Vol. 35, No. 2 (2025) | DOI: 10.37190/ord250203

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Operations Research and Decisions

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OPERATIONS RESEARCH AND DECISIONS QUARTERLY

Solution of an uncertain EPQ model using the Neutrosophic differential equation approach

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Abstract

Time is a very crucial factor in controlling demand patterns for certain products. In manufacturing processes, the production rate must be regulated according to the demand pattern and available stock as part of effective lot size management policies. We incorporate this fundamental idea for constructing the production rate as a function of demand and stock, which is the primary contribution of this paper. Predicting demand patterns and adjusting the production rate inherently involve vagueness. We use neutrosophic logic, an advanced mathematical tool for addressing imprecision in decision planning. Neutrosophic calculus-based analysis of uncertainty involved with the proposed model is the secondary contribution in this paper. Numerical results indicate that the proposed approach yields superior results compared to the crisp environment and traditional neutrosophic approaches for cost minimization. Furthermore, it is worth noting that Case 1 of the proposed Neutrosophic Differential Approach guarantees better results than Case 2.

Keywords: *EPQ model with deterioration, Time impacted demand, Stock and demand dependent product process, Decision making under impreciseness, Neutrosophic ruled uncertainty, Triangular Neutrosophic numbers, Neutrosophic differential equation, Neutrosophic derivative*

1. Introduction

Inventory represents idle resources. Therefore, inventory management involves planning and implementing policies to maintain the optimal stock size, maximizing gains while meeting consumer demand.

Received 8 May 2024, accepted 3 January 2025, published online 8 February 2025
ISSN 2391-6060 (Online)/© 2025 Authors

Inventory management is relevant in both the retail and manufacturing sectors. In manufacturing, production inventory management includes both the manufacturing process and the maintenance of the produced items to optimize average costs. In this context, lot size models concerning order size and production size fulfilled the purpose. In this paper, an economic production quantity (EPQ) model is designed and analyzed using mathematical tools. Demand and production are two important issues impacting the manufacturing-supply-inventory environment. It is noted that demand may increase as time advances in a newly constituted supply enterprise. Therefore, the demand for an EPQ model can be viewed as a function of time. The second notable issue is the dependence of production rates on multiple factors. It is perceived that demand immediately impacts the production rate. Because the managerial body of a manufacturing organization must be bothered about the demand rate while taking measures on production capacity, another important issue is the influence of the present inventory level on the production rate. It is customary to reduce the production rate when the inventory level is already high in the store to avoid carrying costs. Also, the deterioration of production during the inventory inventory-carrying procedure cannot be overlooked. This concern motivates us to design the proposed EPQ model with new insights. Primarily, the model is built in a deterministic environment. However, the consideration of such a deterministic environment may restrict a mathematical model far from the real economic interactions involving uncertainty. Therefore, we find the necessity of uncertainty environments described by aptly fitted mathematical tools. The study of this paper is engaged in finding the answer to the following questions:

(i) Time impacts demand. What will be the overall impact of time on minimizing the average cost ?

(ii) Production rate is influenced by demand and present stock. What are the roles of these issues in cost minimization objectives?

(iii) Deterioration is an unavoidable factor related to the inventory-carrying procedure. What is the subsequent impact of the deterioration factor on cost reduction?

(iv) The impreciseness involved in economic production strategy cannot be ignored. What will be the mathematical tool and approach to encountering such phenomena with the mentioned objective and a meaningful perspective?

With the above-mentioned questionaries, the existing literature has been surveyed, which is detailed in the succeeding section.

The remaining text in this paper is structured in pockets as follows: Section 2 summarizes the literature survey and gaps in the existing literature. Section 3 provides the mathematical preliminaries, which help the reader understand the mathematical foundation of this paper. Section 4 describes the notations and symbols in the paper and their meanings. Also, the same section discusses the assumptions for the mathematical formulation of the proposed model. Section 5 details the proposed model in a crisp environment. Subsequently, Section 6 reconsiders the proposed model under neutrosophic uncertainty. The crispification of the neutrosophic model is described in Section 7. Section 8 is about the numerical results of the proposed model in different environments and approaches. Section 9 lists significant findings and managerial interpretations. The concluding remarks on the investigation and findings of the paper are given in Section 10.

2. Literature review

The literature survey is performed on keywords like recent literature on the EPQ model in a crisp environment, inventory models under different types of uncertainty, and recent advancements in neutrosophic logic and its applications. The subsections corresponding to each mentioned keyword are followed by subsections concerning research gaps, motivations, and the contribution of the paper.

2.1. Recent literature on the EPQ model in a crisp environment

Cárdenas-Barrón [10] proposed an EPQ model incorporating planned backorders. This system produces imperfect-quality items, all of which are reworked within the same production cycle. A deteriorating EPQ model with multiple manufacturing stages and one remanufacturing stage is introduced by Widyadana and Wee [60]. Taleizadeh et al. [57] formulated an EPQ model that incorporates random defective items, repair failures, shortages, and the presence of a single machine, which leads to constrained production capacity. A production-supply model with learning-based production cost is developed by Teng et al. [58] from the seller's perspective to determine the optimal lot size and trade credit tenure simultaneously. Hsu and Hsu [22] presented a defective production-based EPQ model that allows for complete backlog shortages. A learning-based EPQ model is investigated by Khan et al. [25], where learning is imposed on production rate and demand is dependent on variable lead time stochastically. Cunha et al. [13] proposed an EPQ model considering partial backlogged shortages and defective production batches. They demonstrate that selling imperfect items promptly is preferable since the reduction in holding costs leads to an overall cost decrease. Taleizadeh et al. [56] proposed an EPQ model considering sustainability with three different scenarios of shortage, namely, partial backordered shortage, lost sale, and full backorder shortage. They found that a partial backlogged shortage case is the best and most realistic model. A defective production-based EPQ model is presented by Keshavarzfard et al. [24], where the production rate is dependent on the demand pattern. Marchi et al. [30] examined a production inventory model by applying the learning-by-doing approach to the production rate and maintaining the quality of items, which leads to the efficiency of energy and reliability of the system. They demonstrated the interconnectedness between production learning and energy efficiency and how this relationship influences the optimal quantity of lot size. Taleizadeh et al. [55] introduced a reworked-based EPQ system with price-dependent demand-taking pricing, producing quantity, and back-ordered quantity as decision variables. Nobil et al. [38] studied an EPQ model for a defective production system with remanufacturing and shortages under a 100% inspection process. Recently, Rahaman et al. [42] designed an imperfect production-based EPQ model where the demand rate is influenced by the frequency of promotion, the greenness of the item, and the selling price, and the rate of producing defective items is linearly dependent on time. Haque et al. [20] presented a sustainable production inventory model with price- and greenness-dependent demand and demand-dependent remanufacturing rates. In this study, a deteriorating EPQ model is analyzed, where production rate is influenced by demand and on-hand stock level, and demand is influenced by time.

2.2. Inventory models under different types of uncertainty

After fuzzy set theory was introduced, Park [40] was the first to incorporate fuzzy concepts into an Economic Order Quantity (EOQ) model. In the inventory control problem, several research articles are published incorporating fuzzy uncertainty, such as work by Hojati [21], Bag et al. [8], De and Sana [14], Sadeghi et al. [45], Sarkar and Mahapatra [48], Majumder et al. [28], and Mahata et al. [26]. Debnath et al. [17] investigated a sustainable EPQ model in a type-2 fuzzy uncertain environment where the demand pattern is dependent on inventory level and selling price and the production rate is dependent on the demand rate. De and Sana [15, 16] used the notion of intuitionistic fuzzy in controlling strategy. Garai et al. [18] used intuitionistic fuzzy numbers to measure the uncertainty in an inventory system with stock-dependent demand. A good number of studies have been published on inventory models using intuitionistic fuzzy, such as Ali et al. [5], Sahoo et al. [46], Supakar et al. [54], and Giri et al. [19]. Momena et al. [34] discussed an EOQ model with price-dependent demand under all unit price discount policies in a densely fuzzy environment. A sustainable production and rework model is developed in a dense-lock fuzzy environment by Karmakar et al. [23]. Rahaman et al. [44] studied an EPQ model in a lock fuzzy environment where the demand rate is dependent on price, inventory level, and deterioration of the item using a preservation facility. Maiti [27] studied an EPQ model with imperfect production and demand-dependent production rates in an uncertain arena by taking the price of the produced item as a fuzzy cloud number and using PSO to solve the problem. Barman et al. [9] investigated a deteriorating EPQ model with a partial backlogged shortage and time-dependent demand in the cloud fuzzy phenomenon. Manna et al. [29] studied a production inventory model with green-level-dependent demand, considering carbon emissions during production in an uncertain environment. Rahaman et al. [43] interpreted an EOQ model with price and stock-dependent demand in a type-2 interval uncertain environment.

2.3. Recent advancements in neutrosophic logic and its application

The introduction of the neutrosophic philosophy triggers enthusiasm among the researcher communities around the globe. Classification of different types of neutrosophic numbers, namely single-valued (Wang et al. [59]), triangular (Chakraborty et al. [12]), trapezoidal (Ye [61]), pentagonal (Chakraborty [11]), and complex (Ali and Smarandache [6]), neutrosophic numbers, has been done subsequently, along with their possible application in the fields of engineering and management. The intuition and sense of neutrosophic philosophy had been further enriched by several worthy findings, such as the neutrosophic triplet group by Smarandache and Ali [50], the neutrosophic vector space by Agboola and Akinleye [4], and the neutrosophic topological space by Salama and Alblowi [47]. The fuzzy derivative of a fuzzy valued function was first introduced by Puri and Ralescu [41]. Several researchers [1–3, 7] applied this concept to their research. Smarandache [49] introduced the concept of the neutrosophic derivative within the neutrosophic realm as an expansion of the fuzzy derivative. Son et al. [51] gave a novel definition of the neutrosophic derivative, namely, the granular derivative providing the necessary and sufficient state for the granular derivative of a neutrosophic-valued function. Sumathi and Priya [53] and Sumathi and Sweety [52] discussed an NDE based on the parametric representation of the neutrosophic number. However, Moi et al. [31] introduced a new type of neutrosophic derivative, which is known as a generalized neutrosophic derivative and used this definition to discuss a second-order neutrosophic boundary value problem in [32]. Several researchers used neutrosophic logic in various fields, including control theory and decision-making. Mullai and Surya [36, 37] used triangular neutrosophic numbers in the lot-sizing model. Pal and Chakraborty [39] used the area removal technique method to optimize a triangular neutrosophic-based economic order quantity model. Mondal et al. [35] studied a lot-sizing model for deteriorating seasonal products under partial backordering and time-dependent demand. Up to date, the neutrosophic differential equation (NDE) has little use in solving uncertain inventory control problems. Momena et al. [33] used NDE to discuss an uncertain EOQ model where the market demand pattern is dependent on the stock level, price, and warranty time of the item. In this article, the NDE approach is used to solve a production quantity model where demand-related parameters and deterioration rates are taken as neutrosophic numbers.

2.4. Research gaps and motivations

The following Table 1 shows the comparison between the literature in the related keywords and the proposed model:

Table 1. Comparison of the contributions among the literature related to the present paper

References	Model types	Stock-dependent production rate	Demand-dependent production rate	Time-dependent demand	Deterioration	Model environment	Solution Methodology
Khan et al. [25]	EPQ	×	×	√	×	Crisp	Differential equation
Keshavarzfard et al. [24]	EPQ	×	√	√	×	Crisp	Differential equation
Haque et al. [20]	EPQ	×	√	×	×	Crisp	Differential equation
Majumder et al. [28]	EPQ	×	×	√	√	Fuzzy	Fuzzy differential equation
Debnath et al. [17]	EPQ	×	√	×	×	Type-2 fuzzy	Fuzzy differential equation
Rahaman et al. [44]	EPQ	√	×	×	√	Lock fuzzy	Parametric representation of lock fuzzy number
Maiti [27]	EPQ	×	√	×	×	Cloudy fuzzy	Parametric representation of the cloudy fuzzy number
Barman et al. [9]	EPQ	×	×	√	√	Cloudy fuzzy	Parametric representation of the cloudy fuzzy number
Mondal et al. [35]	EOQ	-	-	√	√	Neutrosophic	Parametric representation of the neutrosophic number
Momena et al. [33]	EOQ	-	-	×	×	Neutrosophic	Neutrosophic differential equation
This article	EPQ	√	√	√	√	Neutrosophic	Neutrosophic differential equation

From the detailed research survey on the above-mentioned keywords, we have found the following research gaps which are targeted to be overcome in the present article:

(i) Demand with time dependency was discussed in many existing models. However, the demand-dependent production rate is rarely considered in the existing literature. In this paper, the production rate is assumed to be demand- and stock-dependent.

(ii) Many economic scenarios under impreciseness were discussed using mathematical tools like different fuzzy and interval numbers. Among these studies, only a few adapted uncertain differential equation approaches driven by interval and fuzzy-valued calculus. This lacuna motivates us to consider the proposed EPQ model under an uncertain decision environment and to describe it using uncertain differential equations.

(iii) The philosophy of neutrosophy extends the idea of uncertainty incurred by fuzzy and intuitionistic fuzzy logic. A fuzzy set comprises the notion of membership of elements generalizing the classical set. An intuitionistic fuzzy has the additional feature of a non-membership grade. Furthermore, a neutrosophic set contributes a more structured and generalized mathematical sense of uncertainty with membership, non-membership, and hesitancy grades. Thus, we feel the necessity of discussing the proposed EPQ model under uncertainty using neutrosophic sets and numbers.

(iv) Surveying the applications of the neutrosophic numbers in the inventory models, we find some EOQ and EPQ models in the existing literature. However, in those studies, the de-neutrosophication technique is used very before the optimization. In other words, the dynamics of the neutrosophic-based system in those studies are described in terms of crisp calculus. In this paper, the neutrosophic differential equation is used to describe an inventory control model.

2.5. Novelty

The current article contributes to some novel perspectives on theoretical advancement. They are:

(i) Time has an obvious impact on the demand rate. In many newly organized supply bodies, the enthusiasm increases linearly with time. This concern was addressed in much literature. Also, present stock controls the production rate of a manufacturing body, and this issue has been investigated by many authors. However, we cannot find a single piece of literature where the impact of time on the production rate through demand patterns and the stock has been traced. The proposed model is distinguished in this context.

(ii) The proposed crisp model is analyzed, and the optimality criteria have been found in the cost minimization objective. Then, we introduced uncertainty using the neutrosophic number, which carries a more generalized sense of uncertainty compared to fuzzy and intuitionistic fuzzy numbers. The theory of neutrosophic calculus and differential equations has been employed to analyze the uncertain version of the proposed EPQ model.

(iii) In this current article, a novel de-neutrosophication formula has been introduced.

(iv) Moreover, the new de-neutrosophication formula, along with the neutrosophic differential equation approach, provides better results compared to the crisp and old neutrosophic optimization techniques regarding the cost minimization objective.

3. Mathematical preliminaries

Definition 1. [12] A fuzzy set is denoted by the pair $(x, \mu(x))$, where x is an element in the universal set X, and $\mu(x)$ represents the degree to which x belongs to X, with $\mu(x)$ falling within the range [0, 1].

Definition 2. [12] An intuitionistic fuzzy set is represented by the ordered triplet $(x, \mu(x), \nu(x))$, where x is an element in the universal set X, $\mu(x)$ and $\nu(x)$ represents the degree of belongingness and non-belongingness of x in X, respectively, and both $\mu(x)$ and $\nu(x)$ are within the range [0, 1] satisfying the condition $0 \leq \mu(x) + \nu(x) \leq 1$.

Definition 3. [12] A neutrosophic set is represented by the ordered triplet $(x, T(x), I(x), F(x))$, where x is an element in the universal set X, $T(x)$, $I(x)$ and $F(x)$ respectively signify the degrees of truthiness, indeterminacy, and falsity of x in X. Each of $T(x),\ I(x)$ and $F(x)$ lies in the range [0, 1], fulfilling the condition $0 \leq T(x) + I(x) + F(x) \leq 3$.

Definition 4. [12] A single-valued triangular neutrosophic (SVTN) number of Type 1, denoted as $\tilde{G}_{TN} = (d_1, d_2, d_3; s_1, s_2, s_3; z_1, z_2, z_3)$ is a specific type of neutrosophic set on the real numbers R. It is characterized by its truth, indeterminacy, and falsity membership functions, which are defined as follows:

$$T_{\tilde{G}_{TN}}(x) = \begin{cases} \frac{x-d_1}{d_2-d_1} & when\ d_1 \leq x \leq d_2 \\ 1 & when\ x = d_2 \\ \frac{d_3-x}{d_3-d_2} & when\ d_2 \leq x \leq d_3 \\ 0 & otherwiae \end{cases}$$

$$I_{\tilde{G}_{TN}}(x) = \begin{cases} \frac{s_1-x}{s_2-s_1} & when\ s_1 \leq x \leq s_2 \\ 0 & when\ x = s_2 \\ \frac{x-s_2}{s_3-s_2} & when\ s_2 \leq x \leq s_3 \\ 1 & otherwiae \end{cases}$$

$$F_{\tilde{G}_{TN}}(x) = \begin{cases} \frac{z_1-x}{z_2-z_1} & when\ z_1 \leq x \leq z_2 \\ 0 & when\ x = z_2 \\ \frac{x-z_2}{z_3-z_2} & when\ z_2 \leq x \leq z_3 \\ 1 & otherwiae \end{cases}$$

and $0 \leq T_{\tilde{G}_{TN}}(x) + I_{\tilde{G}_{TN}}(x) + F_{\tilde{G}_{TN}}(x) \leq 3$.

Definition 5. [12] The (α, β, γ)-cut of a neutrosophic set $\tilde{G} = (x, T_{\tilde{G}}(x), I_{\tilde{G}}(x), F_{\tilde{G}}(x)$ over X is indicated by $[\tilde{G}]_{\alpha,\beta,\gamma}$ and is defined by $[\tilde{G}]_{\alpha,\beta,\gamma} = \{\langle x, T_{\tilde{G}}(x), I_{\tilde{G}}(x), F_{\tilde{G}}(x)\rangle : T_{\tilde{G}} \geq \alpha, I_{\tilde{G}}(x) \leq \beta, F_{\tilde{G}} \leq \gamma\}$. It is also known as a parametric representation or parametric form of the neutrosophic set.

Note: The parametric representation of an SVTN number $\tilde{G}_{TN}N = (d_1, d_2, d_3; s_1, s_2, s_3; z_1, z_2, z_3)$ includes six components. These six components are written as $\langle[G_1(\alpha), G_2(\alpha)], [G'_1(\beta), G'_2(\beta)], [G''_1(\gamma), G''_2(\gamma)]\rangle$, where $G_1(\alpha) = d_1 + \alpha(d_2 - d_1)$, $G_2(\alpha) = d_3 - \alpha(d_3 - d_2)$, $G'_1(\beta) = s_2 - \beta(s_2 - s_1)$, $G'_2(\beta) = s_2 + \beta(s_3 - s_2)$, $G''_1(\gamma) = z_2 - \gamma(z_2 - z_1)$ and $G''_2(\gamma) = z_2 + \gamma(z_3 - z_2)$.

Definition 6. [31] Let $\tilde{g} : I \to N$ be a neutrosophic-valued function given in the parametric representation by $\tilde{g}(t) = \langle[g_1(t;\alpha), g_2(t;\alpha)], [g'_1(t;\beta), g'_2(t;\beta)], [g''_1(t;\gamma), g''_2(t;\gamma)]\rangle, \forall t \in I$. The generalized neutrosophic derivative of $\tilde{g}(t)$ at $t = c \in I$ is written as $\dot{\tilde{g}}(c) = \langle \dot{g}_T(c), \dot{g}_I(c), \dot{g}_F(c)\rangle$ in which $\dot{g}_T(c)$, $\dot{g}_I(c)$ and $\dot{g}_F(c)$ are defined as the following

1. $\dot{g}_T(c) = [min\{\dot{g}_1(c;\alpha), \dot{g}_2(c;\alpha)\},\ max\{\dot{g}_1(c;\alpha), \dot{g}_2(c;\alpha)\}]$
2. $\dot{g}_I(c) = [min\{\dot{g}'_1(c;\beta), \dot{g}'_2(c;\beta)\},\ max\{\dot{g}'_1(c;\beta), \dot{g}'_2(c;\alpha)\}]$
3. $\dot{g}_F(c) = [min\{\dot{g}''_1(c;\gamma), \dot{g}''_2(c;\gamma)\},\ max\{\dot{g}''_1(c;\gamma), \dot{g}''_2(c;\gamma)\}]$

provided $\dot{g}_1(c;\alpha)$, $\dot{g}_2(c;\alpha)$, $\dot{g}'_1(c;\beta), \dot{g}'_2(c;\beta)$, $\dot{g}''_1(c;\gamma)$ and $\dot{g}''_2(c;\gamma)$ are all exists. $\dot{\tilde{g}}(c)$ is said to be a type-1 derivative if the parametric representation of $\dot{\tilde{g}}(c)$ is given by

$$\dot{\tilde{g}}(c) = \langle[\dot{g}_1(c;\alpha),\ \dot{g}_2(c;\alpha)], [\dot{g}'_1(c;\beta), \dot{g}'_2(c;\beta)], [\dot{g}''_1(c;\gamma),\ \dot{g}''_2(c;\gamma)]\rangle$$

and type-2 derivative if the parametric representation of $\tilde{g}(c)$ is given by

$$\tilde{g}(c) = \langle [\dot{g}_2(c;\alpha),\ \dot{g}_1(c;\alpha)], [\dot{g}_2'(c;\beta), \dot{g}_1'(c;\beta)], [\dot{g}_2''(c;\gamma),\ \dot{g}_1''(c;\gamma)] \rangle$$

4. Notations and assumptions of the proposed model

To explain the proposed model, the subsequent symbols and presumptions are employed.

4.1. Notations

All the notations related to the proposed model are described in Table 2.

Table 2. Descriptions of the notations and their description units

Notations	Explanation	Units
a	The constant part of the demand pattern	Constant
b	Coefficient of time in the demand pattern	Constant
m	A constant part of the production rate	Constant
n	Coefficient of inventory level in the production rate	Constant
l	Coefficient of the demand pattern in the production rate	Constant
θ	Rate of deterioration	Constant
c_h	Holding cost per unit item per unit time	\$/item/unit time
c_p	Production cost per unit item per unit time	\$/item/unit time
C_0	Ordering cost	\$/cycle
K	Production rate	Unit item/unit time
t_1	Production time (decision variable)	Unit time
T	Total cycle time (decision variable)	Unit time
ϕ	Total average profit (objective function)	\$

4.2. Assumptions

We consider the phenomenon of a newly launched production plant. There are ambiguities about the demand pattern, and the demand for the produced items increases gradually as time passes due to the recognition of the newly built production plan among the consumer communities. The production of items is dependent on the stock of items already produced and on the demand pattern. Also, the deterioration of items in stock is taken into consideration. Mathematically, the proposed model is developed based on the assumptions listed below:

(a) The demand rate of the produced item is dependent on time, i.e., over time, the rate of demand is steadily rising as a linear function of time. $D(t) = a + bt$, where a, b are positive constants.

(b) The production rate is dependent on the hand stock level as well as on the demand rate. i.e., $K = m - nq(t) + lD(t) = (m + al) - nq(t) + lbt$, where m, n, l are positive constants.

(c) The products in stock have deteriorated at a constant rate $\theta (0 \leq \theta \leq 1)$ throughout the whole lot-sizing cycle.

(d) The lead time is zero.

(e) The rate of replenishment is infinite, yet the lot size is finite.

(f) No shortage is considered in the whole lot-sizing cycle.

(g) The time horizon is finite.

5. Formulation of the proposed crisp EPQ model

The production inventory model is started at $t = 0$ with stock- and demand-dependent production rates. The production phase is stopped at $t = t_1$ making the highest possible stocks of the whole production-supply cycle. The lot-sizing cycle has faced decay in the inventory level due to deterioration at a constant rate and a time-dependent demand rate. The whole lot-sizing cycle is terminated at $t = T$ (see Figure 1). The mathematical correspondence of the mentioned set-up can be represented by the set of subsequent two differential equations.

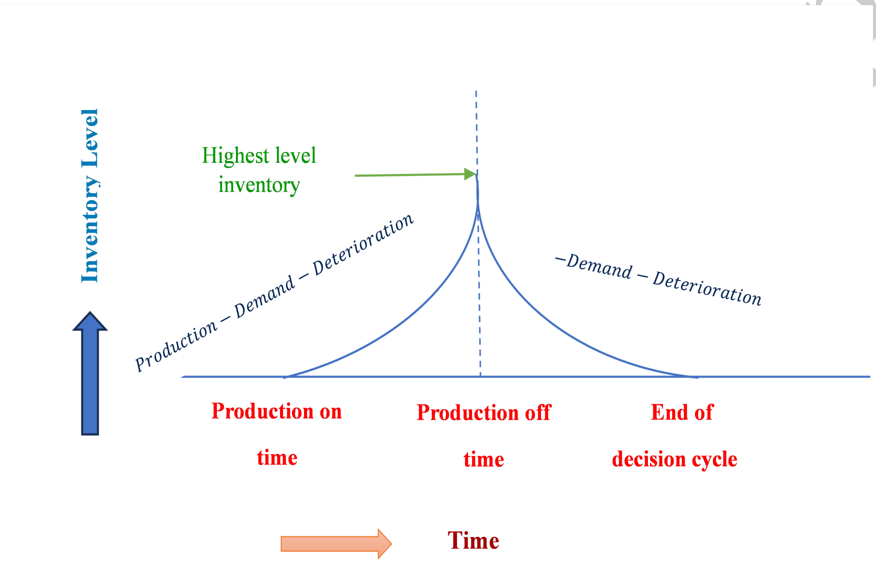

Figure 1. Graphical representation of the proposed EOQ mode

The governing differential equation of the production time $(0 \leq t \leq t_1)$ along with the boundary conditions is given by

$$\frac{dq(t)}{dt} = \{(m + al) - nq(t) + lbt\} - \{a + bt\} - \theta q(t)$$

with $q(0) = 0$, $q(t_1) = Q$. That means,

$$\frac{dq(t)}{dt} = m - (1 - l)a - (1 - l)bt - (\theta + n)q(t) \tag{1}$$

and the governing differential equation of the non-production time $(t_1 \leq t \leq T)$ of the lot cycle is given by

$$\frac{dq(t)}{dt} = -\{a + bt\} - \theta q(t) \tag{2}$$

with $q(T) = 0$.

Solving the systems represented by equations (1) and (2), the following results are obtained: the stock level in the productive time interval $(0 \leq t \leq t_1)$ is obtained as

$$q(t) = \frac{k_1k_3 + k_2}{k_1^2}\{1 - e^{-k_1t}\} - \frac{k_2}{k_1}t \tag{3}$$

where $k_1 = \theta + n$, $k_2 = b(1-l)$ and $k_3 = m - a(1-l)$.
The stock level in the non-productive time interval $(t_1 \leq t \leq T)$ is given by

$$q(t) = \frac{b - \theta(a + bt)}{\theta^2} - \frac{b - \theta(a + bT)}{\theta^2}e^{\theta(T-t)} \tag{4}$$

The highest level of stock at $t = t_1$ is obtained as

$$Q = \frac{k_1k_3 + k_2}{k_1^2}\{1 - e^{-k_1t_1}\} - \frac{k_2}{k_1}t_1 \tag{5}$$

Using the continuity of the stock function at $t = t_1$ from equations (3) and (4), the following constraint is obtained

$$\frac{b - \theta(a + bt_1)}{\theta^2} - \frac{b - \theta(a + bT)}{\theta^2}e^{\theta(T-t_1)} = \frac{k_1k_3 + k_2}{k_1^2}\{1 - e^{-k_1t_1}\} - \frac{k_2}{k_1}t_1 \tag{6}$$

Several relevant costs associated with the model are formulated as follows:

$$\begin{aligned} HC &= c_h\left[\int_0^{t_1} q(t)dt + \int_{t_1}^{T} q(t)dt\right] \\ &= c_h\left[\frac{k_1k_3 + k_2}{k_1^3}\{e^{-k_1t_1} + t_1k_1 - 1\} - \frac{k_2t_1^2}{2k_1} + \frac{b}{2\theta}(t_1^2 - T^2) + \frac{b - \theta(a + bT)}{\theta^3}\{1 - e^{\theta(T-t_1)}\}\right] \end{aligned}$$

Production Cost (PC): the production cost is given by

$$\begin{aligned} PC &= c_p\left[\int_0^{t_1}\{(m + al) - nq(t) + lbt\}dt\right] \\ &= c_p\left[(m + al)t_1 + \frac{lbt_1^2}{2} - n\left\{\frac{k_1k_3 + k_2}{k_1^3}\{e^{-k_1t_1} + t_1k_1 - 1\} - \frac{k_2t_1^2}{2k_1}\right\}\right] \end{aligned}$$

Setup Cost (SC): the setup cost is taken as constant C_0 throughout the inventory cycle.
Therefore, the total average cost of the production system during the entire circle is given by

$$\Phi(t_1, T) = \frac{HC + PC + C_0}{T}$$

$$\begin{aligned}\Phi(t_1,T) &= \frac{1}{T}\Bigg[c_h\Big[\frac{k_1k_3+k_2}{k_1^3}\{e^{-k_1t_1}+t_1k_1-1\}-\frac{k_2t_1^2}{2k_1}+\frac{b}{2\theta}(t_1^2-T^2)+\frac{b-\theta(a+bT)}{\theta^3}\\ &\times\{1-e^{\theta(T-t_1)}\}\Big]+c_p\Big[(m+al)t_1+\frac{lbt_1^2}{2}-n\Big\{\frac{k_1k_3+k_2}{k_1^3}\{e^{-k_1t_1}+t_1k_1-1\}\\ &-\frac{k_2t_1^2}{2k_1}\Big\}\Big]+C_0\Bigg]\end{aligned}$$

Therefore, the cost-minimization problem will be

$$\begin{cases} Minimize & \Phi(t_1,T)=\frac{HC+PC+C_0}{T}\\ Subject\ to & Q=\frac{k_1k_3+k_2}{k_1^2}\{1-e^{-k_1t_1}\}-\frac{k_2}{k_1}t_1\\ & \frac{b-\theta(a+bt_1)}{\theta^2}-\frac{b-\theta(a+bT)}{\theta^2}e^{\theta(T-t_1)}=\frac{k_1k_3+k_2}{k_1^2}\{1-e^{-k_1t_1}\}-\frac{k_2}{k_1}t_1\\ & 0\le t_1\le T\end{cases} \tag{7}$$

Here, the goal is to investigate the convexity of the average cost function $\Phi(t_1,T)$ concerning the decision variables t_1 and T.

Theorem 1. The average cost function $\Phi(t_1,T)$ is strictly pseudo-convex in t_1 and T hence, $\Phi(t_1,T)$ attains the minimum value at the point (t_1^*,T^*), provided

$$\frac{(c_h-nc_p)(k_1k_3+k_2)}{k_1}e^{-k_1t_1}+\frac{c_hb}{\theta}+c_plb>(c_h-nc_p)\frac{k_2}{k_1}+\frac{c_hb}{\theta}e^{\theta(T-t_1)} \tag{8}$$

Proof. For convenience, let us take the average cost function $\Phi(t_1,T)$ as

$$\Phi(t_1,T)=\frac{\Phi_1(t_1,T)}{\Phi_2(t_1,T)}$$

where $\Phi_1(t_1,T)=c_h\Big[\frac{k_1k_3+k_2}{k_1^3}\{e^{-k_1t_1}+t_1k_1-1\}-\frac{k_2t_1^2}{2k_1}+\frac{b}{2\theta}(t_1^2-T^2)+\frac{b-\theta(a+bT)}{\theta^3}\{1-e^{\theta(T-t_1)}\}\Big]+c_p\Big[(m+al)t_1+\frac{lbt_1^2}{2}-n\Big\{\frac{k_1k_3+k_2}{k_1^3}\{e^{-k_1t_1}+t_1k_1-1\}-\frac{k_2t_1^2}{2k_1}\Big\}\Big]+C_0$ and $\Phi_2(t_1,T)=T$.
Now, $\Phi_1(t_1,T)$ can be written as

$$\begin{aligned}\Phi_1(t_1,T) &= C_0+\frac{A}{k_1^3}(e^{-k_1t_1}+t_1k_1-1)+c_p(m+al)t_1+\frac{B}{2}t_1^2-\frac{C}{2}t_1^2-\frac{c_hb}{2\theta}T^2\\ &+\frac{c_h}{\theta^3}\{b-\theta(a+bT)\}\{1-e^(\theta(T-t_1))\}\end{aligned}$$

where $A=(c_h-nc_p)(k_1k_3+k_2)$, $B=\frac{c_hb}{\theta}+c_plb$ and $C=(c_h-nc_p)\frac{k_2}{k_1}$.
To construct the Hessian matrix for $\Phi_1(t_1,T)$, all the first and second-order partial derivatives of $\Phi_1(t_1,T)$ concerning t_1 and T are calculated as

$$\frac{\partial\Phi_1(t_1,T)}{\partial t_1}=\frac{A}{k_1^2}(1-e^{-k_1t_1})+c_p(m+al)+Bt_1-Ct_1+\frac{c_h}{\theta^2}\{b-\theta(a+bT)\}e^{\theta(T-t_1)}$$

$$\frac{\partial \Phi_1(t_1, T)}{\partial T} = -\frac{c_h b}{\theta} T - \frac{c_h}{\theta^2}\{b - \theta(a + bT)e^{\theta(T - t_1)}\}$$

$$\frac{\partial^2 \Phi_1(t_1, T)}{\partial t_1^2} = \frac{A}{k_1} e^{-k_1 t_1} + B - C - \frac{c_h}{\theta}\{b - \theta(a + bT)\}e^{\theta(T - t_1)}$$

$$\frac{\partial^2 \Phi_1(t_1, T)}{\partial t_1 \partial T} = -c_h(a + bT)e^{\theta(T - t_1)} = \frac{\partial^2 \Phi_1(t_1, T)}{\partial T \partial t_1}$$

$$\frac{\partial^2 \Phi_1(t_1, T)}{\partial t_1^2} = c_h(a + bT)e^{\theta(T - t_1)}$$

Therefore, the Hessian matrix corresponding to $\Phi_1(t_1, T)$ can be expressed as

$$H_{ii} = \begin{bmatrix} \frac{\partial^2 \Phi_1(t_1, T)}{\partial t_1^2} & \frac{\partial^2 \Phi_1(t_1, T)}{\partial t_1 \partial T} \\ \frac{\partial^2 \Phi_1(t_1, T)}{\partial T \partial t_1} & \frac{\partial^2 \Phi_1(t_1, T)}{\partial T^2} \end{bmatrix}$$

The first principal minor is

$$\begin{aligned} |H_{11}| &= \frac{A}{k_1} e^{-k_1 t_1} + B - C - \frac{c_h}{\theta}\{b - \theta(a + bT)\}e^{\theta(T - t_1)} \\ &= c_h(a + bT)e^{\theta(T - t_1)} + \frac{A}{k_1} e^{-k_1 t_1} + B - C - \frac{c_h b}{\theta} e^{\theta(T - t_1)} \end{aligned}$$

Clearly, $k_1 = \theta + n > 0$. As $0 < l < 1$, $k_2 = b(1 - l) > 0$. Again, the fixed part of the production rate function is greater than the fixed part of the market demand; therefore, $k_3 = (m - a) + al > 0$. Also, it is assumed that $c_h > nc_p$, therefore, A, B and C are all positive. Hence $|H_{11}| > 0$ if

$$\frac{A}{k_1} e^{-k_1 t_1} + B > C + \frac{c_h b}{\theta} e^{\theta(T - t_1)}$$

The second principal minor is

$$\begin{aligned} |H_{22}| &= \frac{\partial^2 \Phi_1(t_1, T)}{\partial t_1^2} \frac{\partial^2 \Phi_1(t_1, T)}{\partial T^2} - \frac{\partial^2 \Phi_1(t_1, T)}{\partial t_1 \partial T} \frac{\partial^2 \Phi_1(t_1, T)}{\partial T \partial t_1} \\ &= c_h(a + bT)e^{\theta(T - t_1)} \left[\frac{A}{k_1} e^{-k_1 t_1} + B - C - \frac{c_h b}{\theta} e^{\theta(T - t_1)} \right] \end{aligned}$$

Clearly, $|H_{22}|$ is positive when (8) is preserved.

Therefore, $\Phi_1(t_1, T)$ is a positive definite and hence it is a convex function of t_1 and T. Also $\Phi_2(t_1, T)$ is non-negative and differentiable concerning t_1 and T. Furthermore, $\Phi_2(t_1, T)$ is positive and affine function. Consequently, the average cost function $\Phi(t_1, T)$ is strictly a pseudo-convex function in t_1 and T, and it has a unique minimum value. □

The first-order partial derivatives of $\Phi(t_1, T)$ concerning t_1 and T equal to zero yield the necessary conditions for minimizing the total cost $\Phi(t_1, T)$.

$$\frac{\partial \Phi(t_1,T)}{\partial t_1} = \frac{1}{T}\Bigg[\frac{(c_h - nc_p)(k_1k_3+k_2)}{k_1^2}(1-e^{-k_1t_1}) + c_p(m+al) + \Big(\frac{c_hb}{\theta} + c_plb\Big)t_1 - (c_h - nc_p)\frac{k_2}{k_1}t_1 + \frac{c_h}{\theta^2}\{b - \theta(a+bT)\}e^{\theta(T-t_1)}\Bigg] = 0$$

which gives

$$\frac{(c_h - nc_p)(k_1k_3+k_2)}{k_1^2}(1-e^{-k_1t_1}) + c_p(m+al) + \Big(\frac{c_hb}{\theta} + c_plb\Big)t_1 - (c_h - nc_p)\frac{k_2}{k_1}t_1 + \frac{c_h}{\theta^2}\{b - \theta(a+bT)\}e^{\theta(T-t_1)} = 0 \quad (9)$$

and

$$\frac{\partial \Phi(t_1,T)}{\partial T} = -\frac{\Phi_1(t_1,T)}{T^2} + \frac{\partial \Phi_1(t_1,T)}{\partial T} = 0$$

which gives

$$C_0 + \frac{(c_h - nc_p)(k_1k_3+k_2)}{k_1^3}(e^{-k_1t_1} + t_1k_1 - 1) + c_p(m+al)t_1 + \Big(\frac{c_hb}{\theta} + c_plb\Big)\frac{t_1^2}{2} - (c_h - nc_p) \times \frac{k_2t_1^2}{2k_1} - \frac{c_hb}{2\theta}T^2 + \frac{c_h}{\theta^3}\{b - \theta(a+bT)\}\{1 - e^{\theta(T-t_1)}\} + \frac{c_hT}{\theta^2}\{b - \theta(a+bT)e^{\theta(T-t_1)}\} = 0 \quad (10)$$

6. Proposed EPQ model in a neutrosophic environment

In this subsection, we reconstruct the manufacturing-supply scenario in the neutrosophic arena, letting three parameters $\tilde{a}$, $\tilde{b}$ and $\tilde{\theta}$ as neutrosophic numbers. Then, the neutrosophic counterpart of the equations (1) and (2) is given as follows:
For the productive phase $(0 \le t \le t_1)$:

$$\begin{cases} \frac{d\tilde{q}(t)}{dt} = m - (1-l)\tilde{a} - (1-l)\tilde{b}t - (\tilde{\theta}+n)\tilde{q}(t) \\ with\ \ \tilde{q}(0) = 0,\ \tilde{q}(t_1) = \tilde{Q} \end{cases} \quad (11)$$

For the non-productive phase $(t_1 \le t \le T)$:

$$\begin{cases} \frac{d\tilde{q}(t)}{dt} = -\{\tilde{a} + \tilde{b}t\} - \tilde{\theta}\tilde{q}(t) \\ with\ \tilde{q}(t_1) = \tilde{Q},\ \tilde{q}(T) = 0 \end{cases} \quad (12)$$

Now the concept of generalized neutrosophic differentiation is applied to explain the neutrosophic differential equations (11) and (12). Suppose (α, β, γ)-cut of the neutrosophic valued function $\tilde{q}(t)$ is given as

$$\big[\tilde{q}(t)\big]_{(\alpha,\beta,\gamma)} = \langle [q_1(t;\alpha), q_2(t;\alpha)], [q_1'(t;\beta), q_2'(t;\beta)], [q_1''(t;\gamma), q_2''(t;\gamma)]\rangle$$

Also, let the parametric representations, i.e., (α,β,γ)-cut of the neutrosophic number $\tilde{a}$, $\tilde{b}$ and $\tilde{\theta}$ be given as

$$\begin{aligned}
\left[\tilde{\theta}\right]_{(\alpha,\beta,\gamma)} &= \langle[\theta_1(\alpha),\theta_2(\alpha)],[\theta_1'(\beta),\theta_2'(\beta)],[\theta_1''(\gamma),\theta_2''(\gamma)]\rangle\\
\left[\tilde{a}\right]_{(\alpha,\beta,\gamma)} &= \langle[a_1(\alpha),a_2(\alpha)],[a_1'(\beta),a_2'(\beta)],[a_1''(\gamma),a_2''(\gamma)]\rangle\\
\left[\tilde{b}\right]_{(\alpha,\beta,\gamma)} &= \langle[b_1(\alpha),b_2(\alpha)],[b_1'(\beta),b_2'(\beta)],[b_1''(\gamma),b_2''(\gamma)]\rangle
\end{aligned}$$

The following two cases are considered according to the two types of generalized neutrosophic derivatives of the neutrosophic valued function $\tilde{q}(t)$.

Case 1: When $\tilde{q}(t)$ is type-1 neutrosophic differentiable

Then, the differential equation (11) represents the productive phase $(0 \le t \le t_1)$ and is turned into a parametric form as

$$\begin{aligned}
&\langle[\dot{q}_1(t;\alpha),\dot{q}_2(t;\alpha)],[\dot{q}_1'(t;\beta),\dot{q}_2'(t;\beta)],[\dot{q}_1''(t;\gamma),\dot{q}_2''(t;\gamma)]\rangle = m-(1-l)\langle[a_1(\alpha),a_2(\alpha)],\\
&[a_1'(\beta),a_2'(\beta)],[a_1''(\gamma),a_2''(\gamma)]\rangle-(1-l)\langle[b_1(\alpha),b_2(\alpha)],[b_1'(\beta),b_2'(\beta)],[b_1''(\gamma),b_2''(\gamma)]\rangle t\\
&-\left\{\langle[\theta_1(\alpha),\theta_2(\alpha)],[\theta_1'(\beta),\theta_2'(\beta)],[\theta_1''(\gamma),\theta_2''(\gamma)]\rangle+n\right\}\langle[q_1(t;\alpha),q_2(t;\alpha)],\\
&[q_1'(t;\beta),q_2'(t;\beta)],[q_1''(t;\gamma),q_2''(t;\gamma)]\rangle
\end{aligned}$$

The above expression gives a system of crisp differential equations as follows:

$$\dot{q}_1(t;\alpha)=m-(1-l)a_2(\alpha)-(1-l)b_2(\alpha)t-\{\theta_2(\alpha)+n\}q_2(t;\alpha) \tag{13}$$
$$\dot{q}_2(t;\alpha)=m-(1-l)a_1(\alpha)-(1-l)b_1(\alpha)t-\{\theta_1(\alpha)+n\}q_1(t;\alpha) \tag{14}$$
$$\dot{q}_1'(t;\beta)=m-(1-l)a_2'(\beta)-(1-l)b_2'(\beta)t-\{\theta_2'(\beta)+n\}q_2'(t;\beta) \tag{15}$$
$$\dot{q}_2'(t;\beta)=m-(1-l)a_1'(\beta)-(1-l)b_1'(\beta)t-\{\theta_1'(\beta)+n\}q_1'(t;\beta) \tag{16}$$
$$\dot{q}_1''(t;\gamma)=m-(1-l)a_2''(\gamma)-(1-l)b_2''(\gamma)t-\{\theta_2''(\gamma)+n\}q_2''(t;\gamma) \tag{17}$$
$$\dot{q}_2''(t;\gamma)=m-(1-l)a_1''(\gamma)-(1-l)b_1''(\gamma)t-\{\theta_1''(\gamma)+n\}q_1''(t;\gamma) \tag{18}$$

with $q_1(0;\alpha)=q_2(0;\alpha)=q_1'(0;\beta)=q_2'(0;\beta)=q_1''(0;\gamma)=q_2''(0;\gamma)=0$.
Equations (13) and (14) can be simplified as

$$\begin{cases}\dot{q}_1(t;\alpha)=-u_2q_2(t;\alpha)-v_2t+w_2\\ \dot{q}_2(t;\alpha)=-u_1q_1(t;\alpha)-v_1t+w_1\end{cases} \tag{19}$$

where $u_2=\theta_1(\alpha)+n,\ u_1=\theta_2(\alpha)+n,\ v_1=(1-l)b_1(\alpha),\ v_2=(1-l)b_2(\alpha),\ w_1=m-(1-l)a_1(\alpha)$ and $w_2=m-(1-l)a_2(\alpha)$.
The solution of the system (19) is given by

$$\begin{cases}q_1(t;\alpha)=c_1e^{\sqrt{u_1u_2}t}+c_2e^{-\sqrt{u_1u_2}t}+\frac{u_2(w_1-v_1t)+v_2}{u_1u_2}\\ q_2(t;\alpha)=-\sqrt{\frac{u_1}{u_2}}c_1e^{\sqrt{u_1u_2}t}+\sqrt{\frac{u_1}{u_2}}c_2e^{-\sqrt{u_1u_2}t}+\frac{u_1(w_2-v_2t)+v_1}{u_1u_2}\end{cases} \tag{20}$$

Using the initial information $q_1(0;\alpha) = 0$ and $q_2(0;\alpha) = 0$, the values of the constants c_1 and c_2 are obtained as

$$\begin{cases} c_1 = -\frac{\sqrt{u_1}(u_2w_1+v_2)-\sqrt{u_2}(u_1w_2+v_1)}{2\sqrt{u_1}u_1u_2} \\ c_2 = -\frac{\sqrt{u_1}(u_2w_1+v_2)+\sqrt{u_2}(u_1w_2+v_1)}{2\sqrt{u_1}u_1u_2} \end{cases} \tag{21}$$

Similarly, the solution of the remaining system of equations (15)-(16) and (17)-(18) are obtained as

$$\begin{cases} q_1'(t;\beta) = c_3e^{\sqrt{u_3u_4}t} + c_4e^{-\sqrt{u_3u_4}t} + \frac{u_4(w_3-v_3t)+v_4}{u_3u_4} \\ q_2'(t;\beta) = -\sqrt{\frac{u_3}{u_4}}c_3e^{\sqrt{u_3u_4}t} + \sqrt{\frac{u_3}{u_4}}c_4e^{-\sqrt{u_3u_4}t} + \frac{u_3(w_4-v_4t)+v_3}{u_3u_4} \\ q_1''(t;\gamma) = c_5e^{\sqrt{u_5u_6}t} + c_6e^{-\sqrt{u_5u_6}t} + \frac{u_6(w_5-v_5t)+v_6}{u_5u_6} \\ q_2''(t;\gamma) = -\sqrt{\frac{u_5}{u_6}}c_5e^{\sqrt{u_5u_6}t} + \sqrt{\frac{u_5}{u_6}}c_6e^{-\sqrt{u_5u_6}t} + \frac{u_5(w_6-v_6t)+v_5}{u_5u_6} \end{cases} \tag{22}$$

where $u_i,\ v_i$ and w_i $(i = 3,4,5,6)$ are taken as follows

$$\begin{cases} u_3 = \theta_1'(\beta) + n,\ v_3 = b_1'(\beta)(1-l),\ w_3 = m + a_1'(\beta)(l-1) \\ u_4 = \theta_2'(\beta) + n,\ v_4 = b_2'(\beta)(1-l),\ w_4 = m + a_2'(\beta)(l-1) \\ u_5 = \theta_1''(\gamma) + n,\ v_5 = b_1''(\gamma)(1-l),\ w_5 = m + a_1''(\gamma)(l-1) \\ u_6 = \theta_2''(\gamma) + n,\ v_6 = b_2''(\gamma)(1-l),\ w_6 = m + a_2''(\gamma)(l-1) \end{cases}$$

and the constants are $c_3,\ c_4,\ c_5$ and c_4 are given as

$$\begin{cases} c_3 = -\frac{\sqrt{u_3}(u_4w_3+v_4)-\sqrt{u_4}(u_3w_4+v_3)}{2\sqrt{u_3}u_3u_4} \\ c_4 = -\frac{\sqrt{u_3}(u_4w_3+v_4)+\sqrt{u_4}(u_3w_4+v_3)}{2\sqrt{u_3}u_3u_4} \\ c_5 = -\frac{\sqrt{u_5}(u_6w_5+v_6)-\sqrt{u_6}(u_5w_6+v_5)}{2\sqrt{u_5}u_5u_6} \\ c_6 = -\frac{\sqrt{u_5}(u_6w_5+v_6)+\sqrt{u_6}(u_5w_6+v_5)}{2\sqrt{u_5}u_5u_6} \end{cases}$$

Again, the neutrosophic differential equation (12) represents the non-productive phase $(t_1 \le t \le T)$ is turned in the parametric form,

$$\begin{aligned} \langle[\dot q_1(t;\alpha), \dot q_2(t;\alpha)], [\dot q_1'(t;\beta), \dot q_2'(t;\beta)], [\dot q_1''(t;\gamma), \dot q_2''(t;\gamma)]\rangle = -\langle[a_1(\alpha), a_2(\alpha)], [a_1'(\beta), a_2'(\beta)], \\ [a_1''(\gamma), a_2''(\gamma)]\rangle - \langle[b_1(\alpha), b_2(\alpha)], [b_1'(\beta), b_2'(\beta)], [b_1''(\gamma), b_2''(\gamma)]\rangle t - \langle[\theta_1(\alpha), \theta_2(\alpha)], \\ [\theta_1'(\beta), \theta_2'(\beta)], [\theta_1''(\gamma), \theta_2''(\gamma)]\rangle\langle[q_1(t;\alpha), q_2(t;\alpha)], [q_1'(t;\beta), q_2'(t;\beta)], [q_1''(t;\gamma), q_2''(t;\gamma)]\rangle \end{aligned}$$

which gives a system of differential equations as follows:

$$\dot q_1(t;\alpha) = -a_2(\alpha) - b_2(\alpha)t - \theta_2(\alpha)q_2(t;\alpha) \tag{23}$$

$$\dot q_2(t;\alpha) = -a_1(\alpha) - b_1(\alpha)t - \theta_1(\alpha)q_1(t;\alpha) \tag{24}$$

$$\dot q_1'(t;\beta) = -a_2'(\beta) - b_2'(\beta)t - \theta_2'(\beta)q_2'(t;\beta) \tag{25}$$

$$\dot q_2'(t;\beta) = -a_1'(\beta) - b_1'(\beta)t - \theta_1'(\beta)q_1'(t;\beta) \tag{26}$$

$$\dot q_1''(t;\gamma) = -a_2''(\gamma) - b_2''(\gamma)t - \theta_2''(\gamma)q_2''(t;\gamma) \tag{27}$$

$$\dot q_2''(t;\gamma) = -a_1''(\gamma) - b_1''(\gamma)t - \theta_1''(\gamma)q_1''(t;\gamma) \tag{28}$$

with $q_1(T;\alpha) = q_2(T;\alpha) = q_1'(T;\beta) = q_2'(T;\beta) = q_1''(T;\gamma) = q_2''(T;\gamma) = 0$.
By solving the equations (23) and (24) one can get

$$\begin{cases} q_1(t;\alpha) = c_7 e^{\sqrt{\theta_1(\alpha)\theta_2(\alpha)}t} + c_8 e^{-\sqrt{\theta_1(\alpha)\theta_2(\alpha)}t} + \frac{b_2(\alpha)-\theta_2(\alpha)(b_1(\alpha)t+a_1(\alpha))}{\theta_1(\alpha)\theta_2(\alpha)} \\ q_2(t;\alpha) = c_7\sqrt{\frac{\theta_1(\alpha)}{\theta_2(\alpha)}} e^{\sqrt{\theta_1(\alpha)\theta_2(\alpha)}t} + c_8\sqrt{\frac{\theta_1(\alpha)}{\theta_2(\alpha)}} e^{-\sqrt{\theta_1(\alpha)\theta_2(\alpha)}t} + \frac{b_1(\alpha)-\theta_1(\alpha)(b_2(\alpha)t+a_2(\alpha))}{\theta_1(\alpha)\theta_2(\alpha)} \end{cases} \tag{29}$$

Using the initial information $q_1(T;\alpha) = 0$, $q_2(T;\alpha) = 0$ the values of the constants c_7 and c_8 are obtained as

$$\begin{cases} c_7 = -\frac{e^{-\sqrt{\theta_1(\alpha)\theta_2(\alpha)}T}}{2\sqrt{\theta_1(\alpha)}\theta_1(\alpha)\theta_2(\alpha)}\Big[\sqrt{\theta_1(\alpha)}\big\{b_2(\alpha) - \theta_2(\alpha)(b_1(\alpha)T + a_1(\alpha))\big\} - \sqrt{\theta_2(\alpha)}\big\{b_1(\alpha) \\ \quad -\theta_1(\alpha)(b_2(\alpha)T + a_2(\alpha))\big\}\Big] \\ c_8 = -\frac{e^{\sqrt{\theta_1(\alpha)\theta_2(\alpha)}T}}{2\sqrt{\theta_1(\alpha)}\theta_1(\alpha)\theta_2(\alpha)}\Big[\sqrt{\theta_1(\alpha)}\big\{b_2(\alpha) - \theta_2(\alpha)(b_1(\alpha)T + a_1(\alpha))\big\} + \sqrt{\theta_2(\alpha)}\big\{b_1(\alpha) \\ \quad -\theta_1(\alpha)(b_2(\alpha)T + a_2(\alpha))\big\}\Big] \end{cases} \tag{30}$$

Similarly, by solving the remaining equations (25)-(28) one can get,

$$\begin{cases} q_1'(t;\beta) = c_9 e^{\sqrt{\theta_1'(\beta)\theta_2'(\beta)}t} + c_{10} e^{-\sqrt{\theta_1'(\beta)\theta_2'(\beta)}t} + \frac{b_2'(\beta)-\theta_2'(\beta)(b_1'(\beta)t+a_1'(\beta))}{\theta_1'(\beta)\theta_2'(\beta)} \\ q_2'(t;\beta) = c_9\sqrt{\frac{\theta_1'(\beta)}{\theta_2'(\beta)}} e^{\sqrt{\theta_1'(\beta)\theta_2'(\beta)}t} + c_{10}\sqrt{\frac{\theta_1'(\beta)}{\theta_2'(\beta)}} e^{-\sqrt{\theta_1'(\beta)\theta_2'(\beta)}t} + \frac{b_1'(\beta)-\theta_1'(\beta)(b_2'(\beta)t+a_2'(\beta))}{\theta_1'(\beta)\theta_2'(\beta)} \\ q_1''(t;\gamma) = c_{11} e^{\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}t} + c_{12} e^{-\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}t} + \frac{b_2''(\gamma)-\theta_2''(\gamma)(b_1''(\gamma)t+a_1''(\gamma))}{\theta_1''(\gamma)\theta_2''(\gamma)} \\ q_2''(t;\gamma) = c_{11}\sqrt{\frac{\theta_1''(\gamma)}{\theta_2''(\gamma)}} e^{\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}t} + c_{12}\sqrt{\frac{\theta_1''(\gamma)}{\theta_2''(\gamma)}} e^{-\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}t} + \frac{b_1''(\gamma)-\theta_1''(\gamma)(b_2''(\gamma)t+a_2''(\gamma))}{\theta_1''(\gamma)\theta_2''(\gamma)} \end{cases} \tag{31}$$

Where the values of the constants c_9, c_{10}, c_{11} and c_{12} are obtained as

$$c_9 = -\frac{e^{-\sqrt{\theta_1'(\beta)\theta_2'(\beta)}T}}{2\sqrt{\theta_1'(\beta)}\theta_1'(\beta)\theta_2'(\beta)}\Big[\sqrt{\theta_1'(\beta)}\big\{b_2'(\beta) - \theta_2'(\beta)(b_1'(\beta)T + a_1'(\beta))\big\} - \sqrt{\theta_2'(\beta)}\big\{b_1'(\beta) -\theta_1'(\beta)(b_2'(\beta)T + a_2'(\beta))\big\}\Big]$$

$$c_{10} = -\frac{e^{\sqrt{\theta_1'(\beta)\theta_2'(\beta)}T}}{2\sqrt{\theta_1'(\beta)}\theta_1'(\beta)\theta_2'(\beta)}\Big[\sqrt{\theta_1'(\beta)}\big\{b_2'(\beta) - \theta_2'(\beta)(b_1'(\beta)T + a_1'(\beta))\big\} + \sqrt{\theta_2'(\beta)}\big\{b_1'(\beta) -\theta_1'(\beta)(b_2'(\beta)T + a_2'(\beta))\big\}\Big]$$

$$c_{11} = -\frac{e^{-\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}T}}{2\sqrt{\theta_1''(\gamma)}\theta_1''(\gamma)\theta_2''(\gamma)}\Big[\sqrt{\theta_1''(\gamma)}\big\{b_2''(\gamma) - \theta_2''(\gamma)(b_1''(\gamma)T + a_1''(\gamma))\big\} - \sqrt{\theta_2''(\gamma)}\big\{b_1''(\gamma) -\theta_1''(\gamma)(b_2''(\gamma)T + a_2''(\gamma))\big\}\Big]$$

$$c_{12} = -\frac{e^{\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}T}}{2\sqrt{\theta_1''(\gamma)}\theta_1''(\gamma)\theta_2''(\gamma)}\Big[\sqrt{\theta_1''(\gamma)}\big\{b_2''(\gamma) - \theta_2''(\gamma)(b_1''(\gamma)T + a_1''(\gamma))\big\} + \sqrt{\theta_2''(\gamma)}\big\{b_1''(\gamma) -\theta_1''(\gamma)(b_2''(\gamma)T + a_2''(\gamma))\big\}\Big]$$

Some relevant costs:
Therefore, the holding cost, $\tilde{HC} = \langle [HC_1(\alpha), HC_2(\alpha)], [HC_1'(\beta), HC_2'(\beta)], [HC_1''(\gamma), HC_2''(\gamma)]\rangle$ given by

$$\begin{aligned}
HC_1(\alpha) &= c_h\Big[\int_0^{t_1} q_1(t;\alpha)dt + \int_{t_1}^{T} q_2(t;\alpha)dt\Big] \\
&= c_h\Big[\int_0^{t_1}\Big\{c_1 e^{\sqrt{u_1u_2}t} + c_2 e^{-\sqrt{u_1u_2}t} + \frac{u_2(w_1 - v_1 t) + v_2}{u_1u_2}\Big\}dt + \int_{t_1}^{T}\Big\{c_7 e^{\sqrt{\theta_1(\alpha)\theta_2(\alpha)}t} \\
&\quad + c_8 e^{-\sqrt{\theta_1(\alpha)\theta_2(\alpha)}t} + \frac{b_2(\alpha) - \theta_2(\alpha)(b_1(\alpha)t + a_1(\alpha))}{\theta_1(\alpha)\theta_2(\alpha)}\Big\}dt\Big] \\
&= c_h\Big[\Big\{\frac{c_1}{\sqrt{u_1u_2}}\big(e^{\sqrt{u_1u_2}t_1} - 1\big) - \frac{c_2}{\sqrt{u_1u_2}}\big(e^{-\sqrt{u_1u_2}t_1} - 1\big) + \frac{2(u_2w_1 + v_2)t_1 - u_2v_1t_1^2}{2u_1u_2}\Big\} \\
&\quad + \Big\{\frac{c_7}{\sqrt{\theta_1(\alpha)\theta_2(\alpha)}}\Big(e^{\sqrt{\theta_1(\alpha)\theta_2(\alpha)}T} - e^{\sqrt{\theta_1(\alpha)\theta_2(\alpha)}t_1}\Big) + \frac{c_8}{\sqrt{\theta_1(\alpha)\theta_2(\alpha)}}\Big(e^{-\sqrt{\theta_1(\alpha)\theta_2(\alpha)}T} \\
&\quad - e^{-\sqrt{\theta_1(\alpha)\theta_2(\alpha)}t_1}\Big) - \frac{\theta_2(\alpha)b_1(\alpha)(T^2 - t_1^2) + 2(\theta_2(\alpha)a_1(\alpha) - b_2(\alpha))(T - t_1)}{2\theta_1(\alpha)\theta_2(\alpha)}\Big\}\Big]
\end{aligned}$$

Similarly, solving the other components of the holding cost as

$$
\begin{aligned}
HC_2(\alpha) \;=\; & c_h\Bigg[\Bigg\{\frac{c_1}{u_2}\big(1-e^{\sqrt{u_1u_2}t_1}\big)-\frac{c_2}{u_2}\big(1-e^{-\sqrt{u_1u_2}t_1}\big)+\frac{2(u_1w_2+v_1)t_1-u_1v_2t_1^2}{2u_1u_2}\Bigg\} \\
& +\Bigg\{\frac{c_7}{\theta_2(\alpha)}\Bigg(e^{\sqrt{\theta_1(\alpha)\theta_2(\alpha)}t_1}-e^{\sqrt{\theta_1(\alpha)\theta_2(\alpha)}T}\Bigg)+\frac{c_8}{\theta_2(\alpha)}\Bigg(e^{-\sqrt{\theta_1(\alpha)\theta_2(\alpha)}t_1} \\
& -e^{-\sqrt{\theta_1(\alpha)\theta_2(\alpha)}T}\Bigg)-\frac{\theta_1(\alpha)b_2(\alpha)(T^2-t_1^2)+2(\theta_1(\alpha)a_2(\alpha)-b_1(\alpha))(T-t_1)}{2\theta_1(\alpha)\theta_2(\alpha)}\Bigg\}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
HC_1'(\beta) \;=\; & c_h\Bigg[\Bigg\{\frac{c_3}{\sqrt{u_3u_4}}\big(e^{\sqrt{u_3u_4}t_1}-1\big)-\frac{c_4}{\sqrt{u_3u_4}}\big(e^{-\sqrt{u_3u_4}t_1}-1\big)+\frac{2(u_4w_3+v_4)t_1-u_4v_3t_1^2}{2u_3u_4}\Bigg\} \\
& +\Bigg\{\frac{c_9}{\sqrt{\theta_1'(\beta)\theta_2'(\beta)}}\Bigg(e^{\sqrt{\theta_1'(\beta)\theta_2'(\beta)}T}-e^{\sqrt{\theta_1'(\beta)\theta_2'(\beta)}t_1}\Bigg)+\frac{c_{10}}{\sqrt{\theta_1'(\beta)\theta_2'(\beta)}}\Bigg(e^{-\sqrt{\theta_1'(\beta)\theta_2'(\beta)}T} \\
& -e^{-\sqrt{\theta_1'(\beta)\theta_2'(\beta)}t_1}\Bigg)-\frac{\theta_2'(\beta)b_1'(\beta)(T^2-t_1^2)+2(\theta_2'(\beta)a_1'(\beta)-b_2'(\beta))(T-t_1)}{2\theta_1'(\beta)\theta_2'(\beta)}\Bigg\}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
HC_2'(\beta) \;=\; & c_h\Bigg[\Bigg\{\frac{c_3}{u_4}\big(1-e^{\sqrt{u_3u_4}t_1}\big)-\frac{c_4}{u_4}\big(1-e^{-\sqrt{u_3u_4}t_1}\big)+\frac{2(u_3w_4+v_3)t_1-u_3v_4t_1^2}{2u_3u_4}\Bigg\} \\
& +\Bigg\{\frac{c_9}{\theta_2'(\beta)}\Bigg(e^{\sqrt{\theta_1'(\beta)\theta_2'(\beta)}t_1}-e^{\sqrt{\theta_1'(\beta)\theta_2'(\beta)}T}\Bigg)+\frac{c_{10}}{\theta_2'(\beta)}\Bigg(e^{-\sqrt{\theta_1'(\beta)\theta_2'(\beta)}t_1} \\
& -e^{-\sqrt{\theta_1'(\beta)\theta_2'(\beta)}T}\Bigg)-\frac{\theta_1'(\beta)b_2'(\beta)(T^2-t_1^2)+2(\theta_1'(\beta)a_2'(\beta)-b_1'(\beta))(T-t_1)}{2\theta_1'(\beta)\theta_2'(\beta)}\Bigg\}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
HC_1''(\gamma) \;=\; & c_h\Bigg[\Bigg\{\frac{c_5}{\sqrt{u_5u_6}}\big(e^{\sqrt{u_5u_6}t_1}-1\big)-\frac{c_6}{\sqrt{u_5u_6}}\big(e^{-\sqrt{u_5u_6}t_1}-1\big)+\frac{2(u_6w_5+v_6)t_1-u_6v_5t_1^2}{2u_5u_6}\Bigg\} \\
& +\Bigg\{\frac{c_{11}}{\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}}\Bigg(e^{\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}T}-e^{\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}t_1}\Bigg)+\frac{c_{12}}{\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}}\Bigg(e^{-\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}T} \\
& -e^{-\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}t_1}\Bigg)-\frac{\theta_2''(\gamma)b_1''(\gamma)(T^2-t_1^2)+2(\theta_2''(\gamma)a_1''(\gamma)-b_2''(\gamma))(T-t_1)}{2\theta_1''(\gamma)\theta_2''(\gamma)}\Bigg\}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
HC_2''(\gamma) \;=\; & c_h\Bigg[\Bigg\{\frac{c_5}{u_6}\big(1-e^{\sqrt{u_5u_6}t_1}\big)-\frac{c_6}{u_6}\big(1-e^{-\sqrt{u_5u_6}t_1}\big)+\frac{2(u_5w_6+v_5)t_1-u_5v_6t_1^2}{2u_5u_6}\Bigg\} \\
& +\Bigg\{\frac{c_{11}}{\theta_2''(\gamma)}\Bigg(e^{\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}t_1}-e^{\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}T}\Bigg)+\frac{c_{12}}{\theta_2''(\gamma)}\Bigg(e^{-\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}t_1} \\
& -e^{-\sqrt{\theta_1''(\gamma)\theta_2''(\gamma)}T}\Bigg)-\frac{\theta_1''(\gamma)b_2''(\gamma)(T^2-t_1^2)+2(\theta_1''(\gamma)a_2''(\gamma)-b_1''(\gamma))(T-t_1)}{2\theta_1''(\gamma)\theta_2''(\gamma)}\Bigg\}\Bigg]
\end{aligned}
$$

Therefore, the production cost $\tilde{PC} = \langle [PC_1(\alpha), PC_2(\alpha)], [PC_1'(\beta), PC_2'(\beta)], [PC_1''(\gamma), PC_2''(\gamma)]\rangle$ during the entire circle is given by

$$
\begin{aligned}
PC_1(\alpha) &= c_p\Big[\int_0^{t_1}\big\{(m+la_1(\alpha))-nq_1(t;\alpha)+lb_1(\alpha)t\big\}dt\Big]\\
&= c_p\Big[(m+la_1(\alpha))t_1+lb_1(\alpha)\frac{t_1^2}{2}-n\int_0^{t_1}q_1(t;\alpha)dt\Big]\\
&= c_p\Big[(m+la_1(\alpha))t_1+lb_1(\alpha)\frac{t_1^2}{2}-n\Big\{\frac{c_1}{\sqrt{u_1u_2}}\big(e^{\sqrt{u_1u_2}t_1}-1\big)-\frac{c_2}{\sqrt{u_1u_2}}\big(e^{-\sqrt{u_1u_2}t_1}-1\big)\\
&\quad+\frac{2(u_2w_1+v_2)t_1-u_2v_1t_1^2}{2u_1u_2}\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
PC_2(\alpha) &= c_p\Big[\int_0^{t_1}\big\{(m+la_2(\alpha))-nq_2(t;\alpha)+lb_2(\alpha)t\big\}dt\Big]\\
&= c_p\Big[(m+la_2(\alpha))t_1+lb_2(\alpha)\frac{t_1^2}{2}-n\int_0^{t_1}q_2(t;\alpha)dt\Big]\\
&= c_p\Big[(m+la_2(\alpha))t_1+lb_2(\alpha)\frac{t_1^2}{2}-n\Big\{\frac{c_1}{u_2}\big(1-e^{\sqrt{u_1u_2}t_1}\big)-\frac{c_2}{u_2}\big(1-e^{-\sqrt{u_1u_2}t_1}\big)\\
&\quad+\frac{2(u_1w_2+v_1)t_1-u_1v_2t_1^2}{2u_1u_2}\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
PC_1'(\beta) &= c_p\Big[(m+la_1'(\beta))t_1+lb_1'(\beta)\frac{t_1^2}{2}-n\Big\{\frac{c_3}{\sqrt{u_3u_4}}\big(e^{\sqrt{u_3u_4}t_1}-1\big)-\frac{c_4}{\sqrt{u_3u_4}}\big(e^{-\sqrt{u_3u_4}t_1}-1\big)\\
&\quad+\frac{2(u_4w_3+v_4)t_1-u_4v_3t_1^2}{2u_3u_4}\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
PC_1''(\beta) &= c_p\Big[(m+la_2'(\beta))t_1+lb_2'(\beta)\frac{t_1^2}{2}-n\Big\{\frac{c_3}{u_4}\big(1-e^{\sqrt{u_3u_4}t_1}\big)-\frac{c_4}{u_4}\big(1-e^{-\sqrt{u_3u_4}t_1}\big)\\
&\quad+\frac{2(u_3w_4+v_3)t_1-u_3v_4t_1^2}{2u_3u_4}\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
PC_2''(\gamma) &= c_p\Big[(m+la_1''(\gamma))t_1+lb_1''(\gamma)\frac{t_1^2}{2}-n\Big\{\frac{c_5}{\sqrt{u_5u_6}}\big(e^{\sqrt{u_5u_6}t_1}-1\big)-\frac{c_6}{\sqrt{u_5u_6}}\big(e^{-\sqrt{u_5u_6}t_1}-1\big)\\
&\quad+\frac{2(u_6w_5+v_6)t_1-u_6v_5t_1^2}{2u_5u_6}\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
PC_2''(\gamma) &= c_p\Big[(m+la_2''(\gamma))t_1+lb_2''(\gamma)\frac{t_1^2}{2}-n\Big\{\frac{c_5}{u_6}\big(1-e^{\sqrt{u_5u_6}t_1}\big)-\frac{c_6}{u_6}\big(1-e^{-\sqrt{u_5u_6}t_1}\big)\\
&\quad+\frac{2(u_5w_6+v_5)t_1-u_5v_6t_1^2}{2u_5u_6}\Big\}\Big]
\end{aligned}
$$

Therefore, the total average cost of the system during the entire circle can be obtained in the parametric form as $[\tilde{\Phi}]_{\alpha,\beta,\gamma} = \langle[\Phi_1(\alpha),\ \Phi_2(\alpha)], [\Phi_1'(\beta),\ \Phi_2'(\beta)], [\Phi_1''(\gamma),\ \Phi_2''(\gamma)]\rangle$, where $\Phi_1(\alpha) = \frac{HC_1(\alpha)+PC_1(\alpha)+C_0}{T}$, $\Phi_2(\alpha) = \frac{HC_2(\alpha)+PC_2(\alpha)+C_0}{T}$, $\Phi_1'(\beta) = \frac{HC_1'(\beta)+PC_1'(\beta)+C_0}{T}$, $\Phi_2'(\beta) = \frac{HC_2'(\beta)+PC_2'(\beta)+C_0}{T}$, $\Phi_1''(\gamma) = \frac{HC_1''(\gamma)+PC_1''(\gamma)+C_0}{T}$ and $\Phi_2''(\gamma) = \frac{HC_2''(\gamma)+PC_2''(\gamma)+C_0}{T}$.

Therefore, mathematically, the minimization problem concerning the production inventory model in the case of type-1 neutrosophic differentiability of $\tilde{q}(t)$ is obtained as follows:

$$\begin{cases} Minimize\ \Phi_1(\alpha) \\ Minimize\ \Phi_2(\alpha) \\ Minimize\ \Phi_1'(\beta) \\ Minimize\ \Phi_2'(\beta) \\ Minimize\ \Phi_1''(\gamma) \\ Minimize\ \Phi_2''(\gamma) \\ Subject\ to\ T \geq t_1 \geq 0 \\ 0 \leq \alpha, \beta, \gamma \leq 1\ with\ \alpha + \beta + \gamma \leq 3 \end{cases} \tag{32}$$

Case 2: When $\tilde{q}(t)$ is type-2 neutrosophic differentiable

Then, the differential equation (11) representing the productive phase $(0 \leq t \leq t_1)$ is turned into a parametric form,

$$\begin{aligned} &\langle[\dot{q}_2(t;\alpha), \dot{q}_1(t;\alpha)], [\dot{q}_2'(t;\beta), \dot{q}_1'(t;\beta)], [\dot{q}_2''(t;\gamma), \dot{q}_1''(t;\gamma)]\rangle = m - (1-l)\langle[a_1(\alpha), a_2(\alpha)], \\ &[a_1'(\beta), a_2'(\beta)], [a_1''(\gamma), a_2''(\gamma)]\rangle - (1-l)\langle[b_1(\alpha), b_2(\alpha)], [b_1'(\beta), b_2'(\beta)], [b_1''(\gamma), b_2''(\gamma)]\rangle t \\ &-\big\{\langle[\theta_1(\alpha), \theta_2(\alpha)], [\theta_1'(\beta), \theta_2'(\beta)], [\theta_1''(\gamma), \theta_2''(\gamma)]\rangle + n\big\}\langle[q_1(t;\alpha), q_2(t;\alpha)], \\ &[q_1'(t;\beta), q_2'(t;\beta)], [q_1''(t;\gamma), q_2''(t;\gamma)]\rangle \end{aligned}$$

The above expression gives a system of crisp differential equations as follows:

$$\dot{q}_2(t;\alpha) = m - (1-l)a_2(\alpha) - (1-l)b_2(\alpha)t - \{\theta_2(\alpha) + n\}q_2(t;\alpha) \tag{33}$$
$$\dot{q}_1(t;\alpha) = m - (1-l)a_1(\alpha) - (1-l)b_1(\alpha)t - \{\theta_1(\alpha) + n\}q_1(t;\alpha) \tag{34}$$
$$\dot{q}_2'(t;\beta) = m - (1-l)a_2'(\beta) - (1-l)b_2'(\beta)t - \{\theta_2'(\beta) + n\}q_2'(t;\beta) \tag{35}$$
$$\dot{q}_1'(t;\beta) = m - (1-l)a_1'(\beta) - (1-l)b_1'(\beta)t - \{\theta_1'(\beta) + n\}q_1'(t;\beta) \tag{36}$$
$$\dot{q}_2''(t;\gamma) = m - (1-l)a_2''(\gamma) - (1-l)b_2''(\gamma)t - \{\theta_2''(\gamma) + n\}q_2''(t;\gamma) \tag{37}$$
$$\dot{q}_1''(t;\gamma) = m - (1-l)a_1''(\gamma) - (1-l)b_1''(\gamma)t - \{\theta_1''(\gamma) + n\}q_1''(t;\gamma) \tag{38}$$

with $q_1(0;\alpha) = q_2(0;\alpha) = q_1'(0;\beta) = q_2'(0;\beta) = q_1''(0;\gamma) = q_2''(0;\gamma) = 0$.

Now, the equation (34) can be simplified as

$$\begin{cases} \dot{q}_1(t;\alpha) + u_1 q_1(t;\alpha) = w_1 - v_1 t \\ q_1(0;\alpha) = 0 \end{cases} \tag{39}$$

The solution of equation (39) is obtained as

$$q_1(t;\alpha) = \frac{u_1 w_1 + v_1}{u_1^2}(1 - e^{-u_1 t}) - \frac{v_1}{u_1} t \tag{40}$$

Proceeding similarly, the solution of the equations (33) and (35)-(38) are obtained as

$$\begin{cases} q_2(t;\alpha) = \frac{u_2 w_2 + v_2}{u_2^2}(1 - e^{-u_2 t}) - \frac{v_2}{u_2} t \\ q'_1(t;\beta) = \frac{u_3 w_3 + v_3}{u_3^2}(1 - e^{-u_3 t}) - \frac{v_3}{u_3} t \\ q'_2(t;\beta) = \frac{u_4 w_4 + v_4}{u_4^2}(1 - e^{-u_4 t}) - \frac{v_4}{u_4} t \\ q''_1(t;\gamma) = \frac{u_5 w_5 + v_5}{u_5^2}(1 - e^{-u_5 t}) - \frac{v_5}{u_5} t \\ q''_2(t;\gamma) = \frac{u_6 w_6 + v_6}{u_6^2}(1 - e^{-u_6 t}) - \frac{v_6}{u_6} t \end{cases} \tag{41}$$

Again, the neutrosophic differential equation (12) represents the non-productive phase $(t_1 \leq t \leq T)$ in case of type-2 neutrosophic differentiability is turned in the parametric form,

$$\begin{aligned} \langle[\dot{q}_2(t;\alpha), \dot{q}_1(t;\alpha)], [\dot{q}'_2(t;\beta), \dot{q}'_1(t;\beta)], [\dot{q}''_2(t;\gamma), \dot{q}''_1(t;\gamma)]\rangle = -\langle[a_1(\alpha), a_2(\alpha)], [a'_1(\beta), a'_2(\beta)], \\ [a''_1(\gamma), a''_2(\gamma)]\rangle - \langle[b_1(\alpha), b_2(\alpha)], [b'_1(\beta), b'_2(\beta)], [b''_1(\gamma), b''_2(\gamma)]\rangle t - \langle[\theta_1(\alpha), \theta_2(\alpha)], \\ [\theta'_1(\beta), \theta'_2(\beta)], [\theta''_1(\gamma), \theta''_2(\gamma)]\rangle\langle[q_1(t;\alpha), q_2(t;\alpha)], [q'_1(t;\beta), q'_2(t;\beta)], [q''_1(t;\gamma), q''_2(t;\gamma)]\rangle \end{aligned}$$

which gives a system of differential equations as follows:

$$\dot{q}_2(t;\alpha) = -a_2(\alpha) - b_2(\alpha)t - \theta_2(\alpha) q_2(t;\alpha) \tag{42}$$
$$\dot{q}_1(t;\alpha) = -a_1(\alpha) - b_1(\alpha)t - \theta_1(\alpha) q_1(t;\alpha) \tag{43}$$
$$\dot{q}'_2(t;\beta) = -a'_2(\beta) - b'_2(\beta)t - \theta'_2(\beta) q'_2(t;\beta) \tag{44}$$
$$\dot{q}'_2(t;\beta) = -a'_1(\beta) - b'_1(\beta)t - \theta'_1(\beta) q'_1(t;\beta) \tag{45}$$
$$\dot{q}''_2(t;\gamma) = -a''_2(\gamma) - b''_2(\gamma)t - \theta''_2(\gamma) q''_2(t;\gamma) \tag{46}$$
$$\dot{q}''_2(t;\gamma) = -a''_1(\gamma) - b''_1(\gamma)t - \theta''_1(\gamma) q''_1(t;\gamma) \tag{47}$$

with $q_1(T;\alpha) = q_2(T;\alpha) = q'_1(T;\beta) = q'_2(T;\beta) = q''_1(T;\gamma) = q''_2(T;\gamma) = 0$.

The solution of the equation (43) is obtained as

$$q_1(t;\alpha) = \frac{b_1(\alpha) - \theta_1(\alpha)(b_1(\alpha)t + a_1(\alpha))}{(\theta_1(\alpha))^2} - \frac{b_1(\alpha) - \theta_1(\alpha)(b_1(\alpha)T + a_1(\alpha))}{(\theta_1(\alpha))^2} e^{\theta_1(\alpha)(T - t_1)} \tag{48}$$

Proceeding similarly, the solution of the equations (42) and (44)-(47) for the non-productive phase $(t_1 \leq t \leq T)$ are obtained as

$$\begin{cases} q_1(t;\alpha) = \frac{b_2(\alpha)-\theta_2(\alpha)(b_2(\alpha)t+a_2(\alpha))}{(\theta_2(\alpha))^2} - \frac{b_2(\alpha)-\theta_2(\alpha)(b_2(\alpha)T+a_2(\alpha))}{(\theta_2(\alpha))^2} e^{\theta_2(\alpha)(T-t_1)} \\ q_1'(t;\beta) = \frac{b_1'(\beta)-\theta_1'(\beta)(b_1'(\beta)t+a_1'(\beta))}{(\theta_1'(\beta))^2} - \frac{b_1'(\beta)-\theta_1'(\beta)(b_1'(\beta)T+a_1'(\beta))}{(\theta_1'(\beta))^2} e^{\theta_1'(\beta)(T-t_1)} \\ q_2'(t;\beta) = \frac{b_2'(\beta)-\theta_2'(\beta)(b_2'(\beta)t+a_2'(\beta))}{(\theta_2'(\beta))^2} - \frac{b_2'(\beta)-\theta_2'(\beta)(b_2'(\beta)T+a_2'(\beta))}{(\theta_2'(\beta))^2} e^{\theta_2'(\beta)(T-t_1)} \\ q_1''(t;\gamma) = \frac{b_1''(\gamma)-\theta_1''(\gamma)(b_1''(\gamma)t+a_1''(\gamma))}{(\theta_1''(\gamma))^2} - \frac{b_1''(\gamma)-\theta_1''(\gamma)(b_1''(\gamma)T+a_1''(\gamma))}{(\theta_1''(\gamma))^2} e^{\theta_1''(\gamma)(T-t_1)} \\ q_2''(t;\gamma) = \frac{b_2''(\gamma)-\theta_2''(\gamma)(b_2''(\gamma)t+a_2''(\gamma))}{(\theta_2''(\gamma))^2} - \frac{b_2''(\gamma)-\theta_2''(\gamma)(b_2''(\gamma)T+a_2''(\gamma))}{(\theta_2''(\gamma))^2} e^{\theta_2''(\gamma)(T-t_1)} \end{cases} \tag{49}$$

Several relevant costs associated with the model are formulated as follows: Therefore, the holding cost, $\tilde{HC} = \langle [HC_1(\alpha), HC_2(\alpha)], [HC_1'(\beta), HC_2'(\beta)], [HC_1''(\gamma), HC_2''(\gamma)] \rangle$ given by

$$\begin{aligned} HC_1(\alpha) &= c_h\Big[\int_0^{t_1} q_1(t;\alpha)dt + \int_{t_1}^{T} q_2(t;\alpha)dt\Big] \\ &= c_h\Big[\int_0^{t_1} \Big\{\frac{u_1w_1+v_1}{u_1^2}(1-e^{-u_1t}) - \frac{v_1}{u_1}t\Big\}dt + \int_{t_1}^{T}\Big\{\frac{b_1(\alpha)-\theta_1(\alpha)(b_1(\alpha)t+a_1(\alpha))}{(\theta_1(\alpha))^2} \\ &\quad - \frac{b_1(\alpha)-\theta_1(\alpha)(b_1(\alpha)T+a_1(\alpha))}{(\theta_1(\alpha))^2} e^{\theta_1(\alpha)(T-t_1)}\Big\}dt\Big] \\ &= c_h\Big[\frac{u_1w_1+v_1}{u_1^3}\big(e^{-u_1t_1} + u_1t_1 - 1\big) - \frac{v_1}{2u_1}t_1^2 - \frac{b_1(\alpha)}{2\theta_1(\alpha)}(T^2 - t_1^2) \\ &\quad + \frac{b_1(\alpha)-\theta_1(\alpha)(b_1(\alpha)T+a_1(\alpha))}{(\theta_1(\alpha))^3}\big\{1 - e^{\theta_1(\alpha)(T-t_1)}\big\}\Big] \end{aligned}$$

Similarly, calculating the other components of the holding cost as

$$\begin{aligned} HC_2(\alpha) &= c_h\Big[\frac{u_2w_2+v_2}{u_2^3}\big(e^{-u_2t_1} + u_2t_1 - 1\big) - \frac{v_2}{2u_2}t_1^2 - \frac{b_2(\alpha)}{2\theta_2(\alpha)}(T^2 - t_1^2) \\ &\quad + \frac{b_2(\alpha)-\theta_2(\alpha)(b_2(\alpha)T+a_2(\alpha))}{(\theta_2(\alpha))^3}\big\{1 - e^{\theta_2(\alpha)(T-t_1)}\big\}\Big] \end{aligned}$$

$$\begin{aligned} HC_1'(\beta) &= c_h\Big[\frac{u_3w_3+v_3}{u_3^3}\big(e^{-u_3t_1} + u_3t_1 - 1\big) - \frac{v_3}{2u_3}t_1^2 - \frac{b_1'(\beta)}{2\theta_1'(\beta)}(T^2 - t_1^2) \\ &\quad + \frac{b_1'(\beta)-\theta_1'(\beta)(b_1'(\beta)T+a_1'(\beta))}{(\theta_1'(\beta))^3}\big\{1 - e^{\theta_1'(\beta)(T-t_1)}\big\}\Big] \end{aligned}$$

$$\begin{aligned} HC_2'(\beta) &= c_h\Big[\frac{u_4w_4+v_4}{u_4^3}\big(e^{-u_4t_1} + u_4t_1 - 1\big) - \frac{v_4}{2u_4}t_1^2 - \frac{b_2'(\beta)}{2\theta_2'(\beta)}(T^2 - t_1^2) \\ &\quad + \frac{b_2'(\beta)-\theta_2'(\beta)(b_2'(\beta)T+a_2'(\beta))}{(\theta_2'(\beta))^3}\big\{1 - e^{\theta_2'(\beta)(T-t_1)}\big\}\Big] \end{aligned}$$

$$\begin{aligned} HC''_1(\gamma) &= c_h\Big[\frac{u_5w_5+v_5}{u_5^3}\big(e^{-u_5t_1}+u_5t_1-1\big)-\frac{v_5}{2u_5}t_1^2-\frac{b''_1(\gamma)}{2\theta''_1(\gamma)}(T^2-t_1^2) \\ &\quad +\frac{b''_1(\gamma)-\theta''_1(\gamma)(b''_1(\gamma)T+a''_1(\gamma))}{(\theta''_1(\gamma))^3}\big\{1-e^{\theta''_1(\gamma)(T-t_1)}\big\}\Big] \end{aligned}$$

$$\begin{aligned} HC''_2(\gamma) &= c_h\Big[\frac{u_6w_6+v_6}{u_6^3}\big(e^{-u_6t_1}+u_6t_1-1\big)-\frac{v_6}{2u_6}t_1^2-\frac{b''_2(\gamma)}{2\theta''_2(\gamma)}(T^2-t_1^2) \\ &\quad +\frac{b''_2(\gamma)-\theta''_2(\gamma)(b''_2(\gamma)T+a''_2(\gamma))}{(\theta''_2(\gamma))^3}\big\{1-e^{\theta''_2(\gamma)(T-t_1)}\big\}\Big] \end{aligned}$$

Therefore, the production cost $\tilde{PC} = \langle [PC_1(\alpha), PC_2(\alpha)], [PC'_1(\beta), PC'_2(\beta)], [PC''_1(\gamma), PC''_2(\gamma)]\rangle$ during the entire circle is given by

$$\begin{aligned} PC_1(\alpha) &= c_p\Big[\int_0^{t_1}\big\{(m+la_1(\alpha))-nq_1(t;\alpha)+lb_1(\alpha)t\big\}dt\Big] \\ &= c_p\Big[(m+la_1(\alpha))t_1+lb_1(\alpha)\frac{t_1^2}{2}-n\int_0^{t_1}q_1(t;\alpha)dt\Big] \\ &= c_p\Big[(m+la_1(\alpha))t_1+lb_1(\alpha)\frac{t_1^2}{2}-n\Big\{\frac{u_1w_1+v_1}{u_1^3}\big(e^{-u_1t_1}+u_1t_1-1\big)-\frac{v_1}{2u_1}t_1^2\Big\}\Big] \end{aligned}$$

$$\begin{aligned} PC_2(\alpha) &= c_p\Big[\int_0^{t_1}\big\{(m+la_2(\alpha))-nq_2(t;\alpha)+lb_2(\alpha)t\big\}dt\Big] \\ &= c_p\Big[(m+la_2(\alpha))t_1+lb_2(\alpha)\frac{t_1^2}{2}-n\int_0^{t_1}q_2(t;\alpha)dt\Big] \\ &= c_p\Big[(m+la_2(\alpha))t_1+lb_2(\alpha)\frac{t_1^2}{2}-n\Big\{\frac{u_2w_2+v_2}{u_2^3}\big(e^{-u_2t_1}+u_2t_1-1\big)-\frac{v_2}{2u_2}t_1^2\Big\}\Big] \end{aligned}$$

$$PC'_1(\beta) = c_p\Big[(m+la'_1(\beta))t_1+lb'_1(\beta)\frac{t_1^2}{2}-n\Big\{\frac{u_3w_3+v_3}{u_3^3}\big(e^{-u_3t_1}+u_3t_1-1\big)-\frac{v_3}{2u_3}t_1^2\Big\}\Big]$$

$$PC''_2(\beta) = c_p\Big[(m+la'_2(\beta))t_1+lb'_2(\beta)\frac{t_1^2}{2}-n\Big\{\frac{u_4w_4+v_4}{u_4^3}\big(e^{-u_4t_1}+u_4t_1-1\big)-\frac{v_4}{2u_4}t_1^2\Big\}\Big]$$

$$PC''_1(\gamma) = c_p\Big[(m+la''_1(\gamma))t_1+lb''_1(\gamma)\frac{t_1^2}{2}-n\Big\{\frac{u_5w_5+v_5}{u_5^3}\big(e^{-u_5t_1}+u_5t_1-1\big)-\frac{v_5}{2u_5}t_1^2\Big\}\Big]$$

$$PC_2''(\gamma) \;=\; c_p\left[(m + la_2''(\gamma))t_1 + lb_2''(\gamma)\frac{t_1^2}{2} - n\left\{\frac{u_6 w_6 + v_6}{u_6^3}\left(e^{-u_6 t_1} + u_6 t_1 - 1\right) - \frac{v_6}{2u_6}t_1^2\right\}\right]$$

Therefore, the total average cost of the system during the entire circle can be obtained in the parametric form as $[\tilde{\Psi}]_{\alpha,\beta,\gamma} = \langle[\Psi_1(\alpha),\ \Psi_2(\alpha)], [\Psi_1'(\beta),\ \Psi_2'(\beta)], [\Psi_1''(\gamma),\ \Psi_2''(\gamma)]\rangle$, where $\Psi_1(\alpha) = \frac{HC_1(\alpha)+PC_1(\alpha)+C_0}{T}$, $\Psi_2(\alpha) = \frac{HC_2(\alpha)+PC_2(\alpha)+C_0}{T}$, $\Psi_1'(\beta) = \frac{HC_1'(\beta)+PC_1'(\beta)+C_0}{T}$, $\Psi_2'(\beta) = \frac{HC_2'(\beta)+PC_2'(\beta)+C_0}{T}$, $\Psi_1''(\gamma) = \frac{HC_1''(\gamma)+PC_1''(\gamma)+C_0}{T}$ and $\Psi_2''(\gamma) = \frac{HC_2''(\gamma)+PC_2''(\gamma)+C_0}{T}$.

Therefore, mathematically, the minimization problem concerning the production inventory model in the case of type-2 neutrosophic differentiability of $\tilde{q}(t)$ is obtained as follows:

$$\begin{cases} Minimize\ \Psi_1(\alpha) \\ Minimize\ \Psi_2(\alpha) \\ Minimize\ \Psi_1'(\beta) \\ Minimize\ \Psi_2'(\beta) \\ Minimize\ \Psi_1''(\gamma) \\ Minimize\ \Psi_2''(\gamma) \\ Subject\ to\ T \geq t_1 \geq 0 \\ 0 \leq \alpha, \beta, \gamma \leq 1\ with\ \alpha + \beta + \gamma \leq 3 \end{cases} \tag{50}$$

7. Used new De-Neutrosophication Method

To understand the neutrosophic outcome and to compare the results obtained in terms of neutrosophic numbers, it urges to assign a crisp value to the neutrosophic numbers in an appropriate means. The de-neutrosophication technique then comes into the picture. The removal area method [12] is one of the popular methods. In this method, the de-neutrosophication value of a triangular single-valued neutrosophic number $\tilde{P}_{DN} = \langle(u, v, w : \epsilon), (x, y, z : \delta), (h, i, j : \gamma)\rangle$ is

$$D_{New}(\tilde{P}_{DN}, 0) = \frac{u + 2v + w + x + 2y + z + h + 2i + j}{12} \tag{51}$$

In the current article, the following de-neutrosophication technique is introduced:

Suppose $\tilde{A}_{TN} = (a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3)$ be a triangular neutrosophic number whose parametric form can be described by $\langle[A_1(\alpha),\ A_2(\alpha)], [A_1'(\beta),\ A_2'(\beta)], [A_1''(\gamma),\ A_2''(\gamma)]\rangle$, where $A_1(\alpha) = a_1 + \alpha(a_2 - a_1)$, $A_2(\alpha) = a_3 - \alpha(a_3 - a_2)$, $A_1'(\beta) = b_2 - \beta(b_2 - b_1)$, $A_2'(\beta) = b_2 + \beta(b_3 - b_2)$, $A_1''(\gamma) = c_2 - \gamma(c_2 - c_1)$ and $A_2''(\gamma) = c_2 + \gamma(c_3 - c_2)$. Then, the de-neutrosophication value of $\tilde{A}_{TN}$ is denoted by A_{De-Neu} and is given by

$$A_{De-Neu} = \frac{A_1 + A_2 + A_3}{3} \tag{52}$$

where $A_1 = \alpha A_1(\alpha) + (1-\alpha)A_2(\alpha),\ A_2 = \beta A_1'(\beta) + (1-\beta)A_2'(\beta)$ and $A_3 = \gamma A_1''(\gamma) + (1-\gamma)A_2''(\gamma)$.

Using the proposed de-neutrosophication technique, the multi-objective optimization problem (30) for case 1 is transformed into a single objective crisp problem.

$$\begin{cases} Minimize\ \tilde{\Phi}_{De-New} \\ where\ \tilde{\Phi}_{De-New} = \frac{\Phi_{11}+\Phi_{12}+\Phi_{13}}{3} \\ \Phi_{11} = \alpha\Phi_1(\alpha) + (1-\alpha)\Phi_2(\alpha) \\ \Phi_{12} = \beta\Phi_1'(\beta) + (1-\beta)\Phi_2'(\beta) \\ \Phi_{13} = \gamma\Phi_1''(\gamma) + (1-\gamma)\Phi_2''(\gamma) \\ Subject\ to\ T > t_1 > 0,\ 0 \le \alpha,\ \beta,\ \gamma \le 1. \end{cases} \tag{53}$$

For case 2, the multi-objective optimization problem (50) is transformed into the following single objective crisp problem

$$\begin{cases} Minimize\ \tilde{\Psi}_{De-New} \\ where\ \tilde{\Psi}_{De-New} = \frac{\Psi_{11}+\Psi_{12}+\Psi_{13}}{3} \\ \Psi_{11} = \alpha\Psi_1(\alpha) + (1-\alpha)\Psi_2(\alpha) \\ \Psi_{12} = \beta\Psi_1'(\beta) + (1-\beta)\Psi_2'(\beta) \\ \Psi_{13} = \gamma\Psi_1''(\gamma) + (1-\gamma)\Psi_2''(\gamma) \\ Subject\ to\ T > t_1 > 0,\ 0 \le \alpha,\ \beta,\ \gamma \le 1. \end{cases} \tag{54}$$

8. Numerical Simulation

8.1. Algorithm of the Numerical Solution

Step 1 Input the numerical value of the crisp parameter $l, m, n, a, b, \theta, c_p, c_h$ and C_0.

Step 2 Solve the crisp minimization problem (7) and get the optimum average cost TAC, production cycle time t_1 and inventory cycle length T.

Step 3 Take the demand-control parameters a and b and deterioration rate θ as the triangular neutrosophic number and consider the model in the neutrosophic arena. Go to step 4 or go to step 6.

Step 4 Find the de-neutrosophication values of the neutrosophic parameters using the formula given in (51).

Step 5 Solve the model with the de-neutrosophication values of the neutrosophic parameters along with other crisp inputs (called the old method).

Step 6 Solve two minimization problems (53) and (54) (namely, Case 1 and Case 2) corresponding to two types of the generalized neutrosophic differentiability of the neutrosophic valued function $\tilde{q}(t)$.

Step 7 Compare the results of Case 1, Case 2, and the old method for neutrosophic problems with the crisp method and get the optimum average cost.

Step 8 End.

The above-mentioned algorithm can be depicted visually in the following flow chart given in Figure 2.

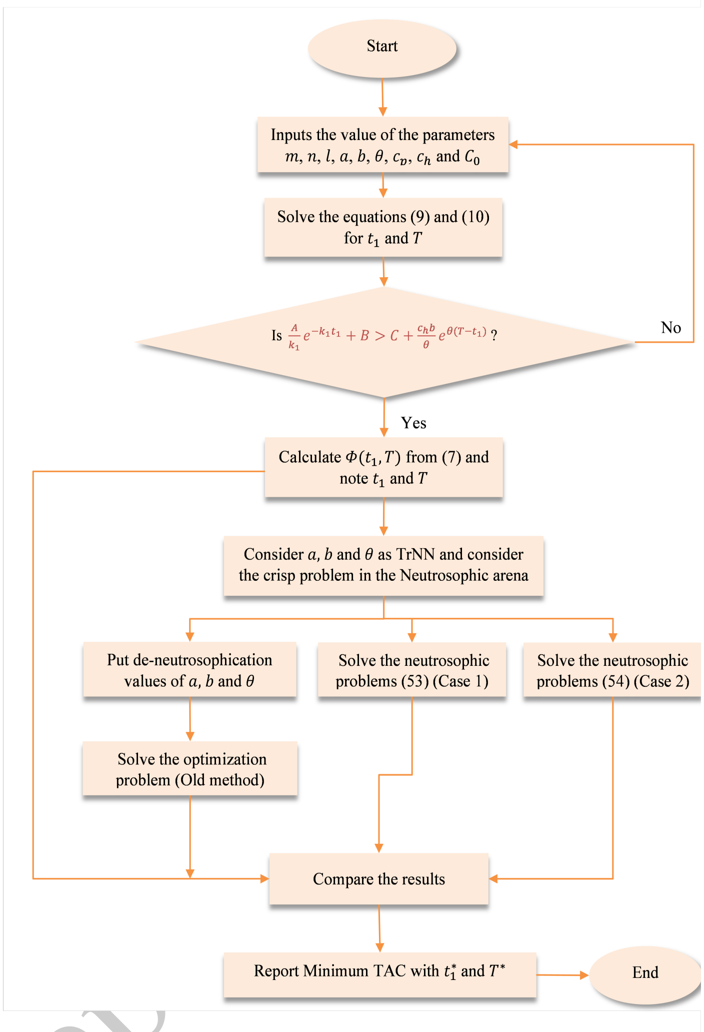

Figure 2. Flowchart for numerical solutions.

8.2. Numerical results and graphical representation

In this subsection, four different problems are set for the numerical manipulation. Besides the crisp problem, three problems are considered from the data with neutrosophic uncertainty. Two types of neutrosophic differentiability are associated with two problems, namely Case 1 and Case 2. The problems named Case 1 and Case 2 are described and analyzed through the neutrosophic differential equation, and the optimum values of the decision variables and objective function are obtained using the proposed de-neutrosophication technique discussed in Section 7. Also, another problem (named the old method) is considered: taking de-neutrosophication of the given neutrosophic parameters before going for crisp-valued calculus-inspired dynamics of the system. For numerical simulation, the following inputs are considered:

1. For the crisp model, we take $m = 90, n = 0.06, l = 0.3, a = 40, b = 0.5, \theta = 0.05, c_p = 6, c_h = 1, C_0 = 140$.

2. For the neutrosophic model, the neutrosophic number $\tilde{a}$, $\tilde{b}$ and $\tilde{\theta}$ are taken as a single-valued triangular neutrosophic number with nine components as
$\tilde{a} = (35, 40, 45; 41, 45, 49; 31, 37, 43)$,
$\tilde{b} = (0.45, 0.50, 0.55; 0.54, 0.55, 0.56; 0.3, 0.45, 0.6)$ and
$\tilde{\theta} = (0.045, 0.050, 0.055; 0.054, 0.055, 0.056; 0.03, 0.045, 0.06)$
and the value of other parameters is taken as the same as in the crisp model.

The optimum value of the average cost (TAC^*) and the decision variables, namely, the total time cycle (T) and production time (t_1) is represented by Table 3. A graphical counterpart of the obtained results is also displayed through the bar diagram given in Figure 3.

Table 3. Optimum results for four different methods for solving the proposed EPQ model in crisp and neutrosophic arena

Model	t_1^*	T^*	TAC^*
Crisp Model	1.867337	3.718347	681.1805
Neutrosophic Model (Case 1)	4.890101	10.28361	309.1551
Neutrosophic Model (Case 2)	2.869033	4.828840	654.0223
Neutrosophic Model (Old Method)	2.070397	3.758374	688.2674

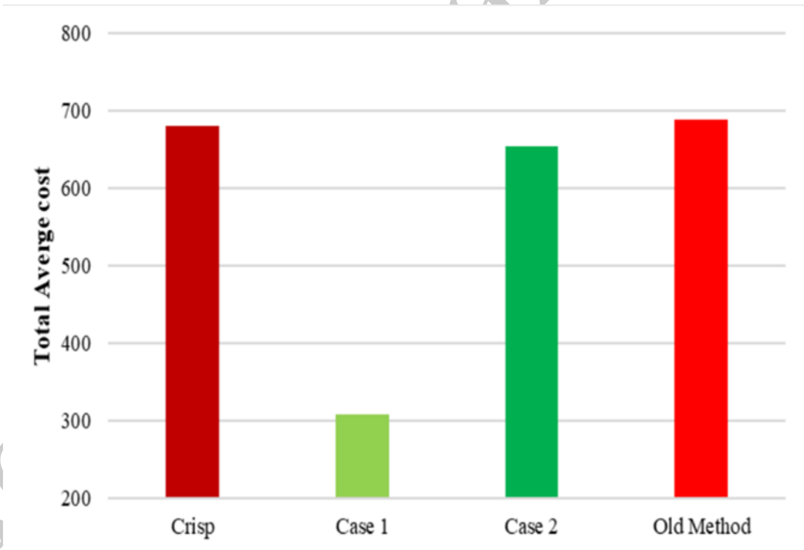

Figure 3. Total average costs in different methods.

From Table 3 and Figure 3, it is perceived that the cost minimization objective is better fulfilled while considering the neutrosophic phenomena with neutrosophic calculus-oriented discussion and proposed de-neutrosophication technique. The old method with the removal of the area de-neutrosophication technique before going for the crisp calculus-oriented approach seems to give the most likely outcome of the crisp model. Therefore, the proposed approach to dealing with the dynamic of the inventory is established through numerical outcomes. Among the two cases of the proposed technique, Case 1 seems to be more effective in minimizing the TAC. However, the inventory cycle phase is seen to be very high in Case 1, which is concerned with the feasibility of the system. Therefore, Case 2 is fitted to the most desired approach in the cost minimization objective with a feasible measure of the productive time and total lot-sizing cycle.

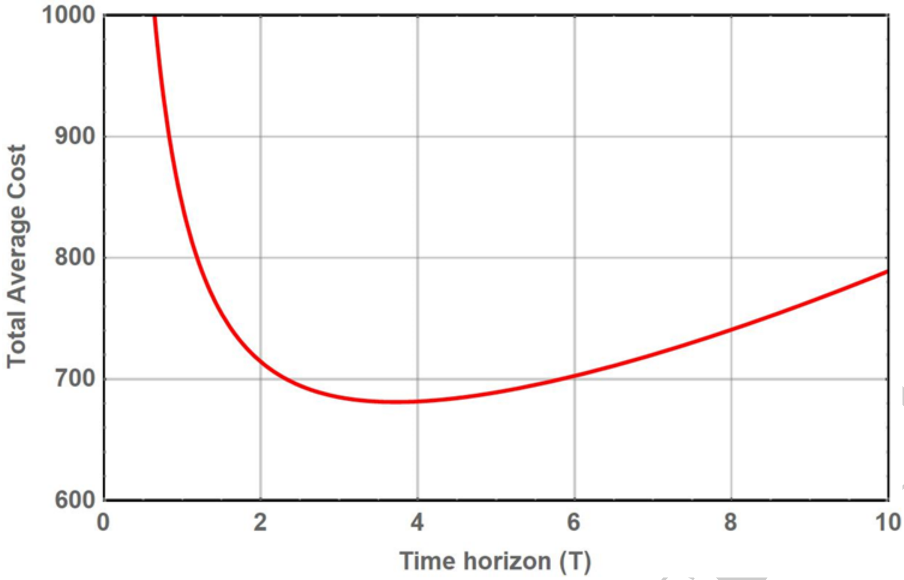

Figure 4. Total average cost versus total inventory cycle time.

Figure 4 shows the graph of the TAC concerning the lot-sizing cycle length. The initial trend of the graph of the TAC is decreasing against the total time cycle, and reaching the lowest value of 681.1805 at time 3.718347 months, the graph again increases.

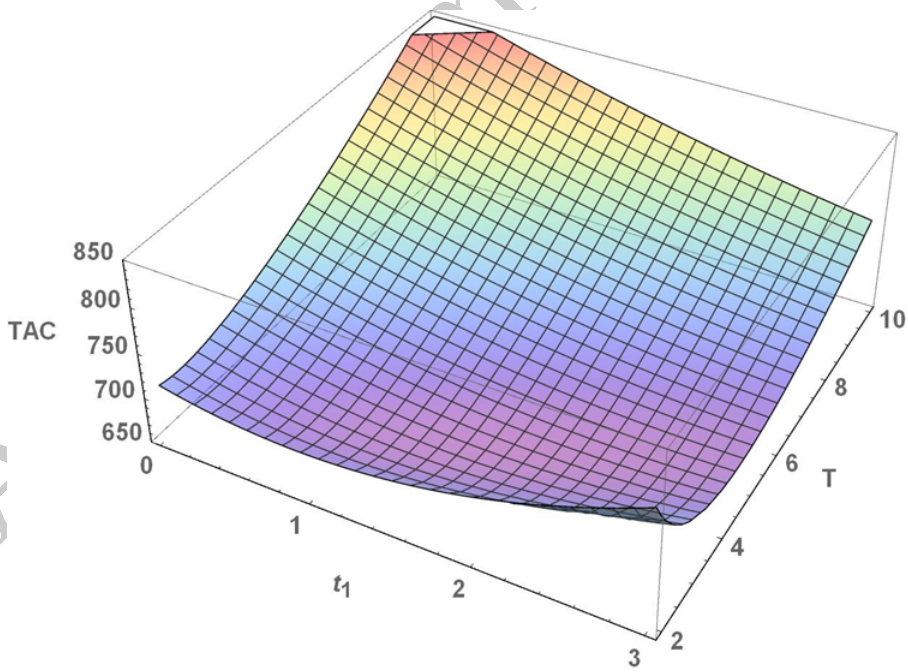

Figure 5. Interdependency of average cost, production time, and time cycle.

Figure 5 shows the three-dimensional inter-dependence among the average cost, total time cycle, and production cycle. The locally convex nature of the graph around (1.867337, 3.718347, 681.1805) is spotted clearly in the figure.

8.3. Sensitivity Analysis with respect to deterministic parameters

In this section, a sensitivity analysis is performed for both cases on the crisp parameter by changing a parameter on a range of -15% to +30% while other parameters kept their original values. The sensitivity of the optimal results against the crisp parameters is given in Table 4. A graphical counterpart of the tabular display is presented in Figures 6 and 7.

Table 4. Sensitivity of the optimum results concerning the crisp input

Crisp parameters	Change in (%)	Case 1 t_1^*	Case 1 T^*	Case 1 TAC^*	Case 2 t_1^*	Case 2 T^*	Case 2 TAC^*
$m = 90$	+15	4.656	10.3219	328.92	-	2.5342	677.14
	+10	-	0.8628	301.39	0.3922	2.8986	676.38
	+5	4.809	10.3031	315.80	1.4687	3.7814	668.75
	-5	-	10.2149	301.3885	46.7081	46.7081	368.63
	-10	5.0546	10.2147	295.72	45.1739	45.1739	335.13
	-15	5.0546	10.2146	295.72	43.5424	43.5424	290.33
n=0.06	+15	4.9276	10.3212	307.01	3.4011	5.3092	652.18
	+10	4.9150	10.3086	307.72	3.1843	5.1119	652.86
	+5	-	0.8628	301.39	3.0116	4.9562	653.47
	-5	4.8777	10.2712	309.89	2.7487	4.7221	654.52
	-10	4.8654	10.2589	310.61	2.6457	4.6312	654.97
	-15	4.8532	10.2466	311.34	2.5568	4.5532	655.38
l=0.3	+15	4.9549	10.4763	312.18	2.0761	4.2152	661.43
	+10	4.9335	10.4124	311.17	2.3046	4.3892	659.17
	+5	-	0.8628	301.39	2.5644	4.5895	656.71
	-5	4.8681	10.2187	308.14	3.2451	5.1328	651.09
	-10	-	0.8628	301.38	3.7611	5.5671	647.85
	-15	4.8237	10.088	306.12	44.522	44.5220	383.48
$c_p = 6$	+15	-	0.8628	301.38	-	2.5342	677.14
	+10	4.8900	10.2836	336.35	-	-	-
	+5	4.8901	10.2836	322.75	1.7362	4.0398	668.66
	-5	4.8901	10.2836	295.56	3.6488	5.2027	636.25
	-10	-	0.8628	301.39	4.0748	5.1733	616.31
	-15	4.8902	10.2836	268.36	4.1871	4.7713	594.25
$c_h = 1$	+15	-	0.8628	322.26	3.7992	4.5065	689.20
	+10	4.8901	10.2835	311.52	3.7494	4.8810	679.32
	+5	4.8901	10.2836	310.34	3.4733	5.0214	667.64
	-5	-	0.8628	294.43	1.8214	4.1753	637.55
	-10	-	0.8628	287.48	0.1646	2.8207	614.75
	-15	-	0.8628	280.52	-	2.7350	583.54
$C_0 = 140$	+15	-	0.8628	325.73	48.1973	48.1974	388.93
	+10	4.8901	10.2836	310.52	3.1793	5.1926	656.82
	+5	-	0.8628	309.50	3.0225	5.0097	655.45
	-5	4.8901	10.2836	308.47	2.7183	4.6493	652.54
	-10	4.8901	10.2836	307.79	2.5695	4.4703	651.01
	-15	4.8901	10.2836	307.11	2.4222	4.2911	649.41

From Table 4, Figure 6, and Figure 7, the following points can be summarized:

(i) In the discussion, m denotes the fixed part of the production rate. It is seen that the average profit in the first case does not follow any monotonicity characteristic when the values of m increases.

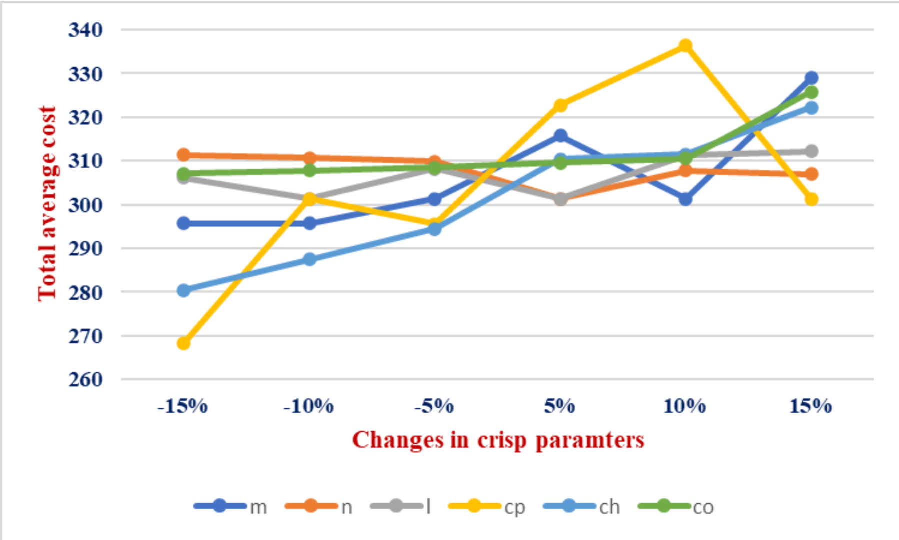

Figure 6. Sensitivity analysis of case 1 of the crisp parameter on total average profit.

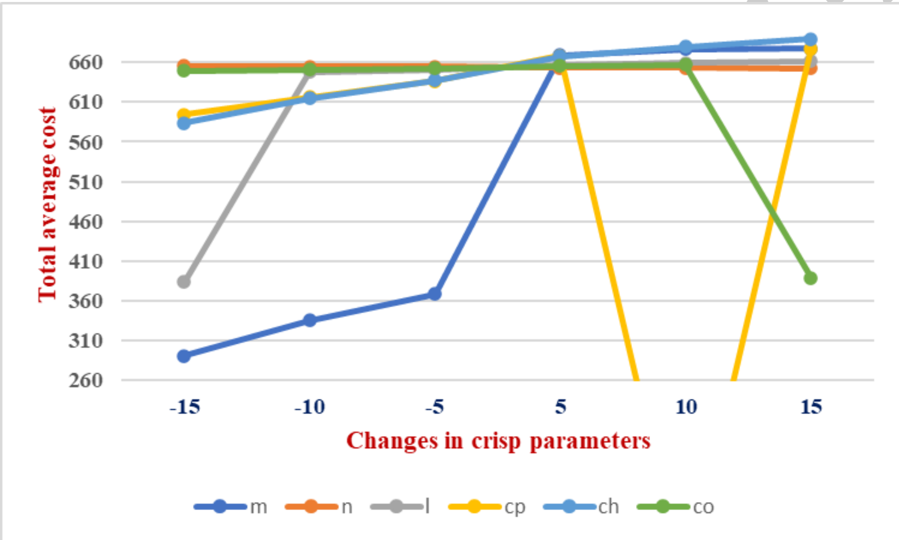

Figure 7. Sensitivity analysis of case 2 of the crisp parameter on total average profit.

Because the optimal solution of variant values of m corresponds to some distinguished production and decision tenures. The second case shows a specific pattern in this regard. It is noted that the average cost can be minimized through the minimization of the production rate.

(ii) In the discussion, n denotes the coefficient of stock dependency of the production rate. As the value of n increases, the production rate should be lowered. However, in the first case, we find some infeasible cases regarding the pattern of optimal values of production and decision cycle to the variance of n. Thus, the impact of n on the cost, minimization is not well perceived in the first case. It is noted that the average cost can be lowered with the enhancement of the values of n in the second case.

(iii) In the discussion, l denotes the coefficient of the demand dependency of the production rate. The model was built with the general perspective that the production rate should be increased as demand increases. Now, in the first case of the above discussion, there is no such straightforward conclusion on the impact of the demand on the minimization of the average profit. A significant result corresponds to the second case of the above discussion. It is seen that the average cost increases with the enhancement of the demand rate.

(iv) The remaining crisp parameters in Table 4 are the production, holding, and setup costs of the items. Therefore, the average cost increases when the individual cost increases.

8.4. Sensitivity Analysis with respect to imprecise parameters

In this section, stability of the best obtained cost with optimal production and decision cycle length is examined in the context of imprecise parameters. The results regarding sensitivity of optimal solution with respect to neutrosophic valued parameters are displayed in Table 5. A graphical counterpart of the tabular display is presented in Figures 8 and 9.

Table 5. Sensitivity of the optimum results concerning the Neutrosophic input

Imprecise parameters	Change in (%)	Case 1 t_1^*	Case 1 T^*	Case 1 TAC^*	Case 2 t_1^*	Case 2 T^*	Case 2 TAC^*
$\tilde{a}$	+15	5.236302	10.45742	331.31	2.871687	4.923177	658.26
	+10	0.000000	0.8412141	332.67	2.871785	4.892951	656.85
	+5	5.014098	10.35621	316.55	2.870835	4.861444	655.44
	-5	0.000000	0.8723352	286.77	2.866525	4.795275	652.59
	-10	0.000000	0.8830415	272.92	2.863420	4.760849	651.15
	-15	4.8901	10.2836	307.11	2.4222	4.2911	649.41
$\tilde{b}$	+15	7.617859	15.06751	316.54	1.869831	4.148519	638.04
	+10	0.000000	0.8792371	286.96	2.267089	4.445040	643.72
	+5	4.842598	10.17940	309.85	2.598877	4.670114	649.01
	-5	0.000000	0.8430415	277.92	3.076379	4.925095	658.62
	-10	0.000000	0.873415	281.92	3.250088	4.974487	663.40
	-15	4.60248	10.19637	312.81	3.375978	4.976268	667.82
$\tilde{\theta}$	+15	4.330256	9.149906	308.22	0.5962931	3.067986	624.63
	+10	4.502700	9.500159	308.43	1.291782	3.652540	636.28
	+5	4.688800	9.877092	308.74	2.036938	4.226846	646.03
	-5	5.108348	10.72296	309.70	3.871620	5.538776	660.22
	-10	5.345510	11.19878	310.41	45.49685	45.49685	415.23
	-15	5.603750	11.71507	311.31	44.12879	44.12879	421.71

In the proposed model, three parameters have been taken imprecise. The demand is the most significant but volatile issue in an economic production quantity model. Therefore, both the demand impacting parameters have been taken triangular neutrosophic numbers for tackling impreciseness. Between these parameters, $\tilde{a}$ denotes the imprecise demand potential, and $\tilde{b}$ represents the coefficient of uncertain reliance of demand on the most crucial independent variable time. Also, we have considered the rate of deterioration as imprecise parameters $\tilde{\theta}$. The best results in an imprecise environment correspond to the imprecise inputs $\tilde{a} = (35, 40, 45; 41, 45, 49; 31, 37, 43)$, $\tilde{b} = (0.45, 0.50, 0.55; 0.54, 0.55, 0.56; 0.3, 0.45, 0.6)$ and $\tilde{\theta} = (0.045, 0.050, 0.055; 0.054, 0.055, 0.056; 0.03, 0.045, 0.06)$ in both cases of neutrosophic differentiation. The sensitivity analysis with respect to these parameters has been performed varying the impacting parameters between -15% and 15% of the mentioned values. The changing the values with given ranges, not only the value is scaled but also dispersion of data due imprecision is also scaled. The results in Table 5 can be interpreted as follows:

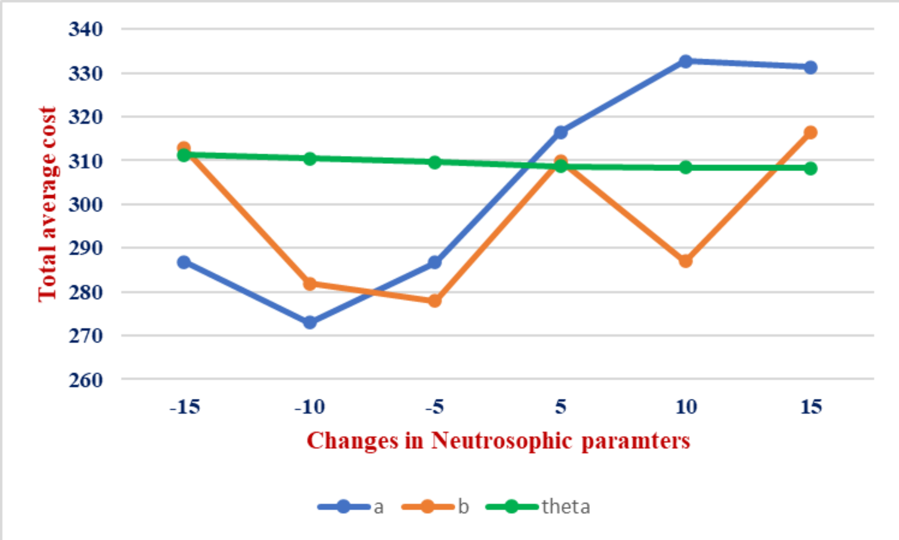

Figure 8. Sensitivity analysis of case 1 of the Neutrosophic parameter on total average profit.

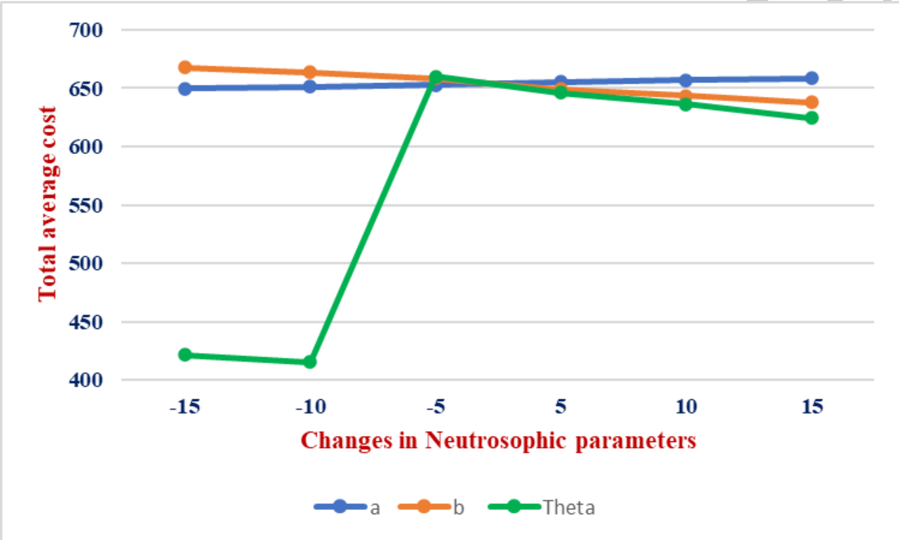

Figure 9. Sensitivity analysis of case 2 of the Neutrosophic parameter on total average profit.

(i) In case of neutrosophic derivative of first type, we did not find any specific pattern of changes in average costs with respect to the demand potential $\tilde{a}$. Furthermore, the productive cycle becomes infeasible in many occasions connected to Case 1. However, in Case 2, a crystal pattern is obtained regarding the impact of the demand potential on average cost. The average cost can be made lowered by decreasing demand potential. In this case, the ambiguities associated with demand prediction increases with production potential resulting negative impression on cost reduction goal. Also, the production cycle and complete lot cycle are increased with the deviation of imprecise data.

(ii) In this imprecise EPQ model, $\tilde{b}$ represents the coefficient of variability of demand on time. In case of neutrosophic derivative of first type, we did not find any specific pattern of changes in average costs against variance of $\tilde{b}$. Furthermore, the productive cycle becomes infeasible in many occasions connected to Case 1. However, in Case 2, the results in Table 5 reflect reverse pattern compared to the pattern obtained for $\tilde{a}$. The cost can be minimized by increasing values of $\tilde{b}$. The observation can be interpreted that increasing time influence of the demand rate favors the cost reduction goal.

The production cycle and whole lot cycle also follow the pattern for average cost.

(iii) The deterioration of products during inventory carrying period is a natural phenomenon. In this paper, deterioration rate is described by an uncertain parameter $\tilde{\theta}$. In both cases of the neutrosophic derivative of inventory function of time, average cost, production and decision lot sizes decrease with the increasing nature of $\tilde{\theta}$ and associated impressions.

9. Major research findings and managerial intuitions

In the first part of this section, we summarize all the research outcomes from the proposed model. The managerial implications corresponding to the outcomes are listed subsequently. Fundamental observations in numerical simulation can be briefed as follows:

(i) Average cost increases robustly accordingly the primary part of the production rate in all the discussed models. However, in both cases of imprecise decision environment, optimal manufacturing cycle and overall decision cycle become infeasible with respect to many values of m. Thus, the obtained solutions are locally optimal and highly sensitive with respect to the demand potential.

(ii) In proposed model, n denotes the coefficient of stock controlling the production rate. The numerical results show that cost can be minimized by increasing the influence of stock on production process.

(iii) The results regarding l shows that average cost increases as the control of demand on the production process increases.

(iv) The imprecise decision-making phenomena with triangular neutrosophic number reduce the cost effectively compared two decision making environment. Among mentioned approaches of neutrosophic decision making phenomena, the neutrosophic differential equation of type 2 is established smarter.

(v) The average cost can be made lowered by decreasing demand potential. However, increasing time influence of the demand rate favors the cost reduction goal.

At the end of this section the managerial implications are decoded as follows:

(i) The decision maker must install cohesion between on hand stock and production process. The average cost can be made significantly reduced by a smart managerial strategy which considers strong influence of the inventory on manufacturing rate.

(ii) The demand is imprecise and indeterministic in nature. As a consequence, much impacts of demand on production process includes additional cost. Therefore, the smart decision maker must set policy such that the production process can be impacted a very little with the imprecisions involved with demand pattern.

(iii) Also, the decision maker can reduce the average cost by increasing the weight of influence of time on the demand pattern.

10. Conclusions

In this paper, we use an EPQ model with a time-dependent demand rate, a demand-, and stock-dependent production rate, and a constant rate of deterioration. The hypothesis was built on a very general perception. In the newly organized manufacturing-supply sector, demand increases as time goes by. Also, the production rate varies according to the demand rate and the stock already in the warehouse. During the inventory-carrying process, the produced items deteriorate gradually as a natural cause. Our objective was to address the impact of the mentioned issues on cost reduction. Furthermore, an additional query was about the involvement of uncertainty regarding the decision environment. Therefore, in this paper, the proposed model has been formulated mathematically in a crisp environment. With the theory of differential equations and calculus, the criteria for the optimality of the objective function have been discussed in a crisp environment. Then, the uncertainty regarding real-world production-inventory management has been tackled using neutrosophic numbers and the theory of the neutrosophic differential equation. In the end, a de-neutrosophication formula has been used for comparing results in distinct cases of uncertain phenomena as well as their crisp counterparts. The numerical results provide some of the finest insights regarding the proposed model. First, the demand enhancement is not favorable for the cost minimization objective. Another significant observation is that stock in the showroom can favor the cost minimization objective. Furthermore, the proposed approach of the neutrosophic differential equation as a tool to describe the EPQ model and the proposed de-neutrosophication formula provide better results compared to the crisp and existing methods to deal with an EPQ model under uncertainty.

At the end of the discussion, it is to be mentioned that the theory of neutrosophic numbers has been limited in some multi-criteria decision-making problems in the recently introduced literature. However, it has a wider scope to describe the uncertainty regarding the dynamics of problems in operational research, especially in inventory theory. In the future, the notions of neutrosophic calculus and neutrosophic differential equations must be improved; therefore, a non-linear inventory model with more realistic assumptions can be fitted to be analyzed through it. Furthermore, the proposed model can be modified with variable rates of deterioration and preservation measures, etc., for more reliable mathematical models in the future.

Acknowledgement

The authors are grateful to two anonymous reviewers for their valuable comments and suggestions made on the previous draft of this manuscript.

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