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Deterministic vs. Stochastic Methods for Frontier Estimation: Update and Illustration

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Abstract

We estimate and compare a deterministic production frontier to a production frontier estimated using stochastic methods. This comparison is illustrated by estimation of the Lerner index of monopoly power for a public sector producer. The Lerner index estimates the percentage mark-up of price over marginal cost. For the deterministic method we use bootstrapping methods to construct confidence intervals for the Lerner index and its price and marginal cost components. Marginal cost estimates are derived from a translog cost function. Since market prices are usually not observed for public sector producers or are distorted because of subsidies, we use duality theory and derive price from observed costs and an estimated translog input distance function. Data from German public theaters' production of performances to attract spectators using artistic staff, administrative staff and operating expenditures are used as an example. We find no evidence of monopoly power.

Keywords: *Lerner index, public sector managerial behavior, Aigner-Chu deterministic method, bootstrapping*

1. Introduction

Aigner and Chu [1] (hereafter AC) pioneered a deterministic method of estimating a frontier production technology and in turn, numerous researchers have used AC to estimate production efficiency, productivity growth, and to recover shadow prices of nonmarket services. Given a set of producers, the AC minimizes the summed log distances of observed input/output vectors for all producers to a best-practice production frontier. The AC uses linear programming and can easily incorporate production theoretical constraints such as monotonicity and feasibility. Unlike DEA (Data Envelopment Analysis-[8]) which constrains the technology to be piecewise linear, the AC allows the researcher to specify a function like the translog or quadratic which allow a second-order representation of the technology.

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The classic paper by Nishimizu and Page [28] used AC to decompose productivity growth into efficiency change (catching up to a frontier) and technological change (shifts in the frontier). An early survey by Coelli [10] on agricultural production identified eleven studies using AC. Other researchers used AC to evaluate producer efficiency and productivity when undesirable outputs are joint by-products of desirable goods' production. Joint production has been applied to the production of pollution and for risk incurred by banks and other financial service producers in producing a portfolio of financial services. Färe, Grosskopf, and Weber [18] studied the environmental costs of pesticide use in US agriculture and estimated that pesticide leaching and runoff into ground and surface water cost about 6% of agricultural revenues. Färe, Grosskopf, and Weber [19] used AC to estimate shadow prices for the desirable environmental services derived from conservation lands. Bostian and Herlihy [6] examined the effects of agricultural productivity on wetland conditions and found that higher agricultural productivity reduced wetland health through greater channelization, greater runoff of agricultural chemicals, and poorer drainage. In the US, the New Madrid Floodway helps reduce water levels in the lower Mississippi River basin during high water events. The Floodway has been a source of conflict between the US Army Corps of Engineers who proposed a levee to augment agricultural production and environmental interests who want to maintain wetlands. Weber [37] used AC and estimated a quadratic directional distance function to examine the trade-off between agricultural production and wetland services and found that the Corps' proposal to build a levee would have reduced wetland services of 31 thousand acres by a range of \$61 to \$106 million. Cross, Färe, Grosskopf, and Weber [11] valued the characteristics of vineyard using AC.

Färe, Grosskopf, Noh and Weber [21] used AC and estimated the efficiency of US electric utilities using a directional output distance function and derived estimates of the environmental costs of sulfur dioxide emissions due to fossil fuel usage. In a study of Swedish power plants, Bonilla, Coria, and Sterner [5] used AC to estimate a directional output distance function to study the trade-offs between power production and emissions of carbon dioxide and nitrogen oxides. Their findings indicated that reductions in emissions were primarily driven by Swedish national policies rather than local emission standards.

Joint production of desirable and undesirable by-products has also been studied for financial services' producers using the AC method. Fukuyama and Weber [22, 24, 39] estimated directional output distance functions for Japanese banks and used these estimates to derive the shadow price of nonperforming loans which are jointly produced along with performing loans and securities investments. Using the same methods, Fukyuama and Weber [23] estimated shadow prices for nonperforming loans for Japanese Shinkin banks and other regional banks and found that the relatively low shadow price for nonperforming loans was driven by the low interest rate environment in Japan during 2001-2004. Färe, Fukuyama, Grosskopf and Weber [15] used AC to measure efficiency in price space for Japanese securities firms and found that securities firms could significantly enhance revenues by increasing prices. Devaney, Morillon, and Weber [12] estimated a directional output distance function for US mutual funds and found that if mutual funds were to operate on the production frontier they could reduce risk and increase returns by about 3.2%. Furthermore, after projecting mutual fund risk and return to the frontier, they found that mutual funds should take less risk to be consistent with the capital market line.

Two other studies used AC measure producer performance in education and knowledge production. Färe, Grosskopf and Weber [20] estimated a budget constrained output distance function to price the non-marketed outputs of community colleges, specifically associates degrees, certificates and full-time equivalent students. Weber [16, 38] constructed a network model of knowledge production where universities produced scientific publications, patents, and Ph.D. students in nanobiotechnology disciplines. The network model included past scientific publications as an intermediate product and a directional output distance function was used to estimate producer efficiency and the shadow prices of the scientific outputs.

Although AC has widely used, the estimates attribute all producer deviation from the frontier technology as inefficiency, when some of this deviation might be due to randomness or mis-measured outputs and inputs. Thus, much of the research has either not investigated the statistical properties of the estimators or has generally performed only nonparametric tests of hypotheses. Here we use the AC to estimate the Lerner index for a group of German public theaters. Two bootstraps are employed to recover 95% confidence intervals of the test statistic using the mean output/input vector as the point of approximation. Estimates of the Lerner index using AC are also compared to estimates from Stochastic Frontier Analysis [32] which allows for random error, shocks, or measurement error.

The Lerner index of monopoly power equals the markup of price over marginal cost as a proportion of price [27]. When firms produce multiple outputs, a theoretically consistent aggregate Lerner index can be derived from the individual output Lerner indexes ([17]. When resources are allocated efficiently, such as would occur in a competitive market, price equals marginal cost and the Lerner index equals zero. Higher values of the index indicate greater monopoly power.

Much research has been devoted to examining the efficiency of public sector producers. Tiebout's [35] pioneering work showed that local public goods' producers are constrained by citizen voters who "vote with their feet" if the allocation of public resources is too far from their most preferred level. In contrast, Caplan [7] argued that even if citizen voters can move freely, bureaucrats in municipalities can still earn monopoly rents when local property taxes are capitalized in land, so that movers pay for monopoly power through lower sales prices of their land. Niskanen [29] showed that the lack of a profit motive can cause bureaucrats to pursue their own perquisites, one of which might be the largest possible budget. Wagner and Weber [36] examined the effects of government consolidation of overlapping services on market power. Overlapping occurs when separate services are provided by different governments, such as when education is provided by an independent school district and police and fire protection are provided by a municipality. Their empirical findings suggested that although consolidation can reduce costs, it also lessens the choices available to citizens as it allows the consolidated government to compel citizens to purchase the full line of government services, which increases the monopoly power of government bureaucrats.

When studying private producers, researchers can compare observed market prices with marginal costs estimates derived from a cost function. However, market prices are often unavailable or biased for public sector producers who give away their products or offer subsidies. In this case Färe, Grosskopf and Margaritis [17], exploited the duality theory of Färe and Primont [13] and showed that price(s) can be recovered from an underlying production technology represented by an input distance function.

We estimate price and marginal cost from a translog input distance function and a translog cost function using AC and stochastic frontier analysis (SFA). The translog form allows for second order effects of inputs and outputs on the technology rather than the piecewise linear representation of the technology as in DEA. The non-convex free disposal hull representation of the technology was shown to be a consistent maximum likelihood estimator by Korostelev, Simar and Tsybakov [25] and these results were extended to the DEA estimator by Korostelev, Simar and Tsybakov [26]. One problem with the original formulation of AC is that the estimators have no statistical properties. Schmidt [31] showed that the AC linear programming estimator is a maximum likelihood estimator when the distribution of inefficiency is exponential and that Ordinary Least Squares could be used to recover unbiased estimates of the slope coefficients of a production function, but not the intercept. However, in conclusion he claimed that the "statistical properties of the estimators remain uncertain[31](p. 239)."

Since the original AC estimator is deterministic, we apply two different bootstraps to construct confidence intervals for the Lerner index: the Simar and Wilson [33] bootstrap and the Simar and Wilson [34] m out of n bootstrap suggested by Amsler et al. [2] in their study of the AC method. Our empirical example uses production and cost data for German public theaters to recover spectator prices to performances, the marginal cost of an additional spectator, and the resulting Lerner index for each estimation method. The findings from the two bootstraps using AC and from SFA are consistent and indicate that German theaters exert no monopoly power.

2. Theory

We follow Färe, Grosskopf and Margaritis [17] and extend the example of Bishop, Färe, Grosskopf, Hayes, Weber and Wetzel [4] for German public theaters. Let $y \in R_+$ represent a single output[1] which is produced from $x \in R_+^N$ variable inputs and $z \in R_+^J$ fixed inputs. The prices of the variable inputs are $w \in R_+^N$.

Given fixed inputs, z, the input requirement set is

$$L(y, z) = \{x : (x, z) \text{ can produce } y\}. \tag{1}$$

This set consists of the variable inputs that, when combined with the fixed inputs z can produce the outputs y.[2] Given $L(y, z)$ and w, the cost function is defined as

$$C(y, w, z) = \min_x \{wx : x \in L(y, z)\}. \tag{2}$$

The cost function is non-negative, linearly homogeneous in input prices, w, and nondecreasing in w and y. These conditions are

$$\begin{aligned} &\text{(i)} \;\; C(y, \gamma w, z) = \gamma C(y, w, z), \\ &\text{(ii)} \;\; C(y, w', z) \geq C(y, w, z) \text{ for } w' \geq w, \\ &\text{(iii)} \;\; (C(y', w, z) \geq C(y, w, z) \text{ for } y' \geq y. \end{aligned} \tag{3}$$

[1]The single output case can be easily extended to multiple outputs as in Bishop et al. [4].

[2]For an axiomatic representation of $L(y)$ see eg., Färe and Primont [13], p. 129.

The Lerner index of monopoly power is

$$L = \frac{p - \mathrm{d}C(y,w,z)/\mathrm{d}y}{p}$$
$$L = \frac{p - MC}{p}, \tag{4}$$

where p is the output price and $\mathrm{d}C(y,w,z)/\mathrm{d}y = MC$ is the marginal cost of producing y. Under perfect competition $p = MC$ and $L = 0$.

Next, we seek a means to recover price when it is unavailable or biased due to subsidies, as is the case with many public sector outputs, including public theaters. Here we rely on the input distance function which is

$$D_i(y,x,z) = \max_{\lambda} \ \{\lambda : (x/\lambda) \in L(y,z)\}. \tag{5}$$

The distance function $D_i(y,x,z)$ measures the maximum proportional contraction of variable inputs x such that those inputs can still produce output y given fixed inputs z. This distance function and the input requirement set are related in that

$$x \in L(y,z) \text{ if and only if } D_i(y,x,z) \geq 1. \tag{6}$$

When production occurs on the frontier of $L(y,z)$, the distance function takes its minimum, $D_i(y,x,z) = 1$. The distance function is linearly homogeneous in x,

$$D_i(y,\gamma x,z) = \gamma D_i(y,x,z) \tag{7}$$

and has monotonicity conditions for x and y such that

$$D_i(y,x',z) \geq D_i(y,x,z), \ \ x' \geq x,$$
$$D_i(y',x,z) \leq D_i(y,x,z), \ \ y' \geq y. \tag{8}$$

Following Färe and Primont [13] and recent work by Färe, Grosskopf and Margaritis [17], the profit function can be written as

$$\pi(w,p,z) = \max_{x,y} py - \frac{wx}{D_i(y,x,z)}. \tag{9}$$

The first-order condition associated with (9) is

$$\frac{\partial \pi(w,p,z)}{\partial y} = p + \frac{wx}{D_i(y,x,z)^2}\frac{\partial D_i(y,x,z)}{\partial y} = 0. \tag{10}$$

Rearranging (10) yields the pricing formula:

$$p = -\frac{wx}{D_i(y,x,z)^2}\frac{\partial D_i(y,x,z)}{\partial y}. \tag{11}$$

2.1. Translog Functional Form

Our data comprise an unbalanced panel of public theaters offering performances during 14 theatrical seasons. Our model includes $N = 3$ variable inputs, $M = 1$ output, $J = 2$ fixed inputs, and indicator variables DT_t for each theatrical season. The cost function is homogeneous of degree +1 in input prices and the input distance function is homogeneous of degree +1 in inputs. Färe and Sung [14] showed that a homogeneous function that is also a flexible functional form (or generalized quadratic) can only take two forms: (1) the mean of order ρ and (2) the translog. The translog function was developed by Christenson, Jorgenson, and Lau [9]. We use the translog form because it includes both first and second order parameters and can be parameterized to satisfy homogeneity. Furthermore, the monotonicity conditions for the input distance function and cost function can be imposed for the translog form using the AC method (see [21] an example).

For our model the translog input distance function is

$$\begin{aligned}\ln D_i(y,x,z) &= \alpha_0 + \alpha_1 \ln y + \sum_{n=1}^{3} \beta_n \ln x_n + \sum_{j=1}^{2} \psi_j \ln z_j \\ &+ 0.5\left(\alpha_{11}\ln y^2 + \sum_{n=1}^{3}\sum_{n'=1}^{3} \beta_{nn'} \ln x_n \ln x_{n'} + \sum_{j=1}^{2}\sum_{j'=1}^{2} \psi_{jj'} \ln z_j \ln z_{j'}\right) \\ &+ \sum_{n=1}^{3} \delta_n \ln x_n \ln y + \sum_{j=1}^{2} \gamma_j \ln z_j \ln y + \sum_{j=1}^{2}\sum_{n=1}^{3} \phi_{jn} \ln z_j \ln x_n \\ &+ \sum_{t=2}^{14} \tau_t DT_t. \end{aligned} \tag{12}$$

In (12), the first row gives the first order effects of changes in x, y, and z; the second row gives the second order effects; the third row gives the cross-product effects; and the last row allows for parallel shifts in the frontier from year to year. Parameter restrictions corresponding to feasibility (6), homogeneity (7), and monotonicity (8) impose constraints on the translog distance function as:

$$\begin{aligned} &\text{(i) } \ln D_i(y,x,z) \geq 0, \\ &\text{(ii) } \sum_{n=1}^{3} \beta_n = 1,\ \sum_{n=1}^{3} \delta_n = 0,\ \sum_{n'=1}^{3} \beta_{nn'} = 0,\ \sum_{j=1}^{2} \phi_{jn} = 0,\ n = 1,2,3, \\ &\text{(iii) } \frac{\partial \ln D_i(y,x,z)}{\partial y} = \alpha_1 + \alpha_{11}\ln y + \sum_{n=1}^{3} \delta_n \ln x_n + \sum_{j=1}^{2} \gamma_j \ln z_j \leq 0, \\ &\text{(iv) } \frac{\partial \ln D_i(y,x,z)}{\partial x_n} = \beta_n + \sum_{n'=1}^{3} \beta_{nn'} \ln x'_n + \delta_n \ln y + \sum_{j=1}^{2} \phi_{jn} \ln z_j \geq 0, n = 1,2,3. \end{aligned} \tag{13}$$

The translog cost function takes the form

$$\begin{aligned}\ln C(y,w,z) &= a_0 + a_1\ln y + \sum_{n=1}^{3} b_n \ln w_n + \sum_{j=1}^{2} c_j \ln z_j \\ &+ 0.5\left(a_{11}\ln y^2 + \sum_{n=1}^{3}\sum_{n'=1}^{3} b_{nn'}\ln w_n \ln w_{n'} + \sum_{j=1}^{2}\sum_{j'=1}^{2} c_{jj'}\ln z_j \ln z_{j'}\right) \\ &+ \sum_{n=1}^{3} d_n \ln w_n \ln y + \sum_{j=1}^{2} e_j \ln z_j \ln y + \sum_{n=1}^{3}\sum_{j=1}^{2} f_{nj}\ln z_j \ln w_n \\ &+ \sum_{t=2}^{14} h_t DT_t. \end{aligned} \tag{14}$$

Like the distance function, translog costs depend on first order effects of output y, input prices, w, and fixed inputs z in the first row of (14); second order and cross-product effects in rows two and three; and shifts in the cost function from year to year in row four. For the translog cost function (14) the cost function restrictions in (3) can be written as

$$\begin{aligned} &\text{(i)}\ \sum_{n=1}^{3} b_n = 1,\ \sum_{n'=1}^{3} b_{nn'} = 0, \sum_{n=1}^{3} d_n = 0,\ \sum_{j=1}^{2} f_{nj} = 0,\ n = 1,2,3, \\ &\text{(ii)}\ \frac{\partial \ln C(y,w,z)}{\partial \ln w_n} = b_n + \sum_{n'=1}^{3} b_{nn'}\ln w_{n'} + d_n \ln y + \sum_{j=1}^{2} f_{nj}\ln z_j \geq 0,\ n = 1,2,3, \\ &\text{(iii)}\ \frac{\partial \ln C(y,w,z)}{\partial \ln y} = a_1 + a_{11}\ln y + \sum_{n=1}^{3} d_n \ln w_n + \sum_{j=1}^{2} e_j \ln z_j \geq 0. \end{aligned} \tag{15}$$

3. Estimation Methods

We estimate the translog cost and input distance functions using four different methods. Three of the methods use the AC deterministic method and the fourth estimates the two functions using stochastic frontier analysis (SFA).

3.1. Stochastic Frontier Analysis

Homogeneity of the input distance function implies $D_i(y, \gamma x, z) = \gamma D_i(y, x, z)$. Our estimation imposes homogeneity by transforming the data using $\gamma = \frac{1}{x_1}$. That is,

$$D_i(y, \frac{x}{x_1}, z) = \frac{1}{x_1} D_i(y, x, z). \tag{16}$$

Taking the natural logarithm yields

$$\ln D_i(y, \frac{x}{x_1}, z) = -\ln x_1 + \ln D_i(y, x, z). \tag{17}$$

Rearranging and adding a residual, yields our estimating equation

$$-\text{ln}x_1 = \text{ln}D_i(y, \frac{x}{x_1}, z) - \text{ln}D_i(y, x, z) + v$$
$$-\text{ln}x_1 = \text{ln}D_i(y, \frac{x}{x_1}, z) + \epsilon, \tag{18}$$

where $\epsilon = v - \mu$ is the two part error term where v has a symmetric distribution with $E(v) = 0$ and μ is a non-negative variable associated with technical inefficiency and $\text{ln}D_i(y, \frac{x}{x_1}, z)$ has the translog form.

Add subscripts, $k = 1, ..., K$ for DMUs (Decision-Making Units) and $t = 1, \ldots, T$ for time. Input technical efficiency (TE_{kt}) equals the reciprocal of the input distance function with $0 \leq TE_{kt} \leq 1$. Estimates of technical efficiency are recovered as

$$\widehat{TE}_{kt} = \frac{1}{\hat{D}_i(y_{kt}, x_{kt}, z_{kt})} = \exp(-\hat{\mu}_{kt}), \quad k = 1, ..., K, \quad t = 1, \ldots, T. \tag{19}$$

To estimate the cost function using SFA we append a residual, ξ_{kt} to the translog cost function given in (14). This residual is decomposed into the sum of two parts, $\xi_{kt} = \psi_{kt} + \omega_{kt}$, where ψ_{kt} is a random error term that has a symmetric distribution with $E(\psi) = 0$ and ω_{kt} is a nonnegative variable that determines cost inefficiency as $ci_{kt} = 1 - \exp(-\omega_{kt})$. We impose homogeneity by normalizing input prices and costs by w_1.

3.2. Aigner-Chu Deterministic Method

We also estimate $\text{ln}D_i(y, x, z)$ and $\text{ln}C(y, w, z)$ using the Aigner and Chu [1] deterministic method (AC). For the translog distance function, the AC chooses the parameters by minimizing the sum of the log distances between each DMU's observed output/input vector and the frontier isoquant. Parameters of the translog cost function are chosen similarly, by minimizing the summed log distances between observed costs and minimum costs for each DMU. Then, we use two different bootstraps to obtain distributions of the parameter estimates and confidence intervals for the Lerner index. First, we implement the bootstrap of Simar and Wilson [33] that was originally used to obtain confidence intervals of technical efficiency for each DMU in a sample. Second, we implement the m out of n bootstrap of Simar and Wilson [34]. This bootstrap takes a large number of samples of m observations from the original sample of n observations. For each sample we estimate the distance function, cost function, and the Lerner index. The two bootstraps give a distribution of the parameter estimates so that confidence intervals for price, marginal cost, and the Lerner index can be obtained.

For the input distance function in (12) the AC chooses parameters (α_0, α_1, β_n, ψ_j, α_{11}, $\beta_{nn'}$, $\psi_{jj'}$, δ_n, γ_j, ϕ_{jn}, τ_t) to minimize the sum of the log radial distances of the observed inputs for each producer to the isoquant frontier. Feasibility requires that $D_i(y, x, z) \geq 1$ for each DMU's observed (y, x, z). For the translog form feasibility requires $\text{ln}D_i(y, x, z) \geq 0$, with frontier firms having $\text{ln}D_i(y, x, z) = 0$. Thus, $\text{ln}D_i(y, x, z)$ is the log distance to the frontier. The objective function minimizes the sum of these log distances over all observations. In addition, we impose homogeneity, monotonicity and symmetry conditions on the parameter estimates. Appending subscripts k and t to the inputs and outputs in (12) the linear programming problem we solve is

$$\text{minimize} \quad \sum_{t=1}^{T}\sum_{k=1}^{K} \ln D_i(y_{kt}, x_{kt}, z_{kt}) \text{ subject to}$$

$$\text{(i) } \ln D_i(y_{kt}, x_{kt}, z_{kt}) \geq 0,\ k = 1, \ldots, K,\ t = 1, \ldots, T,$$

$$\text{(ii) } \sum_{n=1}^{3} \beta_n = 1,\ \sum_{n=1}^{3} \delta_n = 0,\ \sum_{n'=1}^{3} \beta_{nn'} = 0,\ \sum_{j=1}^{2} \phi_{jn} = 0,\ n = 1, 2, 3,$$

$$\text{(iii) } \frac{\partial \ln D_i(y_{kt}, x_{kt}, z_{kt})}{\partial \ln x_{nkt}} = \beta_n + \sum_{n'=1}^{3} \beta_{nn'} \ln x_{n'kt} + \delta_n \ln y_{kt} + \sum_{j=1}^{2} \phi_{jn} \ln z_{jkt} \geq 0,$$

$$n = 1, 2, 3,\quad k = 1, \ldots, K,\ t = 1, \ldots, T,$$

$$\text{(iv) } \frac{\partial \ln D_i(y_{kt}, x_{kt}, z_{kt})}{\partial \ln y_{kt}} = \alpha_1 + \alpha_{11} \ln y_{kt} + \sum_{n=1}^{3} \delta_n \ln x_{nkt} + \sum_{j=1}^{2} \gamma_j \ln z_{jkt} \leq 0,$$

$$k = 1, \ldots, K,\ t = 1, \ldots, T,$$

$$\text{(v) } \beta_{nn'} = \beta_{n'n},\quad \psi_{jj'} = \psi_{j'j}. \tag{20}$$

Following Simar and Wilson [33] we obtain a bootstrap sample of parameter estimates of the input distance function. Let $kt = 1, \ldots, KT$ represent the number of observations in the unbalanced panel. The steps we implement are as follows:

1. Estimate (20) and obtain the estimates of input technical efficiency for each observation: $\hat{\theta}_{kt} = \frac{1}{\hat{D}_i(y_{kt}, x_{kt}, z_{kt})}$, $kt = 1, \ldots, KT$.

2. Generate a random sample of size KT from the distribution of $\hat{\theta}_{kt}$, giving $\hat{\theta}^*_{kt}$, $kt = 1, \ldots, KT$, i.e., $\hat{\theta}^*_1, \hat{\theta}^*_2, \ldots, \hat{\theta}^*_{KT}$.

3. Use equations (4.20), (4.24), (4.25) and (4.27) of Simar and Wilson to obtain the smoothed bootstrap distribution of $\hat{\theta}^*_{kt}$.

4. Compute pseudo inputs $x^*_{kt} = \frac{\hat{\theta}_{kt}}{\hat{\theta}^*_{kt}} x_{kt}$.

5. Use AC to estimate the translog input distance function using outputs y_{kt} and pseudo inputs, x^*_{kt}, to get bootstrap estimates of pseudo efficiency, θ^*_{kt}, $kt = 1, \ldots, KT$ and parameter estimates $\alpha^*_o,\ \alpha^*_1,\ \beta^*_n,\ \alpha^*_{11},\ \beta^*_{nn'},\ \psi^*_{jj'},\ \delta^*_n,\ \gamma^*_j,\ \phi^*_{jn}, \tau^*_t$.

6. Repeat steps 2-5 a large number of times, B, to obtain the bootstrap of technical efficiency for each observation and the bootstrap parameter estimates, $\alpha^{*b}_o,\ \alpha^{*b}_1,\ \beta^{*b}_n,\ \alpha^{*b}_{11},\ \beta^{*b}_{nn'},\ \psi^{*b}_{jj'},\ \delta^{*b}_n,\ \gamma^{*b}_j,\ \phi^*_{jn},\ \tau^{*b}_t,\ b = 1, \ldots, B$.

The translog cost function (14) is also estimated using AC by choosing parameters $(a_0, a_1, b_n, c_j, a_{11}, b_{nn'}, c_{jj'}, d_{\ldots}$ to minimize the sum of the deviations between log actual costs ($\ln c_{kt}$) and log minimum costs ($\ln C(y_{kt}, w_{kt}, z_{kt})$)

of production over all producer observations. This LP problem estimates the cost function parameters:

$$
\begin{aligned}
&\underset{a_0,a_1,a_{11},b_n,b_{nn'},c_{jj'},d_n,e_j,f_{nj},h_t}{\text{minimize}} \quad \sum_{t=1}^{T}\sum_{k=1}^{K}(\ln c_{kt} - \ln C(y_{kt}, w_{kt}, z_{kt})) \quad \text{subject to}\\
&\text{(i) } \ln C(y_{kt}, w_{kt}, , z_{kt}) \leq \ln c_{kt},\ k=1,\ldots,K,\ t=1,\ldots,T,\\
&\text{(ii) } \sum_{n=1}^{3} b_n = 1,\ \sum_{n'=1}^{3} b_{nn'} = 0, \sum_{n=1}^{3} d_n = 0,\ \sum_{j=1}^{2} f_{nj} = 0,\ n=1,2,3,\\
&\text{(iii) } \frac{\partial \ln C(y_{kt}, w_{kt}, , z_{kt})}{\partial \ln w_{nkt}} = b_n + \sum_{n'=1}^{3} b_{nn'} \ln w_{n'kt} + d_n \ln y_{kt} + \sum_{j=1}^{2} f_{nj} \ln z_{jkt} \geq 0,\\
&\qquad n=1,2,3,\ k=1,\ldots,K,\ t=1,\ldots,T\\
&\text{(iv) } \frac{\partial \ln C(y_{kt}, w_{kt}, z_{kt})}{\partial \ln y_{kt}} = a_1 + a_{11}\ln y_{kt} + \sum_{n=1}^{3} d_n \ln w_{nkt} + \sum_{j=1}^{2} e_j \ln z_{kjt} \geq 0,\\
&\qquad k=1,\ldots,K,\ t=1,\ldots,T\\
&\text{(v) } b_{nn'} = b_{n'n},\ \ c_{jj'} = c_{j'j},\ .
\end{aligned}
\tag{21}
$$

For the translog cost function (14) the restrictions in (21) impose feasibility (i), homogeneity (ii), monotonicity (iii and iv) and symmetry (v).

Cost efficiency equals the ratio of minimum costs to actual costs. After solving (21), estimates of cost efficiency are found as

$$
\sigma_{kt} = \frac{C(w_{kt}, y_{kt}, z_{kt})}{c_{kt}},\ k=1,\ldots,K,\ t=1,\ldots,T. \tag{22}
$$

We again follow Simar and Wilson [33] and obtain a bootstrap sample of the parameter estimates of the translog cost function. For the $kt = 1,\ldots,KT$ observations the steps we implement are as follows:

1. Estimate (21) and obtain estimates of cost efficiency for each observation: $\hat{\sigma}_{kt},\ kt = 1,\ldots,KT$.
2. Generate a random sample of size KT from the distribution of $\hat{\sigma}_{kt}$, giving $\hat{\sigma}^*_{kt}$, $kt = 1,\ldots,KT$, i.e., $\hat{\sigma}^*_1, \hat{\sigma}^*_2, \ldots, \hat{\sigma}^*_{KT}$.
3. Use equations (4.20), (4.24), (4.25) and (4.27) of Simar and Wilson to obtain the smoothed bootstrap distribution of $\hat{\sigma}^*_{kt}$.
4. Compute pseudo input prices $w^*_{kt} = \frac{\hat{\sigma}^*_{kt}}{\hat{\sigma}_{kt}} w_{kt}$.
5. Use AC to estimate the translog cost function using output y_{kt} and pseudo input prices, w^*_{kt}, and obtain the parameter estimates, $(a^*_0, a^*_1, b^*_n, c^*_j, a^*_{11}, b^*_{nn'}, c^*_{jj'}, d^*_n, e^*_j, f^*_{nj}, h^*_t)$, and marginal cost, MC^*.
6. Repeat (2)-(5) above a large number of times, B, to obtain the bootstrap distribution of parameter estimates, $(a^{*b}_0, a^{*b}_1, b^{*b}_n, c^{*b}_j, a^{*b}_{11}, b^{*b}_{nn'}, c^{*b}_{jj'}, d^{*b}_n, e^{*b}_j, f^{*b}_{nj}, h^{*b}_t)$, and estimates of marginal cost, MC^{*b}. $b = 1,\ldots,B$.

Our final set of estimates are derived using the m out of n bootstrap which was developed by Politis, Romano, and Wolf [30], extended by Bickel and Sakov [3], adapted for nonparametric efficiency models by Simar and Wilson [34], and used by Amsler, Leonard, and Schmidt [2] in their study of the properties of the AC. This bootstrap samples with replacement m observations from an original set of n observations (in our notation KT is used instead of n, but we follow convention and refer to it as m out of n). We choose the sample size m following Simar and Wilson [34] who used Algorithm 6.1 of Politis, Romano and Wolf [30]. Our parameter of interest is the Lerner index, L. We use AC to estimate the distance and cost functions for different sample sizes: $m_{100}, m_{125}, \ldots, m_{725}$, where the sample size increases in increments of 25. For each sample size m_j, we draw $B = 1000$ bootstrap samples and obtain the 2.5 and 97.5 percentile estimates of L. Next, we obtain the standard deviation of the percentile estimates from the sample sizes $m_{j-k}, \ldots, m_j, \ldots, m_{j+k}$, where k is a small number. Finally, we add the standard deviations for the 2.5 and 97.5 percentile estimates and choose the j that minimizes the summed standard deviations. For $k = 1$ and $k = 2$ the sample size that minimizes the summed standard deviations is $m = 450$. (See Appendix B Table 5.)

4. An Example-German Public Theaters

As an empirical example of the four methods described above we use an unbalanced panel of KT=1791 observations of German public theaters where one observation corresponds to a theatrical season for a particular theater during the fourteen theatrical seasons from 2004–05 to 2017–18. During a theatrical season a theater produces a final output of spectators (y), using variable inputs of artistic staff (x_1), administrative staff (x_2), and real operating expenditures (x_3). In addition, the number of spectators is conditional on two exogenous quasi-fixed variables: the number of performances offered in a season (z_1) and the number of venues the theater has (z_2). We include time indicators, DT_t to allow shifts in the frontier from season to season. We drop the indicator, DT_1, corresponding to the 2004–05 season to avoid exact linear dependence. Descriptive statistics are reported in Table 1.[3]

Although we use the same data as found in Bishop et al. [4] our model and method differs in several ways from that previous research. First, all inputs, outputs, and quasi-fixed variables are in log form and all variables have first order, second order, and interaction effects unlike Bishop et al. (2024). Second, Bishop et al. estimated price and marginal cost of an additional spectator for each theater's inputs, outputs and quasi-fixed variables. Here we estimate price and marginal cost at the mean values of inputs, outputs, and quasi-fixed inputs. Third, Bishop et al. assumed that theater performances, along with spectators, were a final output. In this paper, we do not constrain the effects of an increase in performances to have a negative effect on the input distance function. Relaxing this constraint allows an increase in performances to either increase the input distance function similar to other inputs or decrease the input distance function if performances have an effect similar to the final output of spectators. Fourth, the present paper estimates the model using two bootstraps applied to the AC deterministic method and compares those estimates to estimates derived from SFA.

Applying the pricing formula (11) to the translog input distance function gives the output price for

[3] The nominal price of operating expenditures is 1 and its real price is 1 deflated by the Consumer Price Index for Germany (Statisches Bundersamt D Federal Statistical Office) with a base year of 2015. All money values have also been deflated.

Table 1. Descriptive Statistics

Variable	Mean	Std. Dev.	Minimum	Maximum
spectators y_1	140807	105752	423	636187
artistic staff x_1	134	104	1	679
admin/technical staff x_2	152	132	2	877
operating expenditures x_3	5319999	5522925	97439	43101784
wage of artistic staff w_1	54463	43217	3959	966011
wage of admin/tech staff w_2	42078	10275	7501	80975
price of oper. exp. w_3	1	0	1	1
performances z_1	486	249	5	1618
venues z_2	6	3	1	25
Cost=wx	20010659	18097976	444564	125938746

spectators as

$$p = -\frac{wx}{D_i(y,x,z)^2}\left(\frac{\partial D_i(y,x,z)}{\partial y}\right),$$
$$p = -\frac{wx}{D_i(y,x,z)}\left(\frac{\alpha_1 + \alpha_{11}\ln y + \sum_{n=1}^{3}\delta_n \ln x_n + \sum_{j=1}^{2}\gamma_j \ln z_j}{y}\right). \tag{23}$$

The marginal cost of an additional spectator is estimated as

$$MC = \frac{\partial \ln C(y,w,z)}{\partial \ln y}\frac{C}{y}$$
$$= \left(a_1 + a_{11}\ln y + \sum_{n=1}^{3} d_n \ln w_n + \sum_{j=1}^{2} e_j \ln z_j\right)\frac{C}{y}. \tag{24}$$

For each method we estimate the price of spectators and its marginal cost at the means $\bar{y}$, $\bar{x}$, $\bar{w}$, and $\bar{z}$ given in Table 1. The SFA parameter estimates for the translog cost and distance functions are provided in Appendix A and the Aigner-Chu parameter estimates are reported in Appendix B.

4.1. Estimates

Table 2 reports the Lerner index and its components. The stochastic estimates are from Stata. The 2.5 and 97.5 percentile values for price, marginal cost, and the Lerner index are estimated using the delta method. Evaluated at the means from Table 1 the price is $\bar{p} = 139.5$ and marginal cost is $\bar{MC} = 141.6$. Given that we estimated two equations, the Lerner index for spectators is estimated two ways. First, we hold price constant at p=139.5 and estimate L accounting for variation in marginal cost. Second, we hold marginal cost constant at MC=141.6 and estimate L accounting for price variation derived from variation in the parameter estimates of $\ln D_i(y,x,z)$. Both L indexes give a similar result: the 2.5 to 97.5 percentile range contains 0. Therefore, the stochastic estimates indicate that German theaters are not exercising monopoly power.

The translog distance and cost functions are estimated three ways using AC. First, are the no bootstrap estimates. Second, are the smoothed bootstrap estimates of Simar and Wilson [33]. Third, are m out of n

Table 2. Lerner Index Estimates evaluated at $\bar{y}$, $\bar{x}$, $\bar{w}$, $\bar{z}$

Variable	Method	Mean	2.5 pctl.	97.5 pctl.
p	Stochastic	139.5	134.8	144.2
MC	Stochastic	141.6	135.8	147.4
L	Stochastic, $p = 139.5$	-0.02	-0.57	0.27
L	Stochastic, MC=141.6	0.02	-0.05	0.10
p	AC-no bootstrap	102.5	.	.
MC	AC-no bootstrap	168.6	.	.
L	AC-no bootstrap	-0.65	.	.
p	AC-SW bootstrap	67.5	52.6	83.7
MC	AC-SW bootstrap	80.3	74.7	86.6
L	AC-SW bootstrap	-0.21	-0.54	0.04
p	AC-m of n bootstrap	112.7	70.3	146.8
MC	AC-m of n bootstrap	91.15	63.9	125.2
L	AC-m of n bootstrap	0.18	-0.11	0.38

bootstrap estimates.[4] The two functions are estimated $B = 1000$ times for both the smoothed bootstrap and the m out of n bootstrap. In the m out of n bootstrap the sample size that minimizes the standard deviation of the 2.5–97.5 percentile values is $m = 450$ (See Appendix B.) The smoothed bootstrap uses all 1791 observations while the m out of n bootstrap draws $m = 450$ observations with replacement from the original 1791 observations. Table 2 reports the results.

4.2. Discussion

The stochastic frontier method and AC bootstraps both impose homogeneity on the translog cost and input distance functions. The price of spectators derived from $D_i(y, x, z)$ and marginal cost derived from $C(y, w, z)$ are estimated at the mean values of inputs (x), output (y), and fixed inputs (z) for all estimation methods. The monotonicity conditions for $D_i(y, x, z)$ stated in (8) and monotonicity conditions for $C(y, w, z)$ stated in (3) hold at the point of approximation for both the stochastic and AC estimates. While the AC method imposes monotonicity restrictions on the parameter estimates for all observations, the stochastic estimates are derived without these monotonicity restrictions. In fact, for the stochastic estimates, the monotonicity restriction, $\frac{\partial D_i(y,x,z)}{\partial x_2}$ did not hold for approximately 5% of the observations, although the estimated price of spectators was positive for all observation. For the cost function, the monotonicity restriction, $\frac{\partial C(y,w,z)}{\partial w_3}$, was violated for more than 50% of the observations, although 99% of the observations had positive marginal cost consistent with production theory.

The time indicator variables in the translog distance function (12) and translog cost function (14) allow for shifts in the technological and cost frontiers. The time indicator for the first year of the study (theatrical season 2004–05) is dropped to avoid exact linear dependence. For $\ln D_i(y, x, z)$, positive coefficients for the time indicators in the subsequent years indicate a greater distance from observed inputs, x, to the technological frontier represented by $L(y, z)$ and indicate technological progress, i.e., the same output can be produced with even less input relative to 2004–05. For $\ln C(y, w, z)$, negative coefficients for the time indicators indicate lower costs of production due to technological progress relative to 2004–05. The

[4]The m out of n bootstrap would be m out of KT in our notation.

three AC methods yield positive coefficients for the time indicators in the translog distance function in all years. The time coefficients estimated from the stochastic method are positive in all but the last theatrical season, 2017–2018. In the cost function, negative coefficients occur eight or nine of the years. Similar results are found for estimates from the stochastic method.

Turning to the components of the Lerner index, the stochastic method yielded a higher price for spectators and a higher marginal cost than the estimates from the AC method. One possible reason for the difference in prices for the two estimates can be seen in the pricing formula (11) which shows that price is inversely related to $D_i(y, z, x)$. Technical efficiency equals the reciprocal of the distance function, i.e., $TE = \frac{1}{D_i(y,x,z)}$. At the point of approximation technical efficiency derived using the stochastic method is 0.89, while technical efficiency for the various AC methods ranged from a low of 0.36 for the Simar-Wilson bootstrap to 0.61 for the m out of n bootstrap. Thus, the lower estimated levels of technical efficiency for the AC method yield lower prices relative to the stochastic method.

The estimates of the price of spectators varies from 67.5 for the SW smoothed bootstrap, to 102.5 for the no bootstrap method, to 112.7 for the m out of n bootstrap. The 2.5 to 97.5 percentile range for price from the m out of n bootstrap is 70.3 to 146.8. This range includes the stochastic price estimate of 139.5 and the AC-no bootstrap estimate of 102.5. Marginal cost ranges from 80.3 for the SW smoothed bootstrap to 91.2 for the m out of n bootstrap, to 102.5 for the AC-no bootstrap. The 2.5 to 97.5 percentile range for MC from the m out of n bootstrap includes the MC estimate for the SW bootstrap. Similar to price, the marginal cost estimates are higher for the stochastic method than for the AC methods.

Estimates of the Lerner index range from -0.65 for the AC-no bootstrap to -0.21 for the SW smoothed bootstrap to 0.18 for the m out of n bootstrap. However, the two bootstrap estimates of the Lerner index have 2.5 to 97.5 percentile ranges that include 0. The stochastic estimates also indicate no presence of monopoly power for German theaters.

In Bishop et al. [4], two final outputs—spectators and performances—are assumed. Here we assumed only a single final output of theater spectators. Instead, we impose no constraint on how performances affect the input requirement set. If performances (z_1) are a final output then $\frac{\partial \ln D_i(y,x,z)}{\partial \ln z_1} \leq 0$. If instead, performances are an intermediate product that helps attract spectators to the theater then we would expect $\frac{\partial \ln D_i(y,x,z)}{\partial \ln z_1} \geq 0$. We calculated this derivative using the AC distance function estimates found in Table 6 for the mean values of inputs and outputs and found it to be positive for all three sets of estimates. This result indicates that an increase in performances increases the size of the input requirement set as expected from an intermediate product. The Bishop results found that spectators were under-produced indicated by a price ranging from 76 to 80 with marginal costs ranging from 63 to 68 and a Lerner index of 0.15. Using the two bootstraps we found no evidence of monopoly power.

5. Conclusion

Public sector producers often produce outputs where prices are non-existent or biased because of subsidies. We derive a pricing formula for the output of public sector producers that is consistent with profit maximization. Here, price(s) is(are) derived from an input distance function and observed costs of production. Since public producers might have monopoly power, the pricing formula allows us to derive the Lerner index (L) of monopoly power. The L index measures the markup of price over marginal cost as a proportion of price and equals zero when price equals marginal cost as would occur in a competitive market or if the public sector agents have allocated resources efficiently.

As an empirical example we examine the production of theatrical performances that yield spectators for German public theaters. We specify a translog input distance function and translog cost function and estimate these functions using stochastic frontier analysis and the Aigner-Chu deterministic method (AC) in order to estimate price, marginal cost, and the Lerner index. Using AC, the two functions are estimated without a bootstrap, with the smoothed bootstrap of Simar and Wilson [33], and with the m out of n bootstrap of Simar and Wilson [34]. Although the AC without a bootstrap indicates over-production of spectators ($L < 0$), the stochastic estimates and the two bootstrap estimates have 2.5 to 97.5 percentile ranges for L that includes 0. These results indicate that German public theaters are acting competitively and allocating resources efficiently, evidence in favor of the Tiebout's (1956) theory of local expenditures.

Using the AC along with a bootstrap allows researchers to construct percentile ranges for a statistic of interest. In addition, unlike DEA, the AC allows a second order approximation to the true, but unknown technology. Furthermore, production theoretic constraints, such as monotonicity conditions, can be easily incorporated in the estimation process. However, with AC, the researcher must specify a functional form whereas DEA imposes no functional form to represent the technology. In addition, the chosen functional form determines the number of parameters to estimate and some flexible functional forms, such as the generalized Leontief, are not linear in the parameters when homogeneity is imposed [13]. Investigating how the AC performs with respect to functional forms other than the translog would be a fruitful direction for future research.

A. Appendix-Stochastic Parameter Estimates

B. Appendix-Aigner and Chu Parameter Estimates

The distance and cost function estimates using the Aigner-Chu deterministic method are presented in Tables 6 and 7.

Table 5 reports the 2.5 and 97.5 percentile values of the Lerner index for different size m. We run B=1000 bootstraps and calculate the Lerner index for each bootstrap given a specific value of m. Then, we report the 2.5 percentile ($L_{2.5}$ and 97.5 percentile value ($L_{97.5}$ of the Lerner index from the B=1000 bootstraps for each sample size, m_j. The standard deviation of the two percentiles are reported in the column $s_{2.5}$ and $s_{97.5}$ for $k = 1$. Following Politis, Romanov and Wolf [30] we sum the two standard deviations and choose the value of m that minimizes $s = s_{2.5} + s_{97.5}$. The last column reports the summed standard deviations (s) when $k = 2$. A sample size of $m = 450$ (m_{450}) minimizes s for both $k = 1$ and $k = 2$.

Tables 6 and 7 report the estimates and standard deviations of the parameters of the translog cost and distance functions for the three Aigner-Chu estimation methods. The m of n estimates are reported for $m = 450$.

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Table 3. Translog Distance Function Estimates

Coefficient	Variable	Estimate	Std. Err.	z-value
α_0	cons	-10.862	3.742	-2.90
α_1	ln$y$	0.425	0.334	1.27
β_2	lnx_2	2.431	0.676	3.60
β_3	lnx_3	-0.279	0.556	-0.50
ψ_1	lnz_1	3.630	0.507	7.16
ψ_2	lnz_2	-2.113	0.406	-5.21
α_{11}	ln$y$ln$y$	-0.087	0.036	-2.45
β_{22}	ln$x_2$lnx_2	0.520	0.079	6.59
β_{23}	ln$x_2$lnx_3	-0.251	0.055	-4.57
β_{33}	ln$x_3$lnx_3	0.152	0.049	3.12
ψ_{11}	ln$z_1$lnz_1	-0.243	0.089	-2.74
ψ_{12}	ln$z_1$lnz_2	0.090	0.049	1.84
ψ_{22}	ln$z_2$lnz_2	0.047	0.026	1.81
δ_2	ln$y$lnx_2	0.087	0.037	2.35
δ_3	ln$y$lnx_3	-0.022	0.030	-0.72
γ_1	ln$y$lnz_1	-0.057	0.050	-1.14
γ_2	ln$y$lnz_2	0.023	0.024	0.95
ϕ_{12}	ln$z_1$lnx_2	-0.036	0.048	-0.74
ϕ_{13}	ln$z_1$lnx_3	-0.137	0.040	-3.42
ϕ_{22}	ln$z_2$lnx_2	-0.162	0.043	-3.76
ϕ_{23}	ln$z_2$lnx_3	0.094	0.034	2.75
τ_2	DT_2	0.113	0.044	2.56
τ_3	DT_3	0.100	0.043	2.31
τ_4	DT_4	0.108	0.043	2.50
τ_5	DT_5	0.111	0.043	2.58
τ_6	DT_6	0.083	0.043	1.92
τ_7	DT_7	0.081	0.043	1.90
τ_8	DT_8	0.042	0.043	0.98
τ_9	DT_9	0.054	0.042	1.26
τ_{10}	DT_{10}	0.070	0.042	1.64
τ_{11}	DT_{11}	0.082	0.042	1.93
τ_{12}	DT_{12}	0.088	0.042	2.08
τ_{13}	DT_{13}	0.061	0.042	1.45
τ_{14}	DT_{14}	-0.001	0.042	-0.03
σ_v		0.329	0.016	
σ_u		0.149	0.089	
σ^2		0.130	0.018	

Table 4. Translog Cost Function Estimates

Coefficient	Variable	Estimate	Std. Err.	z-value
a_0	constant	0.134	6.408	0.02
a_1	ln$y$	3.495	0.594	5.88
b_2	lnw_2	1.225	0.872	1.40
b_3	lnw_3	1.006	1.102	0.91
c_1	lnz_1	-6.066	0.937	-6.48
c_2	lnz_2	4.697	0.671	7.00
a_{11}	ln$y$ln$y$	0.219	0.039	5.63
b_{22}	ln$w_2$lnw_2	-0.381	0.106	-3.58
b_{23}	ln$w_2$lnw_3	-0.142	0.191	-0.74
b_{33}	ln$w_3$lnw_3	0.410	0.112	3.66
c_{11}	ln$z_1$lnz_1	-0.027	0.082	-0.32
c_{12}	ln$z_1$lnz_2	0.019	0.040	0.47
c_{22}	ln$z_2$lnz_2	-0.029	0.021	-1.40
d_2	ln$y$lnw_2	-0.267	0.050	-5.35
d_3	ln$y$lnw_3	0.496	0.064	7.81
e_1	ln$y$lnz_1	0.049	0.052	0.95
e_2	ln$y$lnz_2	-0.034	0.026	-1.33
f_{21}	ln$z_1$lnw_2	0.395	0.087	4.54
f_{31}	ln$z_1$lnw_3	-0.507	0.093	-5.47
f_{22}	ln$z_2$lnw_2	-0.590	0.060	-9.80
f_{32}	ln$z_2$lnw_3	0.394	0.069	5.71
h_2	DT_2	-0.055	0.042	-1.32
h_3	DT_3	0.001	0.041	0.03
h_4	DT_4	-0.015	0.041	-0.36
h_5	DT_5	-0.045	0.041	-1.10
h_6	DT_6	-0.017	0.041	-0.41
h_7	DT_7	-0.031	0.040	-0.78
h_8	DT_8	0.016	0.040	0.39
h_9	DT_9	0.003	0.040	0.06
h_{10}	DT_{10}	-0.029	0.040	-0.74
h_{11}	DT_{11}	-0.058	0.040	-1.45
h_{12}	DT_{12}	-0.057	0.040	-1.44
h_{13}	DT_{13}	-0.048	0.039	-1.21
h_{14}	DT_{14}	0.001	0.039	0.04
σ_ψ		0.231	0.014	
σ_ω		0.395	0.027	
σ^2		0.209	0.016	

Table 5. 2.5 and 97.5 percentile values of L index and standard deviations (s) for different sample sizes, m, and k=1,2

			$k=1$			$k=2$
m	$L_{2.5}$	$L_{97.5}$	$s_{2.5}$	$s_{97.5}$	$s=s_{2.5}+s_{97.5}$	s
100	-3.15	0.89				
125	-2.30	0.86	0.621	0.030	0.651	
150	-1.94	0.83	0.307	0.040	0.347	0.644
175	-1.69	0.78	0.131	0.025	0.156	0.321
200	-1.75	0.8	0.075	0.025	0.101	0.229
225	-1.60	0.75	0.170	0.029	0.199	0.224
250	-1.41	0.75	0.152	0.023	0.175	0.277
275	-1.30	0.71	0.120	0.035	0.155	0.230
300	-1.17	0.68	0.081	0.030	0.111	0.165
325	-1.15	0.65	0.025	0.017	0.042	0.164
350	-1.12	0.65	0.119	0.012	0.131	0.181
375	-0.93	0.63	0.152	0.020	0.172	0.177
400	-0.82	0.61	0.064	0.010	0.074	0.158
425	-0.82	0.62	0.012	0.026	0.038	0.072
450	-0.84	0.57	0.012	0.025	0.037	0.041
475	-0.82	0.59	0.025	0.012	0.037	0.093
500	-0.79	0.57	0.068	0.031	0.099	0.094
525	-0.69	0.53	0.055	0.021	0.076	0.119
550	-0.70	0.54	0.055	0.015	0.070	0.093
575	-0.60	0.51	0.050	0.015	0.065	0.059
600	-0.65	0.52	0.032	0.015	0.047	0.062
625	-0.66	0.49	0.021	0.021	0.042	0.064
650	-0.62	0.48	0.061	0.006	0.067	0.065
675	-0.54	0.49	0.040	0.006	0.046	0.061
700	-0.59	0.48	0.029	0.021	0.050	
725	-0.59	0.45				

Table 6. Aigner-Chu Translog Distance Function Estimates

		No bootstrap	SW bootstrap		m of n bootstrap	
Coefficient	Variable	Est.	Est.	Std. Dev.	Est.	Std. Dev.
α_0	cons	10.116	7.431	21.629	4.800	183.347
α_1	ln$y$	-1.711	-1.777	0.181	-1.860	1.811
β_2	lnx_2	0.328	0.006	0.607	0.368	2.660
β_3	lnx_3	0.075	0.400	0.452	0.237	2.375
ψ_1	lnz_1	0.403	0.864	0.260	2.472	3.698
ψ_2	lnz_2	-0.258	-0.308	0.322	-2.231	4.455
α_{11}	ln$y$ln$y$	0.066	0.048	0.002	0.095	0.017
β_{22}	ln$x_2$lnx_2	0.005	-0.036	0.009	0.011	0.029
β_{23}	ln$x_2$lnx_3	0.012	0.034	0.004	0.011	0.017
β_{33}	ln$x_3$lnx_3	0.001	-0.020	0.003	-0.010	0.017
ψ_{11}	ln$z_1$lnz_1	-0.116	-0.203	0.015	-0.190	0.115
ψ_{12}	ln$z_1$lnz_2	0.119	0.142	0.004	0.179	0.062
ψ_{22}	ln$z_2$lnz_2	0.316	0.284	0.005	0.128	0.057
δ_2	ln$y$lnx_2	0.039	0.020	0.002	-0.004	0.009
δ_3	ln$y$lnx_3	-0.006	0.001	0.001	0.016	0.007
γ_1	ln$y$lnz_1	-0.020	0.013	0.005	-0.109	0.032
γ_2	ln$y$lnz_2	-0.128	-0.134	0.002	-0.034	0.010
ϕ_{12}	ln$z_1$lnx_2	-0.041	0.016	0.001	0.044	0.003
ϕ_{13}	ln$z_1$lnx_3	0.042	0.008	0.001	-0.017	0.002
ϕ_{22}	ln$z_2$lnx_2	-0.060	-0.074	0.004	-0.126	0.032
ϕ_{23}	ln$z_2$lnx_3	0.050	0.054	0.002	0.113	0.023
τ_2	DT_2	0.221	0.221	0.003	0.180	0.046
τ_3	DT_3	0.158	0.158	0.003	0.140	0.039
τ_4	DT_4	0.097	0.098	0.003	0.121	0.040
τ_5	DT_5	0.021	0.019	0.003	0.059	0.033
τ_6	DT_6	0.055	0.055	0.003	0.056	0.037
τ_7	DT_7	0.233	0.234	0.003	0.113	0.042
τ_8	DT_8	0.034	0.034	0.002	0.049	0.033
τ_9	DT_9	0.064	0.064	0.002	0.049	0.044
τ_{10}	DT_{10}	0.001	0.003	0.002	0.047	0.037
τ_{11}	DT_{11}	0.296	0.298	0.002	0.128	0.080
τ_{12}	DT_{12}	0.092	0.094	0.002	0.083	0.047
τ_{13}	DT_{13}	0.047	0.046	0.002	0.057	0.047
τ_{14}	DT_{14}	0.047	0.045	0.003	0.011	0.039

Table 7. Aigner-Chu Translog Cost Function Estimates

		No bootstrap	SW bootstrap		m of n bootstrap	
Coefficient	Variable	Est.	Est.	Std. Dev.	Est.	Std. Dev.
a_0	constant	-2.789	-0.918	8.945	-1.424	48.138
a_1	ln$y$	-0.225	-0.295	0.116	0.712	0.930
b_2	lnw_2	-0.316	-0.317	0.352	0.169	0.825
b_3	lnw_3	0.000	0.208	0.202	0.037	0.191
c_1	lnz_1	0.587	0.542	0.284	-1.397	3.040
c_2	lnz_2	0.995	1.038	0.221	0.666	2.316
a_{11}	ln$y$ln$y$	0.181	0.209	0.002	0.015	0.013
b_{22}	ln$w_2$lnw_2	-0.075	-0.010	0.005	-0.031	0.008
b_{23}	ln$w_2$lnw_3	0.000	-0.018	0.002	0.003	0.002
b_{33}	ln$w_3$lnw_3	0.000	0.013	0.002	-0.004	0.002
c_{11}	ln$z_1$lnz_1	0.276	0.372	0.013	0.151	0.118
c_{22}	ln$z_2$lnz_2	-0.349	-0.323	0.004	-0.193	0.051
c_{12}	ln$z_1$lnz_2	-0.350	0.147	0.284	-0.080	0.057
e_1	ln$y$lnz_1	-0.152	-0.197	0.005	0.035	0.034
e_2	ln$y$lnz_2	0.138	0.147	0.001	0.031	0.009
d_2	ln$w_2$ln$y$	-0.001	0.011	0.004	0.038	0.004
d_3	ln$w_3$ln$y$	0.000	-0.001	0.000	-0.004	0.000
f_{21}	ln$w_2$lnz_1	0.192	0.128	0.006	0.051	0.014
f_{31}	ln$w_3$lnz_1	0.000	-0.002	0.001	-0.002	0.001
f_{22}	ln$w_2$lnz_2	-0.164	-0.138	0.004	-0.188	0.010
f_{23}	ln$w_3$lnz_2	0.000	0.000	0.001	0.009	0.001
h_2	DT_2	-0.110	-0.115	0.003	-0.057	0.053
h_3	DT_3	-0.050	-0.065	0.003	-0.067	0.044
h_4	DT_4	-0.020	-0.025	0.003	-0.045	0.044
h_5	DT_5	0.046	0.041	0.003	-0.009	0.035
h_6	DT_6	0.037	0.032	0.003	0.007	0.042
h_7	DT_7	-0.180	-0.183	0.003	-0.023	0.054
h_8	DT_8	0.047	0.042	0.003	0.031	0.035
h_9	DT_9	0.042	0.039	0.003	0.019	0.036
h_{10}	DT_{10}	-0.120	-0.127	0.003	-0.041	0.056
h_{11}	DT_{11}	-0.010	-0.015	0.003	-0.004	0.045
h_{12}	DT_{12}	-0.070	-0.075	0.003	-0.051	0.043
h_{13}	DT_{13}	-0.050	-0.056	0.003	-0.058	0.044
h_{14}	DT_{14}	-0.050	-0.051	0.003	0.020	0.037