ_n with an expectation $\mathbf{p}$ and a continuously differentiable density $f_{\mathcal{V}}$ vanishing at the faces of Δ_n;

- let district-level vote share vectors $\mathbf{P}_1, \ldots, \mathbf{P}_c$ be independent random variables identically distributed according to $\mathcal{V}$.

Assume further that M_k and $\mathbf{P}_k$ are independent for every $k = 1, \ldots, c$.

Among the assumptions of this model, the one regarding the density of $\mathcal{V}$ appears most in need of an additional justification. It comes from an empirical observation that in most elections, a relevant party almost never comes close to winning all (or no) votes in any district. Hence, the assumption that $f_{\mathcal{V}}$ vanishes at the faces of Δ_n is consistent with real-life voting patterns.

Note that by the strong law of large numbers, the average district magnitude $\langle M_k \rangle_{k=1}^c \overset{\text{a.s.}}{\to} m$, and the average vote share vector $\langle \mathbf{P}_k \rangle_{k=1}^c \overset{\text{a.s.}}{\to} \mathbf{p}$ as $c \to \infty$.

Let Θ be an interval-valued function of $(a, \mathbf{b}) \in \mathbb{N}_+ \times \Delta_n$, mapping a district magnitude a and a vote share vector $\mathbf{b} := (b_1, \ldots, b_n)$ to a multiplier interval $[1/Q_a^\downarrow, 1/Q_{a+1}^\downarrow)$, where $(Q_n)_{n \in \mathbb{N}_+}$ is given by $Q_{(j-1)n+i} := b_i/j$ for every $i = 1, \ldots, n$ and $j \in \mathbb{N}_+$. Let θ be a selection of Θ, i.e., a function $\mathbb{N}_+ \times \Delta_n \to \mathbb{R}_+$ such that $\theta(a, \mathbf{b}) \in \Theta(a, \mathbf{b})$ for every $(a, \mathbf{b}) \in \mathbb{N}_+ \times \Delta_n$. Finally, within each district $k = 1, \ldots, c$, let U_k be a random variable given by $U_k := \theta(M_k, \mathbf{P}_k)$. For **A1** to hold, we need to demonstrate that there exists such θ that $\mathbb{E}\{P_i^k U_k\} = 1/2$ and $\operatorname{Cov}(P_i^k, U_k) = 0$ for every $i = 1, \ldots, n$ and every $k = 1, \ldots, c$. This we are unable to do without further assumptions as to $\mathcal{V}$. However, using Theorem 2 below we demonstrate that regardless of the choice of θ, assumption **A1a** is satisfied approximately in the limit of $c \to \infty$. We leave the question of demonstrating that **A1b** is also satisfied approximately for future work, although we note that the chequered pattern of the discrepancies on the probability simplex (as illustrated for $n = 3$ by Fig. 1 above) reveals one promising avenue of approach.

Theorem 2. If:

(**B1**) P is a random variable with an absolutely continuous distribution supported on a subset of $[0, 1]$ with a continuously differentiable density f_P;

(**B2**) U is a random variable with a mixed discrete-continuous distribution supported on a subset of $[u, \infty)$, where $u \in \mathbb{R}_+$, with a cumulative distribution function Ψ, density ψ, and a finite set of probability atoms A, such that $\psi(a) = 0$ for every $a \in A$; and

(**B3**) $f_P(0) = 0$ and $f_P(1) = 0$;

then

$$\left| \mathbb{E}(\{PU\}) - \frac{1}{2} \right| \leq \frac{1}{12u} V(P), \tag{5.2}$$

where $V(P) := \int_0^1 |f_P'(x)| \, dx$ is the total variation of f_P.

Proof. Let $f_{P|U}$ be the conditional density of P with respect to U and let $F_{P|U}$ be the conditional cumulative distribution function of P with respect to U. Let $R := \{PU\}$ and consider its conditional distribution with respect to U. Clearly,

$$\Pr(R \leq x | U = y) = \sum_{l=0}^{\lfloor y \rfloor} \Pr(PU \in [l, l+x] \,|\, U = y). \tag{5.3}$$

Thus, the conditional density of R with respect to U, $\varphi_{R|U} : [0,1) \to \mathbb{R}_{\geq 0}$, is given by

$$\varphi_{R|U}(x|y) := \frac{1}{y} \sum_{l=0}^{\lfloor y \rfloor} f_{P|U}\left(\frac{l+x}{y} \,\Big|\, y \right). \tag{5.4}$$

Accordingly, the unconditional density of R is given by $\varphi : [0,1) \to \mathbb{R}_{\geq 0}$ defined as

$$\varphi_R(x) := \int_u^\infty \frac{1}{y} \sum_{l=0}^{\lfloor y \rfloor} f_{P|U}\left(\frac{l+x}{y} \,\Big|\, y \right) d\Psi(y). \tag{5.5}$$

In consequence, the expected value of R equals

$$\mathbb{E}(R) = \int_0^1 \int_u^\infty \frac{x}{y} \sum_{l=0}^{\lfloor y \rfloor} f_{P|U}\left(\frac{l+x}{y} \,\Big|\, y \right) d\Psi(y) \, dx. \tag{5.6}$$

Let us substitute $z := (l+x)/y$ and change the order of integration and summation in (5.6) to obtain

$$\mathbb{E}(R) = \int_u^\infty \sum_{l=0}^{\lfloor y \rfloor} \int_{\frac{l}{y}}^{\frac{l+1}{y}} \frac{yz - l}{y} f_{P|U}(z|y) \, y \, dz \, d\Psi(y)$$

$$= \int_u^\infty \int_0^{\frac{\lfloor y \rfloor + 1}{y}} yz \, f_{P|U}(z|y) \, dz \, d\Psi(y) - \int_u^\infty \sum_{l=0}^{\lfloor y \rfloor} \int_{\frac{l}{y}}^{\frac{l+1}{y}} l \, f_{P|U}(z|y) \, dz \, d\Psi(y)$$

$$= \mathbb{E}(PU) - \int_u^\infty \sum_{l=0}^{\lfloor y \rfloor} l \left(F_{P|U}\left(\frac{l+1}{y} \,\Big|\, y \right) - F_{P|U}\left(\frac{l}{y} \,\Big|\, y \right) \right) d\Psi(y)$$

$$= \mathbb{E}(PU) - \int_u^\infty \left(\lfloor y \rfloor \, F_{P|U}\left(\frac{\lfloor y \rfloor + 1}{y} \,\Big|\, y \right) - \sum_{l=0}^{\lfloor y \rfloor} F_{P|U}\left(\frac{l}{y} \,\Big|\, y \right) \right) d\Psi(y)$$

$$= \mathbb{E}(PU) - \mathbb{E}(\lfloor U \rfloor) + \int_u^\infty \sum_{l=0}^{\lfloor y \rfloor} F_{P|U}\left(\frac{l}{y} \,\Big|\, y \right) d\Psi(y)$$

$$= \mathbb{E}(PU) - \mathbb{E}(\lfloor U \rfloor) - 1 + \int_u^\infty \sum_{l=0}^{\lfloor y \rfloor + 1} F_{P|U}\left(\frac{l}{y} \,\Big|\, y \right) d\Psi(y). \tag{5.7}$$

Let $\Phi_{P|U}(l|y) := F_{P|U}(l/y|y)$ for every $0 \leq l \leq \lfloor y \rfloor + 1$. If $\Phi_{P|U}$ is smooth of class C^{2h}, $h \in \mathbb{N}_+$, from the Euler-Maclaurin summation formula we obtain

$$\sum_{l=0}^{\lfloor y \rfloor + 1} \Phi_{P|U}(l|y) = \int_0^{\lfloor y \rfloor + 1} \Phi_{P|U}(x|y)\,dx + \frac{\Phi_{P|U}(\lfloor y \rfloor + 1|\,y) - \Phi_{P|U}(0|y)}{2}$$
$$+ \sum_{j=1}^{h} \frac{B_{2j}}{(2j)!} \left(\Phi_{P|U}^{(2j-1)}(\lfloor y \rfloor + 1|\,y) - \Phi_{P|U}^{(2j-1)}(0|y) \right) + \omega_y^{2h}, \tag{5.8}$$

where, for every $k = 2, \ldots, h$, B_k is the k-th *Bernoulli number*, $B_k(x)$ is the k-th *Bernoulli polynomial*, $\Phi_{P|U}^{(k)}$ is the k-th derivative of $\Phi_{P|U}$, and ω_y^{2h} is an error term given by

$$\omega_y^{2h} := -\frac{1}{(2h)!} \int_0^{\lfloor y \rfloor + 1} \Phi_{P|U}^{(2h)}(x|y) B_{2h}\left(\{x\}\right)\,dx \tag{5.9}$$

[6, Corollary 9.2.3 (2)]. Note that

$$\left| \omega_y^{2h} \right| \leq \frac{1}{(2h)!} \left| \max_{x \in (0,1)} B_{2h}(\{x\}) \right| \int_0^{\lfloor y \rfloor + 1} \left| \Phi_{P|U}^{(2h)}(x|y) \right|\,dx. \tag{5.10}$$

Lehmer [26, Thm. 1] has shown that

$$\left| \max_{x \in (0,1)} B_{2h}(\{x\}) \right| \leq |B_{2h}| = \frac{(2h)!}{(2\pi)^{2h}} 2\zeta(2h), \tag{5.11}$$

where ζ is the Riemann zeta function. Accordingly,

$$\left| \omega_y^{2h} \right| \leq \frac{2\zeta(2h)}{(2\pi)^{2h}} \int_0^{\lfloor y \rfloor + 1} \left| \Phi_{P|U}^{(2h)}(x|y) \right|\,dx. \tag{5.12}$$

Substituting $F_{P|U}(l/y|y)$ for $\Phi_{P|U}(l|y)$ in successive terms of (5.8) and in (5.12), we get:

$$\int_0^{\lfloor y \rfloor + 1} \Phi_{P|U}(x|y)\,dx = \int_0^{\lfloor y \rfloor + 1} F_{P|U}\left(\frac{x}{y}\Big|\,y\right) dx = y \int_0^{\frac{\lfloor y \rfloor + 1}{y}} F_{P|U}(z|y)\,dz$$
$$= y \int_0^1 F_{P|U}(z|y)\,dz + y \int_1^{\frac{\lfloor y \rfloor + 1}{y}} F_{P|U}(z|y)\,dz$$
$$= y\big(1 - \mathbb{E}(P|U = y)\big) + (\lfloor y \rfloor + 1 - y), \tag{5.13}$$

$$\frac{\Phi_{P|U}(\lfloor y \rfloor + 1|\,y) - \Phi_{P|U}(0|y)}{2} = \frac{1}{2} F_{P|U}\left(\frac{\lfloor y \rfloor + 1}{y}\Big|\,y\right) = \frac{1}{2}, \tag{5.14}$$

$$\Phi_{P|U}^{(j)}(x|y) = y^{-j} F_{P|U}^{(j)}\left(\frac{x}{y}\Big|\,y\right) \text{ for every } j = 1, \ldots, 2h. \tag{5.15}$$

Therefore,

$$
\sum_{l=0}^{\lfloor y \rfloor + 1} F_{P|U}\left(\frac{l}{y}\bigg| y\right) = \lfloor y \rfloor - y\mathbb{E}(P|U = y) + \frac{3}{2}
$$
$$
+ \sum_{j=1}^{h} \frac{B_{2j}y^{-(2j-1)}}{(2j)!}\left(F_{P|U}^{(2j-1)}\left(\frac{\lfloor y \rfloor + 1}{y}\bigg| y\right) - F_{P|U}^{(2j-1)}(0|y)\right) + \omega_y^{2h}, \tag{5.16}
$$

where

$$
\left|\omega_y^{2h}\right| \le \frac{2y\zeta(2h)}{(2\pi y)^{2h}}\int_0^{\frac{\lfloor y \rfloor + 1}{y}}\left|F_{P|U}^{(2h)}(z|y)\right|\,dz. \tag{5.17}
$$

By **B3** for $h = 1$ we obtain

$$
\sum_{j=1}^{h}\frac{B_{2j}y^{-(2j-1)}}{(2j)!}\left(F_{P|U}^{(2j-1)}\left(\frac{\lfloor y \rfloor + 1}{y}\bigg| y\right) - F_{P|U}^{(2j-1)}(0|y)\right)\Bigg|_{h=1}
$$
$$
= \frac{B_2}{2y}\left(f_{P|U}\left(\frac{\lfloor y \rfloor + 1}{y}\bigg| y\right) - f_{P|U}(0|y)\right) = 0. \tag{5.18}
$$

Therefore,

$$
\sum_{l=0}^{\lfloor y \rfloor + 1} F_{P|U}\left(\frac{l}{y}\bigg| y\right)\Bigg|_{h=1} = \lfloor y \rfloor - y\mathbb{E}(P|U = y) + \frac{3}{2} + \omega_y^2, \tag{5.19}
$$

where

$$
\left|\omega_y^2\right| \le \frac{2y\zeta(2)}{(2\pi y)^2}\int_0^{\frac{\lfloor y \rfloor + 1}{y}}\left|F_{P|U}^{(2)}(z|y)\right|\,dz = \frac{1}{12y}\int_0^1\left|f_{P|U}'(z|y)\right|\,dz. \tag{5.20}
$$

By incorporating the foregoing result into (5.7) we obtain

$$
\mathbb{E}(R) = \mathbb{E}(PU) - \mathbb{E}(\lfloor U \rfloor) - 1 + \int_u^\infty\left(\lfloor y \rfloor - y\mathbb{E}(P|U = y) + \frac{3}{2} + \omega_y^2\right)d\Psi(y)
$$
$$
= \mathbb{E}(PU) - \mathbb{E}(\lfloor U \rfloor) - 1 + \mathbb{E}(\lfloor U \rfloor) - \mathbb{E}(PU) + \frac{3}{2} + \mathbb{E}(\omega_U^2) = \frac{1}{2} + \mathbb{E}(\omega_U^2). \tag{5.21}
$$

Thus we arrive at

$$
\left|\mathbb{E}(R) - \frac{1}{2}\right| \le \mathbb{E}\left(\frac{1}{12U}\int_0^1\left|f_{P|U}'(z|U)\right|\,dz\right), \tag{5.22}
$$

where $\int_0^1 |f_{P|U}'(z|U)|\,dz$ is the total variation of $f_{P|U}$. Since $U \ge u$, it follows that

$$
\left|\mathbb{E}\left(R\right) - \frac{1}{2}\right| \le \frac{1}{12u}\mathbb{E}\left(\int_0^1\left|f_{P|U}'(z|U)\right|\,dz\right). \tag{5.23}
$$

For $y \in \operatorname{supp}\psi$, we have

$$
f_{P|U}'(z|y) = \frac{\partial}{\partial z}\frac{f_{P,U}(z,y)}{\psi(y)} = \frac{1}{\psi(y)}\frac{\partial}{\partial z}f_{P,U}(z,y), \tag{5.24}
$$

where $f_{P,U}$ is the joint density of P and U, and

$$
\begin{aligned}
\int_{\mathrm{supp}\,\psi} \int_0^1 \left| f'_{P|U}(z|y) \right| \, dz \, d\Psi(y) &= \int_u^\infty \frac{\psi(y)}{\psi(y)} \int_0^1 \left| \frac{\partial}{\partial z} f_{P,U}(z,y) \right| \, dz \, dy \\
&= \int_0^1 \int_u^\infty \left| \frac{\partial}{\partial z} f_{P,U}(z,y) \right| \, dy \, dz \\
&= \int_0^1 |f'_P(z)| \, dz = V(P).
\end{aligned}
\tag{5.25}
$$

On the other hand, for $y \in A$, we have

$$
f'_{P|U}(z|y) = \frac{\partial}{\partial z} \frac{f_{P,U}(z,y)}{\Pr(U=y)} = \frac{1}{\Pr(U=y)} \frac{\partial}{\partial z} f_{P,U}(z,y),
\tag{5.26}
$$

and

$$
\begin{aligned}
\int_A \int_0^1 \left| f'_{P|U}(z|y) \right| \, dz \, d\Psi(y) &= \sum_{y \in A} \frac{\Pr(U=y)}{\Pr(U=y)} \int_0^1 \left| \frac{\partial}{\partial z} f_{P,U}(z,y) \right| \, dz \\
&= \int_0^1 \sum_{y \in A} \left| \frac{\partial}{\partial z} f_{P,U}(z,y) \right| \, dz \\
&= \int_0^1 |f'_P(z)| \, dz = V(P).
\end{aligned}
\tag{5.27}
$$

Accordingly,

$$
\mathbb{E}\left(\int_0^1 \left| f'_{P|U}(z|U) \right| \, dz \right) = V(P),
\tag{5.28}
$$

and thus

$$
\left| \mathbb{E}(R) - \frac{1}{2} \right| \le \frac{1}{12u} V(P),
\tag{5.29}
$$

as desired. $\qquad\square$

Under the probabilistic model of elections described above, let P_i^k, $k = 1, \ldots, c$, be the i-th barycentric coordinate of $\mathbf{P}_k$ (i.e., the vote share of the i-th party). Since $\mathbf{P}_k$ has a continuously differentiable density vanishing at the faces of Δ_n, it follows that for every $i = 1, \ldots, n$ the density of P_i^k is also continuously differentiable and vanishes at 0 and 1. Thus, we can equate P_i^k with P in Theorem 2, as it satisfies **B1** and **B3**. To satisfy **B2**, fix an arbitrarily small $\varepsilon > 0$ and a selection θ given by:

$$
\theta(a, \mathbf{b}) := \begin{cases} a + n/2 & \text{for } a + n/2 \in \Theta(a, \mathbf{b}), \\ \inf \Theta(a, \mathbf{b}) & \text{for } a + n/2 < \inf \Theta(a, \mathbf{b}), \\ \sup \Theta(a, \mathbf{b}) - \varepsilon & \text{for } \sup \Theta(a, \mathbf{b}) \le a + n/2. \end{cases}
\tag{5.30}
$$

Note that U_k defined as $\theta(M_k, \mathbf{P}_k)$ is absolutely continuous over $[M_k, \infty) \setminus \{M_k + n/2\}$, and has a single probability atom at $M_k + n/2$, thus satisfying **B2.**

It follows from Theorem 2 that $\left| \mathbb{E}(\{P_i^k U_k\}) - \frac{1}{2} \right|$ is bounded from the above by $\frac{1}{12} M_k^{-1} V(P_i^k)$. Since $\mathbf{P}_1, \ldots, \mathbf{P}_c$ are identically distributed, and so are $M_1, \ldots, M_c$, from the strong law of large numbers it

then follows that

$$\left\langle \left| \{P_i^k U_k\} - \frac{1}{2} \right| \right\rangle_{k=1}^c \overset{\text{a.s.}}{\to} \left| \mathbb{E}(\{P_i^l U_l\}) - \frac{1}{2} \right| \le \frac{1}{12} \mathbb{E}(M_l^{-1}) V(P_i^l) \tag{5.31}$$

for every $l = 1, \ldots, c$.

Remark 15. If the distribution of $\mathbf{P}_l$ is unimodal, the total variation of P_i^l equals twice the value of its marginal density, $f_{P_i^l}(x)$, at its mode, i.e., $V(P_i^l) = 2 f_{P_i^l}(\max_{x \in (0,1)} f_{P_i^l}(x))$ for every $i = 1, \ldots, n$.

For vote distributions encountered in real-life elections, $V(P_i^l)$ rarely exceeds 3. Meanwhile, for typical district magnitudes, $\mathbb{E}(M_l^{-1}) \le 1/5$. Thus, $\frac{1}{12}\mathbb{E}(M_l^{-1})V(P_i^l) \le 1/20$, ensuring that **A1a** is satisfied approximately in the limit of $c \to \infty$.

6.   Relevant Parties and Natural Thresholds

Note that formulae (4.1) and (4.2) can only be applied to relevant parties. This restriction is connected to another important consequence of the Jefferson–D'Hondt method, namely the existence of a *threshold of representation*, i.e., such $\tau \in (0,1)$ that $p_i < \tau$ implies $q_i = 0$. To distinguish it from statutory thresholds which are present in some electoral systems, and which operate independently of the Jefferson–D'Hondt method, threshold τ is frequently referred to as the *natural threshold*.

At a district level, the natural threshold follows from Definition 7. To see how, recall that by (2.9), $s_i^k = \lfloor p_i^k \mu_k \rfloor$. Hence, it is evident that if $p_i^k \le 1/\mu_{\sup}^k$, then s_i^k, and thus also q_i^k, must necessarily be 0. But $1/\mu_{\sup}^k = (Q^k)_{m+1}^\downarrow$ depends on the vector of vote shares. It would be useful to have an estimate of the natural threshold that depends only on n and m_k. Several authors [see, e.g., 9, 32, 36] have provided estimates of the lower and upper bounds of the interval in which the natural threshold must fall, known respectively as the *threshold of inclusion* $\tau_k^- := (n + m_k - 1)^{-1}$ and the *threshold of exclusion* $\tau_k^+ := (m_k + 1)^{-1}$. Those can be used to estimate the *aggregate thresholds of exclusion / inclusion*, viz.

$$\tau^+ := \sum_{k=1}^c \frac{1}{m_k + 1} \frac{v_k}{v}, \tag{6.1}$$

and

$$\tau^- := \frac{1}{n + m_{\min} - 1} \frac{v_{\min}}{v}, \tag{6.2}$$

where $m_{\min}$ and $v_{\min}$ are, respectively, the number of seats and the number of votes cast for all parties in the district with the fewest seats, and, if there are multiple such districts, in the one with the fewest votes.

Once we posit that the relation between renormalized vote shares and seat shares is to satisfy (1.1), it is clear that the renormalized vote shares cannot be arbitrarily small. In particular, as q_i is by definition non-negative, it is necessary that

$$\hat{p}_i \left(1 + \frac{\hat{n}}{2m} \right) \ge \frac{1}{2m}. \tag{6.3}$$

By transforming it, we arrive at the following condition:

Condition 1. If we require that $q_i > 0$, it is necessary that

$$\hat{p}_i > \frac{1}{2m + \hat{n}}, \tag{6.4}$$

where $1/(2m + \hat{n})$ is the *threshold of relevance*, denoted by t.

Note that Condition 1 is equivalent to the condition for relevance in (1.2), i.e., the i-th party is relevant if and only if it satisfies (6.4). It can also be demonstrated that even if we were to modify the definition of $\hat{n}$, parties which do not satisfy (6.4) would not obtain positive seat shares:

Observation 4. Fix any $\tau^* \in [0, 1)$, and let $\Pi^* := \{i = 1, \ldots, n : p_i > \tau^*\}$. Note that

$$q_i^* := \frac{v_i}{\sum_{j \in \Pi^*} v_j} \left(1 + \frac{|\Pi^*|}{2m}\right) - \frac{1}{2m} \tag{6.5}$$

is positive for every $i \in \Pi^*$ and every $\mathbf{p} := (p_1, \ldots, p_n) \in \Delta_n$ if and only if $\Pi^* \setminus \{1, \ldots, \hat{n}\} = \varnothing$.

Remark 16. Condition 1 can also be expressed in equivalent terms of potential interest to some readers:

$$\hat{n} > \frac{\sum_{j=1}^{\hat{n}} v_j}{v_i} - 2m, \tag{6.6}$$

$$2m > \sum_{j=1}^{\hat{n}} \frac{v_j - v_i}{v_i}. \tag{6.7}$$

Remark 17. The threshold of relevance can be used to express (4.1) in yet another form:

$$q_i = \frac{1}{2mt}(\hat{p}_i - t), \tag{6.8}$$

which demonstrates that the seat shares are proportional not to the renormalized vote shares, but only to their excess over the threshold of relevance.

Remark 18. Note that for the single-district case $(m = s)$, our definition of the threshold of relevance is in accord with the earlier findings about the thresholds of exclusion and inclusion. It is easy to show that (4.1) gives at least $1/2$ seat for a party fulfilling $p_i > (m + 1)^{-1} \geq (2m + n)^{-1}$, and at most $\frac{1}{2}(m + 1)/(n + m - 1) \leq 1/2$ for a party satisfying $\hat{p}_i < (n + m - 1)^{-1}$, though the latter is not necessarily non-relevant, as it cannot be ruled out that $t(\sum_{j=1}^{\hat{n}} \hat{p}_j) < (n + m - 1)^{-1}$. Given that seats are integer, a party exceeding the threshold of exclusion is guaranteed to obtain at least one, and a party below the threshold of inclusion is guaranteed to obtain none.

Remark 19. Note that τ^- is approximately of the order of s^{-1}. However, from (1.1) we obtain:

$$q_i|_{\hat{p}_i = 1/s} > 0 \quad \text{if and only if} \quad \frac{\hat{n}}{2m} \geq s - 1, \tag{6.9}$$

which is a rather unrealistic condition, as even for $m = 1$ that would require the number of relevant parties to be at least twice the number of districts minus 2. Thus, the aggregate threshold of inclusion is usually lower than the threshold of relevance, demonstrating that **A2** is not redundant.

Remark 20. To the extent (1.1) is an approximation of the seat allocation under the Jefferson–D'Hondt method, the threshold of relevance is an estimate of the aggregate natural threshold.

6.1. Continuity of threshold effects

Given the discreteness of the number of relevant parties, one could prima facie expect that if **p** changes so that a previously non-relevant party's vote share exceeds the threshold of relevance or a previously relevant party's vote share no longer does, the number of seats for every other party would change discontinuously. This would obviously constitute a significant obstacle to applying Theorem 1 to estimate seat allocations under circumstances where $p_1, \ldots, p_n$ are known only approximately (for instance, obtained from opinion polls) and some parties are in the vicinity of the natural threshold. Fortunately, this is not the case:

Proposition 4. For every $k < n$ if the first k parties are relevant, then $(p_1, \ldots, p_{k+1}) \mapsto (q_1, \ldots, q_{k+1})$ is continuous in p_{k+1} throughout $[0, p_k]$.

Proof. Let $t' := (2m + k + 1)^{-1}$ and $p' := \frac{t'}{1-t'} \sum_{j=1}^{k} p_j$. Note that $p' = t' \left(\sum_{j=1}^{k} p_j + p' \right)$ and that the $(k+1)$-th party is relevant if and only if $p_{k+1} > p'$.

Fix $p_1, \ldots, p_k$ and let $p_{k+1} = p \in [0, p_k]$ be variable. For $p \leq p'$ we have

$$q_i\big|_{p_{k+1}=p} = \begin{cases} \dfrac{2m+k}{2m} \dfrac{p_i}{\sum_{j=1}^{k} p_j} - \dfrac{1}{2m} & \text{for } i = 1, \ldots, k, \\ 0 & \text{for } i = k+1, \ldots, n. \end{cases} \tag{6.10}$$

On the other hand, for $p > p'$ and $i = 1, \ldots, k, k+1$ we obtain

$$q_i\big|_{p_{k+1}=p} = \frac{2m+k+1}{2m} \frac{p_i}{p + \sum_{j=1}^{k} p_j} - \frac{1}{2m}. \tag{6.11}$$

Thus, we only need to consider the case of $p \searrow p'$. For $i = 1, \ldots, k$ we have

$$\lim_{p \searrow p'} q_i\big|_{p_{k+1}=p} = \frac{2m+k+1}{2m} \frac{p_i}{p' + \sum_{j=1}^{k} p_j} - \frac{1}{2m} \tag{6.12}$$

$$= \frac{p_i}{2mp'} - \frac{1}{2m} = \frac{1-t'}{2mt'} \frac{p_i}{\sum_{j=1}^{k} p_j} - \frac{1}{2m} \tag{6.13}$$

$$= \frac{2m+k}{2m} \frac{p_i}{\sum_{j=1}^{k} p_j} - \frac{1}{2m}, \tag{6.14}$$

and for $i = k + 1$ we obtain

$$\lim_{p \searrow p'} q_i\big|_{p_{k+1}=p} = \frac{p'}{2mp'} - \frac{1}{2m} = 0, \tag{6.15}$$

as desired. $\qquad\square$

Funding

References

[1] BALINSKI, M. L., AND YOUNG, H. P. The Quota Method of Apportionment. *American Mathematical Monthly 82*, 7 (Aug. 1975), 701–730.

[2] BALINSKI, M. L., AND YOUNG, H. P. The Jefferson Method of Apportionment. *SIAM Review 20*, 2 (1978), 278–284.

[3] BALINSKI, M. L., AND YOUNG, H. P. *Fair Representation: Meeting the Ideal of One Man, One Vote*, 2nd ed ed. Brookings Institution Press, Washington, D.C., 2001.

[4] BOCHSLER, D. Who Gains from Apparentments under D'Hondt? *Electoral Studies 29*, 4 (Dec. 2010), 617–627.

[5] BURNITZ, G., AND VARRENTRAPP, G. *Methode, Bei Jeder Art von Wahlen Sowohl Der Mehrheit Als Den Minderheiten Die Ihrer Stärke Entsprechende Zahl von Vertretern Zu Sichern*. I. D. Sauerländer, Frankfurt am Main, 1863.

[6] COHEN, H. *Number Theory, Volume II: Analytic and Modern Tools*. No. 240 in Graduate Texts in Mathematics. Springer, New York, NY, 2007. OCLC: ocm77795788.

[7] DANČIŠIN, V. Notes on the Misnomers Associated with Electoral Quotas. *European Electoral Studies 8*, 2 (2013), 160–165.

[8] D'HONDT, V. *Système Pratique et Raisonné de Représentation Proportionnelle*. Librairie C. Muquardt, Bruxelles, 1882.

[9] D'HONDT, V. Formule du minimum dans la représentation proportionnelle. Moyen facile de trouver le diviseur. *Représentation proportionnelle. Revue mensuelle 2* (1883), 117–128, 129–130.

[10] D'HONDT, V. *Exposé Du Système Pratique de Représentation Proportionnelle. Adopté Par Le Comité de l'Association Réformiste Belge*. Eug. Vanderhaeghen, Gand, 1885.

[11] DRTON, M., AND SCHWINGENSCHLÖGL, U. Asymptotic Seat Bias Formulas. *Metrika 62*, 1 (Sept. 2005), 23–31.

[12] EQUER, M. Relation entre la méthode d'Hondt et la proportionnalité. *La Grande Revue, Deuxieme série 31* (Jan. 1911), 130–137.

[13] FLIS, J., SŁOMCZYŃSKI, W., AND STOLICKI, D. Pot and Ladle: A Formula for Estimating the Distribution of Seats Under the Jefferson–D'Hondt Method. *Public Choice 182* (2020), 201–227.

[14] GFELLER, J. Du transfert des suffrages et de la répartition des sièges complémentaires. *Représentation proportionnelle. Revue mensuelle 9* (1890), 120–131.

[15] HAGENBACH-BISCHOFF, E. *Die Frage Der Einführung Einer Proportionalvertretung Statt Des Absoluten Mehres*. H. Georg, Basel, 1888.

[16] HAPPACHER, M. The Discrepancy Distribution of Stationary Multiplier Rules for Rounding Probabilities. *Metrika 53*, 2 (May 2001), 171–181.

[17] HAPPACHER, M., AND PUKELSHEIM, F. Rounding Probabilities: Unbiased Multipliers. *Statistics & Decisions 14*, 4 (Jan. 1996), 373–382.

[18] HAPPACHER, M., AND PUKELSHEIM, F. Rounding Probabilities: Maximum Probability and Minimum Complexity Multipliers. *Journal of Statistical Planning and Inference 85*, 1-2 (Apr. 2000), 145–158.

[19] HEINRICH, L., PUKELSHEIM, F., AND WACHTEL, V. The Variance of the Discrepancy Distribution of Rounding Procedures, and Sums of Uniform Random Variables. *Metrika 80*, 3 (Apr. 2017), 363–375.

[20] HUNTINGTON, E. V. The Apportionment of Representatives in Congress. *Transactions of the American Mathematical Society 30*, 1 (Jan. 1928), 85–110.

[21] JAMES, E. J. The First Apportionment of Federal Representatives in the United States. *Annals of the American Academy of Political and Social Science 9*, 1 (July 1897), 1–41.

[22] JANSON, S. Euler-Frobenius Numbers and Rounding. Tech. Rep. arXiv: 1305.3512 [math.PR], May 2013.

[23] JANSON, S. Asymptotic Bias of Some Election Methods. *Annals of Operations Research 215*, 1 (Oct. 2014), 89–136.

[24] JEFFERSON, T. Opinion on Apportionment Bill. In *Papers of Thomas Jefferson, Digital Edition*, B. Oberg and J. J. Looney, Eds., 2008 ed. University of Virginia Press, Rotunda, Charlottesville, 1792.

[25] JOACHIM, V. K otázce poměrného zastoupení. *Správní obzor 9*, 8 (1917), 289–298.

[26] LEHMER, D. H. On the Maxima and Minima of Bernoulli Polynomials. *American Mathematical Monthly 47*, 8 (Oct. 1940), 533–538.

[27] MARSHALL, A. W., OLKIN, I., AND PUKELSHEIM, F. A Majorization Comparison of Apportionment Methods in Proportional Representation. *Social Choice and Welfare 19*, 4 (Oct. 2002), 885–900.

[28] MORA GINÉ, X. La regla de Jefferson – d'Hondt i les seves alternatives. *Materials matemàtics 2013* (2013), 1–34.

[29] MULLIGAN, C. B., AND HUNTER, C. G. The Empirical Frequency of a Pivotal Vote. *Public Choice 116*, 1-2 (2003), 31–54.

[30] NYMANN, J. E. On the Probability That k Positive Integers Are Relatively Prime. *Journal of Number Theory 4*, 5 (Oct. 1972), 469–473.

[31] OYAMA, T., AND ICHIMORI, T. On the Unbiasedness of the Parametric Divisor Method for the Apportionment Problem. *Journal of the Operations Research Society of Japan 38*, 3 (1995), 301–321.

[32] PALOMARES, A., AND RAMÍREZ GONZÁLEZ, V. Thresholds of the Divisor Methods. *Numerical Algorithms 34*, 2-4 (Dec. 2003), 405–415.

[33] PÓLYA, G. Sur la représentation proportionnelle en matière électorale. *L'Enseignement Mathématique 20* (1918), 355–379.

[34] PÓLYA, G. Proportionalwahl und Wahrscheinlichkeitsrechnung. *Zeitschrift für die gesamte Staatswissenschaft 74* (1919), 297–322.

[35] PUKELSHEIM, F. *Proportional Representation: Apportionment Methods and Their Applications*, first ed. Springer International, Cham–Heidelberg, 2014.

[36] RAE, D. W., HANBY, V. J., AND LOOSEMORE, J. Thresholds of Representation and Thresholds of Exclusion. An Analytic Note on Electoral Systems. *Comparative Political Studies 3*, 4 (1971), 479–488.

[37] SAINTE-LAGUË, A. La représentation proportionnelle et la méthode des moindres carrés. *Annales scientifiques de l'École Normale Supérieure, Série 3 27* (1910), 529–542.

[38] SCHUSTER, K., PUKELSHEIM, F., DRTON, M., AND DRAPER, N. R. Seat Biases of Apportionment Methods for Proportional Representation. *Electoral Studies 22*, 4 (Dec. 2003), 651–676.

[39] TAAGEPERA, R., AND LAAKSO, M. Proportionality Profiles of West European Electoral Systems. *European Journal of Political Research 8*, 4 (Dec. 1980), 423–446.

[40] TUKEY, J. W. On the Distribution of the Fractional Part of a Statistical Variable. *Recueil Mathématique 46*, 3 (1938), 561–562.

Appendix. Approximation Errors in the Seat Allocation Formula

Theorem 1 is very much akin to many theorems appearing in other branches of applied mathematics, like mathematical statistics, in that most of its assumptions, especially **A1** and **A3**, are idealizations which we would not expect to be satisfied with empirical data. We can, however, quantify both deviations from the assumptions, and the relationship between those deviations and errors in (1.1).

Assume that **A1** to **A3** are not satisfied. In that case, (4.7) still holds, so the error in (1.1) can be expressed as:

$$\varepsilon_i := \hat{p}_i \left(1 + \frac{\hat{n}}{2m} \right) - \frac{1}{2m} - q_i = \tag{6.16}$$

$$= \left(\hat{p}_i + \hat{p}_i \frac{\hat{n}}{2m} - \frac{1}{2m} \right) - \frac{1}{s} \left(\sum_{k=1}^{c} \hat{p}_i^k \hat{\mu}_k - \sum_{k=1}^{c} \hat{r}_i^k(\hat{\mu}_k) \right). \tag{6.17}$$

To simplify further analysis, we decompose that error into two components: $\varepsilon_i = \varepsilon_i^A + \varepsilon_i^B$, where

$$\varepsilon_i^A := \frac{1}{m} \left(\hat{p}_i \left(m + \frac{\hat{n}}{2} \right) - \left\langle \hat{p}_i^k \hat{\mu}_k \right\rangle_{k=1}^{c} \right), \tag{6.18}$$

$$\varepsilon_i^B := \frac{1}{m} \left(\left\langle \hat{r}_i^k(\hat{\mu}_k) \right\rangle_{k=1}^{c} - \frac{1}{2} \right). \tag{6.19}$$

Note that

$$\left\langle \hat{p}_i^k \hat{\mu}_k \right\rangle_{k=1}^{c} = \left\langle \hat{p}_i^k \right\rangle_{k=1}^{c} \left\langle \hat{\mu}_k \right\rangle_{k=1}^{c} + \mathrm{Cov}\left(\hat{p}_i^k, \hat{\mu}_k \right), \tag{6.20}$$

and that

$$\left\langle \hat{p}_i^k \right\rangle_{k=1}^{c} \left\langle \hat{w}_k \right\rangle_{k=1}^{c} = \left\langle \hat{p}_i^k \hat{w}_k \right\rangle_{k=1}^{c} - \mathrm{Cov}\left(\hat{p}_i^k, \hat{w}_k \right), \tag{6.21}$$

whence

$$\left\langle \hat{p}_i^k \right\rangle_{k=1}^{c} = \hat{p}_i - \mathrm{Cov}\left(\hat{p}_i^k, \frac{\hat{w}_k}{\hat{w}} \right), \tag{6.22}$$

where $\hat{w} := \langle \hat{w}_k \rangle_{k=1}^{c} = \hat{v}/c$. By incorporating (6.22) and (6.20) into (6.18), from Corollary 3 we obtain

$$\varepsilon_i^A = \frac{1}{m} \left(\hat{p}_i \left(m + \frac{\hat{n}}{2} - \left\langle \hat{\mu}_k \right\rangle_{k=1}^{c} \right) + \left\langle \hat{\mu}_k \right\rangle_{k=1}^{c} \mathrm{Cov}\left(\hat{p}_i^k, \frac{\hat{w}_k}{\hat{w}} \right) - \mathrm{Cov}\left(\hat{p}_i^k, \hat{\mu}_k \right) \right) = \tag{6.23}$$

$$\frac{1}{m} \left(\hat{p}_i \left(\frac{\hat{n}}{2} - \frac{1}{c} \sum_{k=1}^{c} \sum_{j=1}^{\hat{n}} \hat{r}_j^k(\hat{\mu}_k) \right) + \left\langle \hat{\mu}_k \right\rangle_{k=1}^{c} \mathrm{Cov}\left(\hat{p}_i^k, \frac{\hat{w}_k}{\hat{w}} \right) - \mathrm{Cov}\left(\hat{p}_i^k, \hat{\mu}_k \right) \right), \tag{6.24}$$

which can be again disaggregated into two components, $\varepsilon_i^A = \varepsilon_i^C + \varepsilon_i^D$, where:

$$\varepsilon_i^C = \frac{\hat{p}_i}{m} \left(\frac{\hat{n}}{2} - \frac{1}{c} \sum_{k=1}^{c} \sum_{j=1}^{\hat{n}} \hat{r}_j^k(\hat{\mu}_k) \right), \tag{6.25}$$

and

$$\varepsilon_i^D = \frac{\langle \hat{\mu}_k \rangle_{k=1}^c}{m} \operatorname{Cov}\left(\hat{p}_i^k, \frac{\hat{w}_k}{\hat{w}} \right) - \operatorname{Cov}\left(\hat{p}_i^k, \frac{\hat{\mu}_k}{m} \right) = \tag{6.26}$$

$$= \frac{\langle \hat{\mu}_k \rangle_{k=1}^c}{m} \operatorname{Cov}\left(\hat{p}_i^k, \frac{\hat{w}_k}{\hat{w}} - \frac{\hat{\mu}_k}{\langle \hat{\mu}_k \rangle_{k=1}^c} \right). \tag{6.27}$$

.

The three basic error components thus derived, ε_i^B, ε_i^C, and ε_i^D, each have a natural interpretation in terms of the assumptions of Theorem 1: ε_i^D depends on deviations from **A1b** and **A3**, while the former two depend on deviations from **A1a** (per-party in the case of ε_i^B and aggregate in the case of ε_i^C).

To approximate the magnitude of the approximation errors in seat allocation formula occasioned by deviations from the assumptions, we will consider two cases. First, we will estimate upper error bounds for the typical case, which is also the intended use-case for Theorem 1, viz., a system with multi-member districts, i.e., $m \gg 1$. We note that real-life values of m in countries using the Jefferson–D'Hondt method for parliamentary elections vary between 6 and 200, with most of the distribution concentrated between 10 and 20.

We can use Theorem 2 to approximate the upper bounds for the absolute values of ε_i^B and ε_i^C. By (5.31), we have

$$\left| \frac{1}{2} - \hat{r}_i^k(\hat{\mu}_k) \right| \leq \frac{1}{12} \left\langle m_k^{-1} \right\rangle_{k=1}^c V(\hat{P}_i), \tag{6.28}$$

where $V(\hat{P}_i)$ is the total variation of the probability distribution fitted to $\{\hat{p}_i^1, \ldots, \hat{p}_i^k\}$. If, as we can safely assume from the available empirical data, the distribution of district magnitudes can be approximated by a Poisson distribution, i.e., we can think of $m_1, \ldots, m_c$ as k independent realizations of a random variable $M - 1$, where $M \sim \operatorname{Pois}(\lambda)$ and $\lambda = m - 1$. Thus,

$$\mathbb{E}(M^{-1}) = \sum_{j=0}^{\infty} \frac{1}{j+1} \frac{\lambda^k e^{-\lambda}}{j!} = \frac{1 - e^{-\lambda}}{\lambda} < \lambda^{-1}. \tag{6.29}$$

As $\lambda = m - 1$, we then obtain from (6.19)

$$\left| \varepsilon_i^B \right| \leq \frac{1}{12m(m-1)} V(\hat{P}_i), \tag{6.30}$$

and by (6.25)

$$\left| \varepsilon_i^C \right| \leq \frac{\hat{n}}{12s(m-1)} \max_{i=1,\ldots,\hat{n}} V(\hat{P}_i). \tag{6.31}$$

If, as it is likely in real-life data, $m > V(\hat{P}_i)$ and $m > \hat{n}$, both error terms are on the order of magnitude of m^{-2}.

On the other hand, the above reasoning fails if m is small. Thus, let us consider the worst case – that

of single-member districts, i.e., $m = 1$. Then,

$$\left|\varepsilon_i^B\right| \le \frac{1}{12} V(\hat{P}_i), \tag{6.32}$$

and

$$\left|\varepsilon_i^C\right| \le \frac{\hat{n}}{12c} \max_{i=1,\dots,\hat{n}} V(\hat{P}_i). \tag{6.33}$$

Since usually $c \gg \hat{n}$, the first error component, ε_i^B, predominates, and its magnitude depends on the total variation of the vote share distribution, which tends to decrease with the average vote share. Still, the error bounds are much less optimistic than in the multi-district case, suggesting that Theorem 1 is likely to provide a rougher approximation.

The last error component, ε_i^D, cannot be estimated purely mathematically, as it depends on deviations from **A1b** and **A3**. However, as the multipliers, $\mu_1, \dots, \mu_c$, tend to be correlated with district magnitudes, $m_1, \dots, m_c$, and as in real-life electoral systems the latter are in turn highly correlated with the number of voters, $w_1, \dots, w_c$ (to preserve the principle of equality for individual voters), the two components of ε_i^D will in practice tend to cancel each other out. As is the case with ε_i^B and ε_i^C, the error bounds are smaller for larger values of m.