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Optimizing Pharmaceutical Inventory and Investment Strategies During Pandemics: A Dynamic Approach Integrating Environmental Emission Rates and Advanced Optimization Algorithms

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Abstract

This study presents a strategy for managing pharmaceutical inventory during pandemics, focusing on optimizing investment in COVID-19 medicines while ensuring product preservation. A customized inventory model considers critical factors such as price, infection rate, and preservation, adaptable to various pandemic scenarios. Optimal control theory is applied for dynamic investment adjustments, enhancing resource allocations and decision-making. The study addresses a complex replenishment problem involving joint pricing, environmental costs, order costs, preservation technology, and replenishment schedules for non-instantaneous deteriorating items, aiming to maximize retailer's profit. Advanced optimization algorithms, including Ant Colony and Cuckoo Search, determine optimal pricing, investment costs, and replenishment schedules. Theoretical analysis and numerical experiments under a fuzzy learning environment provide a robust foundation for the model. Sensitivity analysis offers practical insights, guiding decision-makers in adapting strategies to real-world challenges. From a managerial perspective, this study provides actionable solutions for balancing profitability with sustainability, ensuring efficient resource use during crises. It also highlights the importance of integrating environmental costs and preservation technology into inventory decisions, particularly in dynamic and uncertain environments like a pandemic. The study delivers comprehensive guidance for effective pandemic response planning, helping managers to make informed decisions that align with both economic and public health goals.

Keywords: *Dynamic investment, Metaheuristic Algorithm, Preservation technology, Infection awareness investments, Fuzzy learning, Environment cost, Trapezoidal-type demand.*

1. Introduction

In December 2019, the world faced the emergence of the novel coronavirus disease (COVID-19) in Wuhan, China, triggering unprecedented global challenges. On March 11, 2020, the World Health Organization (WHO) declared COVID-19 a global pandemic. Within a few months, the virus had rapidly

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spread worldwide, infecting nearly 2.5 million people by April 23, 2020, Kulkarni et al. [20]. This pandemic has not only caused widespread illness but has also profoundly disrupted global economies, particularly impacting the pharmaceutical industry. As a result, pharmaceutical companies are struggling to maintain market stability, with the pandemic hindering access to essential, affordable medicines—a fundamental goal of every pharmaceutical system [1].

Beyond the global challenges, it is crucial to conduct situational analyses of the pharmaceutical industry in developing countries with diverse markets to understand the pandemic's broader impacts. COVID-19 presents a unique opportunity for the pharmaceutical sector due to the increased demand for prescription medicines, vaccines, and medical devices. Amid the pandemic, the importance of effective pharmaceutical inventory management has been underscored by the dynamic demand for medicines, vaccines, masks, and medical devices, coupled with the urgent need for healthcare resources (Akhtar et al.[2]).

1.1. Motivations

Our research is driven by the need to understand and model the unique trapezoidal demand pattern observed in pharmaceutical inventory during the COVID-19 pandemic. During the initial stages of the pandemic, the demand for pharmaceutical products increased gradually as the infection rate surged within the population. This rise was largely driven by the urgent need for medical devices and treatments to manage the growing number of COVID-19 cases. As the pandemic advanced to its middle stage, the infection rate stabilised, leading to a steady demand for pharmaceutical inventory. This plateau was mainly due to the widespread implementation of vaccination campaigns, which played a critical role in controlling the virus's spread and mitigating its impact on public health. In the final stage of the pandemic, the demand for pharmaceutical products started to decline, reflecting the reduced need for COVID-19-related medical resources. This decrease was closely tied to the population's increasing immunity through successful vaccination efforts.

The trapezoidal demand pattern highlights the dynamic interplay between the progression of the pandemic, the effectiveness of vaccination strategies, and the resulting fluctuations in pharmaceutical inventory demand. Additionally, our research integrates the concept of learning, which is vital for enhancing the economic sustainability of the system. This integration is crucial for developing more adaptive and resilient inventory management strategies in the face of future public health crises.

1.2. Objectives

The proposed model seeks to achieve the following objectives:

- We will assess the potential profitability and feasibility of investing in preservation technology within the pharmaceutical industry, focusing on its ability to extend product shelf life and enhance market competitiveness.

- We will develop a sophisticated pricing framework that adapts to dynamic market conditions, i.e. (S_{p_i}) and including promotion efforts ($F_s(t)$) influenced by the fluctuating COVID-19 infection rates.

[1] https://www.who.int/publications/i/item/924154547X

- We will construct a comprehensive model where the demand for pharmaceutical products dynamically responds to changes in the COVID-19 infection rate (ϵ).
- We plan to conduct an in-depth analysis to understand how increasing vaccination rates influence consumer behaviour and subsequently affect demand patterns within the pharmaceutical sector.
- We will explore sustainable inventory management strategies, emphasizing the reduction of environmental emissions and preventing the damage, expiration, or degradation of pharmaceutical products through targeted investments. This includes a focus on controllable environmental carbon emissions, which has gained significant interest from both practitioners and researchers.

In inventory management, key factors such as demand patterns, promotional efforts, and the deterioration rates of pharmaceutical products play crucial roles in determining optimal inventory policies. Retailers often employ various strategies to boost demand for pharmaceutical inventory, with promotional efforts being particularly effective (Manna et al. [23]). These efforts are often aimed at achieving specific sales targets, with promotions offered as incentives once those targets are met (Blom et al. [5]). However, deciding to invest in promotional activities during a pandemic presents significant challenges. Such investments are dynamic rather than deterministic, making it difficult to model them mathematically.

Despite these challenges, the literature highlights the importance of promotions in stimulating consumer demand and staying competitive (Goli et al. [11]). Promotions serve various objectives, such as boosting pharmaceutical sales Goli et al. [12]). As pharmaceutical products move through different stages of their lifecycle, retailers must adapt their promotional strategies to the evolving market dynamics. In the early stages of the COVID-19 pandemic, promotions focused on raising awareness and building brand recognition, similar to initial efforts to educate the public on safety measures. As the pandemic progressed, the focus shifted to persuading consumers, akin to promoting vaccine uptake. In the final stages, promotions aimed to clear the remaining inventory, maximizing revenue as the pandemic wound down.

The demand for pharmaceutical products is influenced by multiple factors, including price, expiration dates, stock levels, and product quality. This is particularly true for perishable items like pharmaceuticals, where demand often spikes during a pandemic. Traders must, therefore, employ effective strategies to minimize deterioration and extend shelf life, as expired pharmaceutical products can pose significant health risks. Governments often establish cold storage facilities for the long-term preservation of these items. Research has underscored the importance of preservation technology in managing the inventory of perishable goods, including pharmaceuticals (Dye et al. [9]).

The volatility in demand creates significant challenges for industries in balancing costs, availability, and consumer needs. These costs include inventory holding, ordering, promotions, and potential stockouts, all of which directly impact profitability (Braglia et al. [6]). Optimizing pharmaceutical inventory management during the COVID-19 pandemic is essential to meet healthcare demands while maximizing profit. Pontryagin's Maximum Principle offers a framework for dynamically adjusting promotional efforts, preservation technologies, pharmaceutical products, and storage strategies in response to fluctuating demand and urgent healthcare needs (Harrison et al. [10]). This approach considers constraints such as vaccine production capacities, storage facilities, and regulatory requirements, enabling businesses to effectively allocate resources and respond to the pandemic (Supakar et al. [29]). This comprehensive strategy allows pharmaceutical industries and healthcare systems to address the challenges of the COVID-19 pandemic while ensuring the availability of essential medical supplies (Almurisi [3]).

2. Literature Review

This literature review is organized around the following key areas of research:

2.1. Trapezoidal-Type Demand for Dynamics of COVID-19

During COVID-19 pandemic has led to a significant rise in the demand for pharmaceuticals inventory as shown in Figure 1, especially medications used to treat COVID-19 and its related issues like pneumonia. This surge is mainly due to more people being hospitalized with COVID-19 and experiencing pneumonia, as well as an increased need for ventilators to support patients. As a result, there has been a shortage of these crucial medicines. Regulatory bodies worldwide have recognized this problem and have published lists confirming the shortages, which mostly include drugs for treating COVID-19 and pneumonia. For instance, the FDA in the United States has noted shortages in medications specifically meant for COVID-19 treatment, including potential therapies like hydroxychloroquine and chloroquine, along with commonly used drugs for COVID-19 patients in critical care, such as azithromycin, dopamine, dobutamine, fentanyl, heparin, midazolam, propofol, and dexmedetomidine, as highlighted by Arati et al. [4].

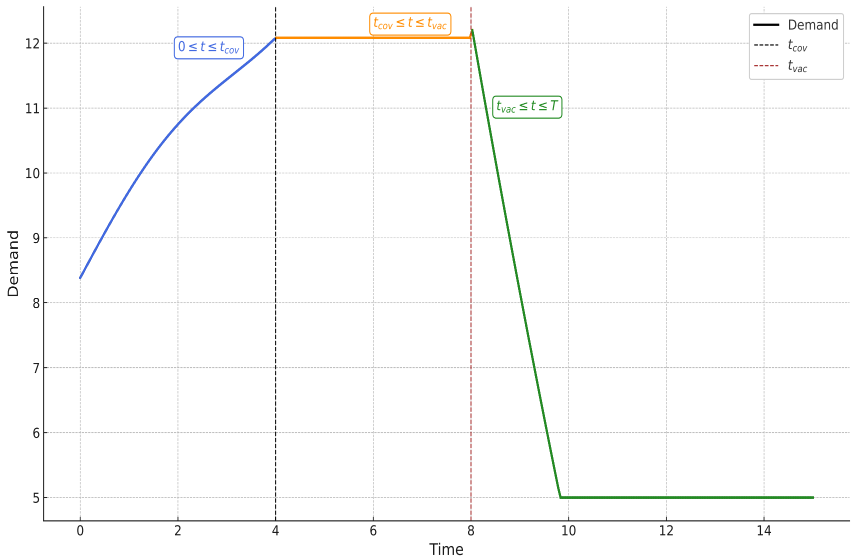

Figure 1. Trajectory of Demand Function with Key Intervention Points t_{cov} and t_{vac}.

2.2. Marketing and Promotional Policy

Pricing strategies and replenishment schedules are fundamental elements of inventory management. Promotional activities significantly influence demand and, consequently, pricing schemes. Ullah et al. [32] investigated dynamic pricing policies within a multi-period inventory model, while Yu et al. [34] explored various types of promotional efforts. Despite these contributions, a notable gap exists in the literature regarding the integration of promotional strategies with the dynamic nature of pandemics, particularly under a trapezoidal demand pattern. Afshar-Nadjafi et al. [1] developed a mathematical model addressing time-dependent demand policies, while Dwivedi [8] highlighted how brand image, staff behaviour, and customer loyalty influence profitability. These studies underscore the importance of integrating promotional strategies with dynamic market conditions to optimize retail performance.

Promotional efforts exert a substantial impact on both the retail industry and consumer behaviour. Research indicates that coordination between manufacturers and retailers can be highly advantageous. For instance, Nguyen et al. [27] examined cooperative advertising policies between retailers and manufacturers. Additionally, factors such as brand image play a critical role in shaping retailer profitability.

2.3. Preservation Technology

In addition to trapezoidal-type demand, the literature relevant to this study primarily focuses on several key research areas, including dynamic investment in preservation technology. In the retail industry, product perishability is a critical concern. Beyond natural deterioration, perishability can be exacerbated by factors such as improper handling, inadequate maintenance, suboptimal display techniques, irregular disposal, and adverse environmental conditions. Investing in preservation technology can significantly reduce deterioration rates, minimize product losses, and help retailers optimise profit. This need has become even more urgent during the global pandemic, particularly due to the sharp increase in product deterioration resulting from lockdowns and disruptions in routine activities. Therefore, investing in preservation technology is crucial to extending product lifespan.

Tiwari et al. [30] examined an optimal pricing policy for deteriorating items within a warehouse setting, considering warehouse capacity in their model. In another study, Tiwari et al. [31] explored a joint pricing and inventory model for deteriorating items with expiration dates, incorporating a partial trade-credit policy with partial backlogging. Hsu et al. [13] introduced a preservation technology strategy aimed at enhancing the profitability of a monopolistic retailer. More recently, various inventory replenishment models for deteriorating items have been discussed, including Dye et al. [9], who proposed a preservation strategy to identify the optimal replenishment policy.

2.4. Sustainable Inventory Model

Recent advancements in sustainable supply chain management highlight a focus on integrating emissions reduction with profit maximization. Datta [7] pioneered a model that balances profit with emissions reduction by incorporating technology equipment costs under carbon tax policies. Lin [22] further advanced this by exploring investment strategies to lower transportation emissions in sustainable inventory systems that accommodate backordering. Meanwhile, Keswani and Khedlekar [16] developed an optimal inventory system that addresses both product deterioration and emission rates and also created an item green retailing model that combines two-phase advance sales with a discount policy. khedlekar et al. [19] expanded on these concepts by designing a inventory model that integrates carbon emissions considerations in a green production model in an interval-valued framework. In this paper, deterioration occurs due to the physical nature of the item. To address this, an investment is made in preservation technology to control the deterioration rate, while cost parameters are adjusted to account for the effects of learning rate.

This proposed research addresses the identified research gap. Several scholars have considered the effects of deterioration without incorporating investment in preservation technology. This study specifically examines the impact of such investment under trapezoidal-type demand. The paper is structured as follows: Sections 1 and 2 provide an introduction and a literature review, which aims to identify the research gap and highlight contributions. Section 3 outlines the problem description, while Section 4 details the simulation approach, including estimating necessary medicine quantities, mathematical modeling, notations, and resilience concepts. Section 5 discusses the ant colony and cuckoo search algorithms. Section 6 presents a numerical example, and Section 7 offers a sensitivity analysis. Section 8 provides managerial insights, and Section 9 concludes with the major findings, limitations, and suggestions for future research.

2.5. Fuzzy Learning Curve

The learning curve represents a geometric progression that demonstrates the consistent cost reduction as repetitive tasks are performed. According to the theory, as production doubles, the per-unit cost decreases by a constant percentage. Although there has been considerable debate among researchers regarding the appropriate form of the learning curve, whether it should follow a power or exponential model, a consensus has emerged in favor of the power form. The most widely accepted model, as proposed by Kazemi et al. [15], Khatua et al. [18], and Kumar et al. [21], is illustrated in Figures 2, 3, 4, 5, 6 and 7. This model outlines three key phases: the initial incipient phase, the learning phase, and the maturity phase.

The graphical representation of the fuzzy learning rate of holding cost, ordering cost and purchasing cost.

3. Comparison with Existing Work

As shown in Table ?? and through our literature review, no previous research has given priority to human learning alongside financial considerations, environmental, and social sustainability in the way this study does. Consequently, a direct numerical comparison with earlier studies is not possible. However, the core model of this study shares similarities with previous models when certain assumptions are relaxed.

- If we relax the assumptions regarding social sustainability and human learning, and instead consider price-dependent demand, holding costs, carbon tax, and a crisp environment, the base model of this study aligns with that of Mishra et al. [26].
- By relaxing the assumptions related to environmental and social sustainability, human learning, and considering fixed holding costs, no deterioration, selling price-dependent demand, shortages without partial backordering, and a crisp environment, the base model of this study corresponds to Kumar et al. [21].
- If we remove the assumptions regarding environmental and social sustainability, omit human learning, and include shortages, Weibull distribution for the deterioration rate, selling price-dependent demand, and a crisp environment, the base model of this study is comparable to Keswani et al. [17].

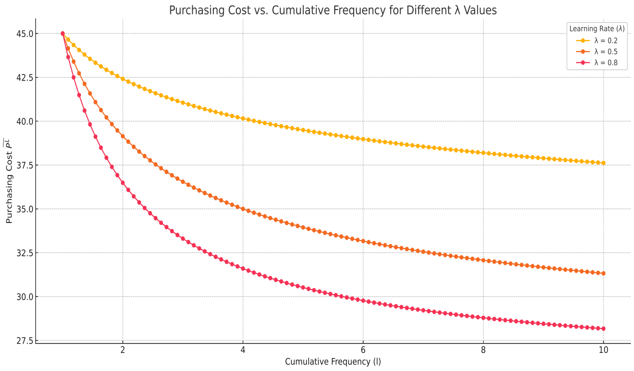

Figure 2. Effect of Learning Rate in Purchasing Cost by Varying λ

Figure 3. Effect of Learning Rate in Purchasing Cost by Varying l

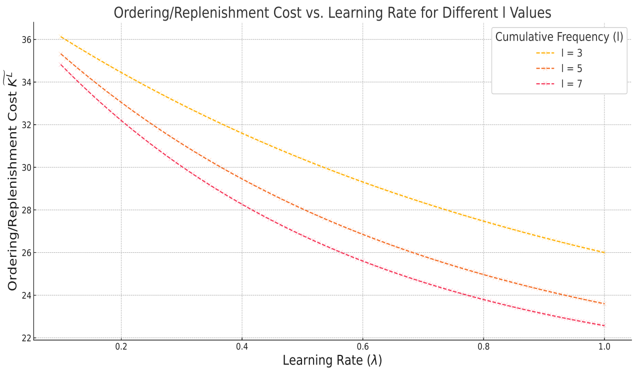

Figure 4. Effect of Learning Rate in Ordering Cost by Varying λ

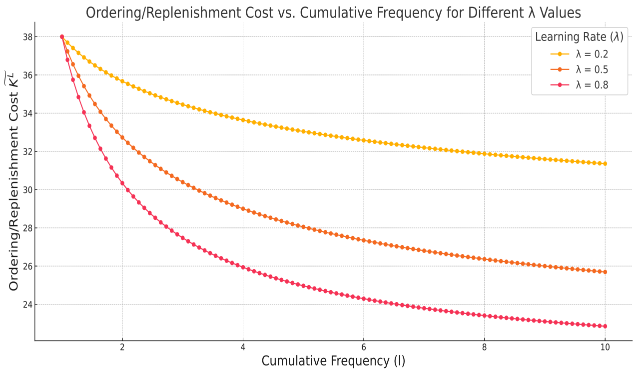

Figure 5. Effect of Learning Rate in Ordering Cost by Varying l

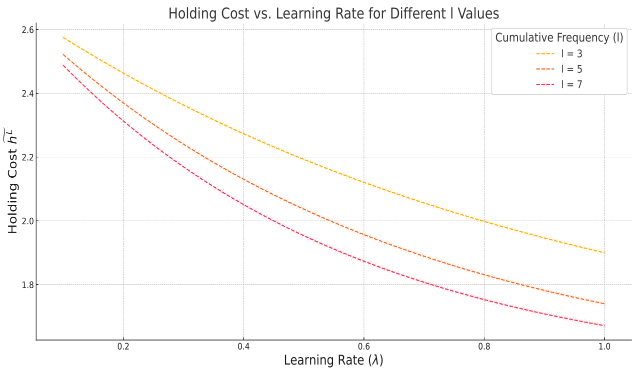

Figure 6. Effect of Learning Rate in Holding Cost by Varying λ

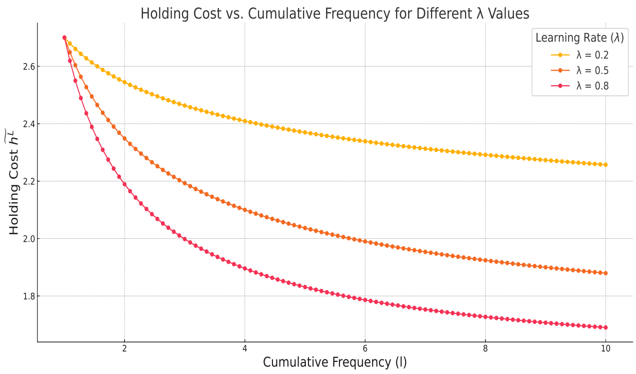

Figure 7. Effect of Learning Rate in Holding Cost by Varying l

4. Problem Formulation, Notation and Assumptions

This study examines how demand for pharmaceutical inventory changes during the different stages of the COVID-19 pandemic. It identifies patterns in demand, from an initial increase during the early stages of the pandemic to a stabilizing phase as vaccination efforts progress, and finally to a decline as cases of COVID-19 decrease. The research also looks at how factors like promotional strategies and preservation techniques impact inventory management during the pandemic. By understanding these dynamics, the study aims to develop effective strategies for pricing, investment in preservation technology, and responding to changes in demand influenced by COVID-19 infection rates and vaccination efforts. Ultimately, the goal is to provide practical insights to help the pharmaceutical industry effectively manage inventory and ensure availability of essential medical supplies while maximizing profitability throughout the pandemic.

This section presents the mathematical model notation and listed the problem's primary assumptions for our work.

4.1. Notation

Index	
I	consecutive time period, $i = 1, 2, 3$
Decision variables	
$f_{s_i}(t)$	dynamic investment rate for promotion at $t \in [t_{cov}, t_{vac}]$
S_{p_i}	selling price per unit time $t \in [t_{cov}, t_{vac}]$
I^0	initial inventory level, $I^0 > 0$
$u(t)$	preservation technology investment, $u \geq 0$
T	length of the replenishment cycle
Parameters	
$I(t)$	inventory holding position, $t \in (0, T]$
$F_s(t)$	impact of promotion on demand rate at time $t \in (0, T]$
$\widetilde{K^L}$	$K + \frac{K_l}{l^\lambda}, \quad 0 < \lambda < 1$ ordering/replenishment cost K per order (\$/order), K_l is constant, λ is the learning rate and l is the cumulative frequency.
$\widetilde{P_c^L}$	$P_c + \frac{P_l}{l^\lambda}, \quad 0 < \lambda < 1$ purchasing cost per unit (\$/unit), P_l is constant, λ is the learning rate and l is the cumulative frequency.
$\widetilde{h^L}$	$h + \frac{h_l}{l^\lambda}, \quad 0 < \lambda < 1$ inventory holding cost per unit per unit time (\$/ unit/unit time), h_l is constant, λ is the learning rate and l is the cumulative frequency.
$\widetilde{c_d^L}$	$c_d + \frac{c_{dl}}{l^\lambda}, \quad 0 < \lambda < 1$ disposal cost per unit (\$/unit), c_{dl} is constant, λ is the learning rate and l is the cumulative frequency.
θ_{d_1}	deterioration rate under the natural condition
θ_{d_0}	reduced feasible deterioration rate under preservation technology investment
η_i	rate of selling price of products at any time $t \in [t_{cov}, t_{vac}]\, ; i = 1, 2, 3$
α_i	marginal cost for investment for promotion at time $t \in [t_{cov}, t_{vac}]\, ; i = 1, 2, 3$
A_Π	total profit per unit time (\$/time)
n	total pharmaceutical inventory needs per person
I_{cov}	number of infected population due to COVID
V_{cov}	number of vaccinated population
ϵ	rate of infected population due to COVID
γ_{vac}	rate of vaccinated population
N	total number of populations
ν_i	rate of promotion in different interval ; i=1, 2, 3
ξ	efficiency of preservation investment

4.2. Assumptions

The following assumptions are used to formulate the mathematical model.

1. The model considers a single-type of items with an attractive dynamic rate in promotion and infection of COVID-19 and a Ant Colony Optimization and Cuckoo Search Algorithm is developed to solve the model by using Prontryagin's maximum principle.

2. The market demand pattern $D\left(t, S_{p_1}, S_{p_2}, S_{p_3}, F_s(t)\right)$ is a price-promotion and trapezoidal-type and

its functional form is as follows:

$$D\left(t, S_{p_1}, S_{p_2}, S_{p_3}, F_s(t)\right) = \begin{cases} a_1 + b_1 t - \eta_1 S_{p_1} + \nu_1 F_s(t) + n\epsilon e^{I_{cov}} & \text{if } 0 \le t \le t_{cov} \\ D_0 - \eta_2 S_{p_2} + \nu_2 F_s(t) + \frac{n\epsilon I_{cov}}{N} - \gamma_{vac} V_{cov} & \text{if } t_{cov} \le t \le t_{vac} \\ a_2 e^{-b_2 t} - \eta_3 S_{p_3} + \nu_3 F_s(t) + n\epsilon e^{-I_{cov}} & \text{if } t_{vac} \le t \le T \end{cases} \tag{1}$$

where a_1 and a_2 are market-fixed demand values selected from historical survey reviews related to consumption, b_1 and b_2 are scaling factors related to the time period, and ν_1 and ν_2 are scaling factors related to service investments during that particular period.

The demand pattern is closely related with Wu et al. [33]. The demand increases directly with respect to time through the initial and growth stage, i.e. $[0, t_{cov}]$, where $e^{I_{cov}}$ implies that the rate of infection increases exponentially during pandemic. In the maturity stage $[t_{cov}, t_{vac}]$, it becomes steady with respect to time and ultimately starts declining in the final stage $[t_{vac}, T]$. In all three stages, the additional fluctuation occurs due to prices, rate of infection of COVID-19 and investment in promotion. The promotion is used to accomplish objectives like, building product awareness, attract new customers, providing information about the store location, stimulating demand, increase sales in off-seasons. Therefore, the impact of the promotion cannot be ignored if the retailer would like to determine tangible the optimal replenishment policy and investment decision. Moreover, it is commonly observed that the retailer sets different prices during different stages of the product lifespan, and effective pricing strategy always helps the retailer to maximise profits on sales. This motivates the proposed study to integrate the effects of promotion and price under a trapezoidal-type demand curve, allowing us to examine the impact of promotion and price on the demand for pharmaceutical inventory across three different intervals.

3. It is assumed that the rate of investment in promotion is not uniform throughout different stages of the product's lifespan. The retailer can adjust the investment rate and price at each stage. Moreover, it is natural that the impact of the promotion diminishes as time progresses. The following differential equation is for the time evolution of promotional investment and its impact is considered as follows:

$$\dot{F}_s(t) = \begin{cases} f_{s_1}(t) - \delta_1 F_s(t) & \text{if } 0 \le t < t_{cov} \\ f_{s_2}(t) - \delta_2 F_s(t) & \text{if } t_{cov} \le t \le t_{vac} \\ f_{s_3}(t) - \delta_3 F_s(t) & \text{if } t_{vac} \le t, \end{cases} \tag{2}$$

where $\delta_i, i = 1, 2, 3$, represent the decay rate of the promotional effect in three consecutive periods, where $\delta_1 \le \delta_2 \le \delta_3$. Further, the retailer's investment in promotion involves quadratic instantaneous cost functions at each stage measured by the following form:

$$C\left(F_{s_i}(t)\right) = \frac{\alpha_i f_{s_i}^2(t)}{2} \tag{3}$$

where $\alpha_i > 0$ implies an increasing marginal cost of service investment. With limited resources, the firm makes joint investment decisions with respect to preservation technology and service. It is assumed that the resource capacity is upper bounded by U, i.e.,

$$\frac{\alpha_i f_{s_i}^2(t)}{2} + u(t) \leq U \tag{4}$$

4. The rate of deterioration is

$$\theta_d = \theta_d(u) = \theta_{d_0} + (\theta_{d_1} - \theta_{d_0})\, e^{-u\xi} \tag{5}$$

under the influence of preservation technology investment u. The existing literature gave the concept that the deterioration rate tends to zero if the retailer puts in immense investment. It is commonly observed that products like medicine, vaccines and pharmaceutical inventory deteriorate naturally, and it is impossible to eliminate the deterioration rate fully. Therefore, a threshold value θ_{d_0} is considered for representing the permissible rate of deterioration under the influence of feasible investment in preservation technology. Note that, for $\theta_{d_0} = 0$, it is similar to existing literature. It is another important note that $\theta_d \to \theta_{d_1}$ if $u \to 0$, i.e. the rate of deterioration remains unchanged if the retailer does not invest in preservation technology. Similarly, $\theta_d \to \theta_{d_0}$ if $u \to \infty$, i.e. the deterioration rate reaches its threshold value under the large investment of the retailer.

5. The replenishment rate is instantaneous. Due to the deteriorating nature of items, inventory levels become zero at the end of a replenishment cycle. Thus, the shortage is not allowed. Deteriorating items should not be repaired or reworked.

6. The environmental emission cost E_R is added to the proposed model as a decision variable. If $i(E_R)$ is an investment for reduction of environmental emissions cost, then $i(E_R) = \frac{1}{\delta}\log\left[\frac{E_R^0}{E_R}\right]T$, for $0 < E_R \leq E_R^0$, where δ is a decrease in the cost of environmental emission E_R per dollar to increase $i(E_R)$ and T is the total cycle time. The environmental emission cost $i(E_R)$ is a one-time investment that will provide better returns in the future. Hence, the annual investment cost is $\rho i(E_R) = \frac{\rho}{\delta}\log\left[\frac{E_R^0}{E_R}\right]T$, where ρ is an opportunity cost (Mishra et al. [25]).

7. The impact of learning on ordering, holding, and purchasing costs is considered (refer to Kumar et al. [21]).

5. Model Formulation

Based on the assumptions, the mathematical formulation of the proposed model is developed in this section. The demand and deterioration are the causes of the depletion of inventory, and the governing differential equations represent the inventory level as follows:

$$\dot{I}(t) = \begin{cases} -a_1 - b_1 t + \eta_1 S_{p_1} - \nu_1 F_s(t) - \theta_d I(t) - n\epsilon e^{I_{cov}} & \text{if } 0 \leq t \leq t_{cov} \\ -D_0 + \eta_2 S_{p_2} - \nu_2 F_s(t) - \theta_d I(t) - \frac{n\epsilon I_{cov}}{N} + \gamma_{vac} V_{cov} & \text{if } t_{cov} \leq t \leq t_{vac} \\ -a_2 e^{-b_2 t} + \eta_3 S_{p_3} - \nu_3 F_s(t) - n\epsilon e^{-I_{cov}} - \theta_d I(t) & \text{if } \mu \leq t \leq T, \end{cases} \tag{6}$$

where $I(0) = Q_0$ is initial stock and $I(T) = 0$. Using the rate of change of inventory during several time intervals, it is easy to calculate the revenues and other system costs, which are needed to decide the dynamic investment in promotion. By that, the management of the industry would decide how much investment they can allow for the promotion. Due to the physical nature of the item, deterioration occurs over time. To manage this deterioration, investments are made in preservation technology. Additionally, the impact of learning is factored into the cost parameters.

- Sells revenue (SR) in the cycle $[0,\ \mathrm{T}]$

  Within any supply chain, the customer is the retailer's source of revenue. Therefore, revenue plays a very important role in any decision-making process. Basic revenue can be found in the multiplication of demand and selling price. However, nowadays, due to the complexity of the business market, it is not always possible to fix demand. It may be possibly different in several cases. The demand and selling prices are different at different intervals. Thus, the sales revenue (SR) in the cycle $[0,\ \mathrm{T}]$ can be calculated as follows:

$$\begin{aligned} SR = \frac{1}{T}\Bigg[& S_{p_1}\int_0^{t_{cov}} \left[a_1 + b_1 t - \eta_1 S_{p_1} + \nu_1 F_s(t) + n\epsilon e^{I_{cov}}\right] dt \\ & + S_{p_2}\int_{t_{cov}}^{t_{vac}} \left[D_0 - \eta_2 S_{p_2} + \nu_2 F_s(t) + \frac{n\epsilon I_{cov}}{N} - \gamma_{vac} V_{cov}\right] dt \\ & + S_{p_3}\int_{t_{vac}}^{T} \left[a_2 e^{-b_2 t} - \eta_3 S_{p_3} + \nu_3 F_s(t) + n\epsilon e^{-I_{cov}}\right] dt \Bigg] \end{aligned} \tag{7}$$

- Ordering cost (OC)

  Generally, the ordering cost is used time for ordering products by the retailer or many times but is considered constant. In this model, however, the ordering cost for a sustainable inventory system, accounting for the effect of learning (per unit time), is given by:

$$OC = \frac{K^L}{T} \tag{8}$$

- Inventory holding cost (HC) in the cycle $[0,\ \mathrm{T}]$

  The average inventory varies across different time intervals. The total holding cost for a sustainable inventory system, considering the effect of learning (per unit time), is given by:

$$HC = \frac{h^L}{T}\left[\int_0^{t_{cov}} I(t)dt + \int_{t_{cov}}^{t_{vac}} I(t)dt + \int_{t_{vac}}^{T} I(t)dt\right]. \tag{9}$$

- Purchasing cost (PC)

  To obtain more profit, it is important that the purchasing cost is low. To purchase basic items, the industry has to pay the purchasing cost, which allows it to hold the inventory for the future. The cost for a sustainable inventory system, accounting for the effect of learning (per unit time), is given by:

$$PC = \frac{P_c^L Q_0}{T} \tag{10}$$

- Disposal cost (DC) in the cycle [0, T]

Due to deterioration, the managers have to invest some funds to dispose of items. The management never prefers to use this fund, but it is very difficult to change it to a zero level. This is the reason that the preservation technology costs are used to reduce this disposal cost.

However, the disposal cost can be calculated as

$$DC = \frac{\theta_d c_d}{T}\left[\int_0^{t_{cov}} I(t)dt + \int_{t_{cov}}^{t_{vac}} I(t)dt + \int_{t_{vac}}^{T} I(t)dt\right]. \tag{11}$$

- Investment in preservation technology (IPT)

  To reduce the disposal cost of the whole system, management always invests in preservation technology. Thus, the cost can be calculated as $IPT = \frac{u}{T}$

- Investment in promotion (IP)

  To motivate more product sales, the common strategy of the management system is investing more in promotion, but it increases the total system cost. The optimum way is to use the dynamic investment in promotion purposes. The investment can be calculated as follows:

$$IP = \frac{1}{T}\left[\int_0^{t_{cov}} \frac{\alpha_1 f_{s_1}^2(t)}{2}dt + \int_{t_{cov}}^{t_{vac}} \frac{\alpha_2 f_{s_2}^2(t)}{2}dt + \int_{t_{vac}}^{T} \frac{\alpha_3 f_{s_3}^2(t)}{2}dt\right]. \tag{12}$$

Hence, the total profit per unit of time, Π is given by

$$\begin{aligned}
A_\Pi = \frac{1}{T}\Bigg[& S_{p_1}\left(\left(a_1 - \eta_1 S_{p_1} + n\epsilon e^{I_{cov}}\right) t_{cov} + b_1 \frac{t_{cov}^2}{2}\right) \\
&+ S_{p_2}\left(t_{vac} - t_{cov}\right)\left(D_0 - \eta_2 S_{p_2} + \frac{n\epsilon I_{cov}}{N} - \gamma_{vac} V_{cov}\right) \\
&+ S_{p_3}\left(\frac{a_2\left(e^{-b_2 t_{vac}} - e^{-b_2 T}\right)}{b_2} - \left(\eta_3 S_{p_3} - n\epsilon e^{-I_{cov}}\right)\left(T - t_{vac}\right)\right) \\
&+ \int_0^{t_{cov}} \left[\nu_1 S_{p_1} F_s(t) - \left(h^L + \theta_d c_d\right) I(t) - \frac{\alpha_1 f_{s_1}^2(t)}{2}\right] dt \\
&+ \int_{t_{cov}}^{t_{vac}} \left[\nu_2 S_{p_2} F_s(t) - \left(h^L + \theta_d c_d\right) I(t) - \frac{\alpha_2 f_{s_2}^2(t)}{2}\right] dt \\
&+ \int_{t_{vac}}^{T} \left[\nu_3 S_{p_3} F_s(t) - \left(h^L + \theta_d c_d\right) I(t) - \frac{\alpha_3 f_{s_3}^2(t)}{2}\right] dt \\
&- \left(K + P_c^L Q_0 + u\right)\Bigg]
\end{aligned} \tag{13}$$

Therefore, one can obtain the solution to the above optimization problem to get optimal dynamic investment rates, selling prices, replenishment time, and preservation technology investment, which maximize the retailer's total profit. The simplified form of the discussed problem is

$$\mathrm{Max}\,\Pi\left(S_{p_1}, S_{p_2}, S_{p_3}, u, f_{s_1}(t), f_{s_2}(t), f_{s_3}(t), T\right) \tag{14}$$

subject to

$$
\dot{F}_s(t) = \begin{cases} f_{s_1}(t) - \delta_1 F_s(t) & \text{if } 0 \le t < t_{cov} \\ f_{s_2}(t) - \delta_2 F_s(t) & \text{if } t_{cov} \le t \le t_{vac} \\ f_{s_3}(t) - \delta_3 F_s(t) & \text{if } t_{vac} \le t \end{cases}
$$

$$
\dot{I}(t) = \begin{cases} -a_1 - b_1 t + \eta_1 S_{p_1} - \nu_1 F_s(t) - n\epsilon e^{I_{cov}} - \theta_d I(t) & \text{if } 0 \le t \le t_{cov} \\ -D_0 + \eta_2 S_{p_2} - \nu_2 F_s(t) - \frac{n\epsilon I_{cov}}{N} + \gamma_{vac} V_{cov} - \theta_d I(t) & \text{if } t_{cov} \le t \le t_{vac} \\ -a_2 e^{-b_2 t} + \eta_3 S_{p_3} - \nu_3 F_s(t) - n\epsilon e^{-I_{cov}} - \theta_d I(t) & \text{if } t_{vac} \le t \le T \end{cases} \tag{15}
$$

$$
I(T) = 0,\ S(0) = S_0, u(t) \ge 0, \frac{\alpha_i f_{s_i}^2(t)}{2} + u(t) \le U,\ T \ge t_{vac}, \text{ and } S_{p_i} > P_c
$$

where $s_i(t)$ represents the control variables, $I(t)$ and $F_s(t)$ represent the state variables and S_{p_1}, S_{p_2}, S_{p_3}, u and T represent static variables. Generally, the initial replenishment quantity Q_0 is considered as a decision variable in the above optimization problem. This study considers the replenishment quantity Q_0 as a dependent decision variable. Once the decision variables S_{p_1}, S_{p_2}, S_{p_3}, u, T are obtained, the replenishment quantity Q_0 can be determined. For analytical tractability, several authors assumed that purchase cost is zero under a dynamic environment, this restriction is relaxed. Now, Pontryagin's maximum principle is employed to obtain control variables for the optimal dynamic investment rates (Mathur & Dwivedi [24]). Then, Ant Colony Optimization and Cuckoo Search Algorithm are applied to find ultimate decisions.

6. Proposed Methodology

The optimization is done through two important theorems, which are discussed in the following section.

6.1. Dynamic Investment Strategy

This subsection assumes that $S_{p_1},\ S_{p_2},\ S_{p_3},\ u,\ and\ T$ are given and reformulate the optimization problem Equation (15) as given below:

$$
\begin{aligned}
\Pi_1 = & \int_0^{t_{cov}} \left[\nu_1 S_{p_1} F_s(t) - (h + \theta_d c_d)\, I(t) - \frac{\alpha_1 f_{s_1}^2(t)}{2} \right] dt \\
& + \int_{t_{cov}}^{t_{vac}} \left[\nu_2 S_{p_2} F_s(t) - (h + \theta_d c_d)\, I(t) - \frac{\alpha_2 f_{s_2}^2(t)}{2} \right] dt \\
& + \int_{t_{vac}}^{T} \left[\nu_3 S_{p_3} F_s(t) - (h + \theta_d c_d)\, I(t) - \frac{\alpha_3 f_{s_3}^2(t)}{2} \right] dt
\end{aligned} \tag{16}
$$

subject to the same set of constraints. The following Theorem 6.1 gives the nature of the investment.

Theorem 6.1 The Hamiltonian function of the dynamic problem Equation (16) is concave with respect to the dynamical investment $f_{s_i}(t)$, whereas the dynamical investments depend on the adjoint variables of the Hamiltonian function.

proof It is assumed that all the functional forms in the problem Equation (16) are non-negative, continuous, and differentiable on $[0,\ T]$. The Hamiltonian function H for the above optimization problem Equation (16) is formulated as follows.

$$H=\begin{cases}\nu_1 S_{p_1}F_s(t)-\left(h^L+\theta_d c_d\right)I(t)-\frac{\alpha_1 f_{s_1}^2(t)}{2}+\lambda_1\left[f_{s_1}(t)-\delta_1 F_s(t)\right] \text{ if } 0\le t\le t_{cov}\\ \quad+\lambda_2\left[-a_1-b_1t+\eta_1 S_{p_1}-\nu_1 F_s(t)-n\epsilon e^{I_{cov}}-\theta_d I(t)\right]\\ \nu_2 S_{p_2}F_s(t)-\left(h^L+\theta_d c_d\right)I(t)-\frac{\alpha_2 f_{s_2}^2(t)}{2}+\lambda_1\left[f_{s_2}(t)-\delta_2 F_s(t)\right] \text{ if } t_{cov}\le t\le t_{vac}\\ \quad+\lambda_2\left[-D_0+\eta_2 S_{p_2}-\nu_2 F_s(t)-\frac{n\epsilon I_{cov}}{N}+\gamma_{vac}V_{cov}-\theta_d I(t)\right]\\ \nu_3 S_{p_3}F_s(t)-\left(h^L+\theta_d c_d\right)I(t)-\frac{\alpha_3 f_{s_3}^2(t)}{2}+\lambda_1\left[f_{s_3}(t)-\delta_3 F_s(t)\right] \text{ if } t_{vac}\le t\le T\\ \quad+\lambda_2\left[-ae^{-bt}+\eta_2 S_{p_2}-\nu_3 F_s(t)-n\epsilon e^{-I_{cov}}-\theta_d I(t)\right],\end{cases} \tag{17}$$

where λ_1 and λ_2 are adjoint variables associated with states equations $\dot{I}(t)$ and $\dot{S}(t)$, respectively. It can be found from Equation (17) that each component of the Hamiltonian function is composed of two parts: the first part is the integrant of objective functional, and the second part consists of the right-hand side of the state equations, which denotes the indirect contribution to the objective functional from the value of the changes $F_s(t)$ and $I(t)$. Note that, the initial condition $I(0)$ and the terminal conditions of $S(T)$ remain free, which introduce the following transversality conditions as $\lambda_1(T)=0$ and $\lambda_2(0)=0$ and the adjoint variables λ_1 and λ_2, must satisfy the following differential equations

$$\dot{\lambda}_1=-\frac{\partial H}{\partial S}=\begin{cases}\delta_1\lambda_1+\nu_1\lambda_2-S_{p_1}\nu_1 & \text{if } 0\le t\le t_{cov}\\ \delta_2\lambda_1+\nu_2\lambda_2-S_{p_2}\nu_2 & \text{if } t_{cov}\le t\le t_{vac}\\ \delta_3\lambda_1+\nu_3\lambda_2-S_{p_3}\nu_3 & \text{if } t_{cov}\le t\le t_{vac}\end{cases} \tag{18}$$

$$\dot{\lambda}_2=-\frac{\partial H}{\partial I}=\begin{cases}\left(h^L+\theta_d c_d\right)+\lambda_2\theta_d & \text{if } 0\le t\le t_{cov}\\ \left(h^L+\theta_d c_d\right)+\lambda_2\theta_d & \text{if } t_{cov}\le t\le t_{vac}\\ \left(h^L+\theta_d c_d\right)+\lambda_2\theta_d & \text{if } t_{vac}\le t\le T.\end{cases} \tag{19}$$

Solving the differential Equation (19), one yields

$$\lambda_2^{opt}(t)=\frac{h^L+\theta_d c_d}{\theta_d}\left(e^{\theta_d t}-1\right),0\le t\le T \tag{20}$$

Substituting Equation (20) into Equation (18) and solving, one can obtain

$$\lambda_1^{opt}(t)=\begin{cases}\nu_1\left(S_{p_1}+\frac{h^L+\theta_d c_d}{\theta_d}\right)\frac{1-e^{-\delta_1(t_{cov}-t)}}{\delta_1}+\frac{\nu_1(h^L+\theta_d c_d)e^{\theta_d t}}{\theta_d(\theta_d-\delta_1)}\left(1-e^{(\theta_d-\delta_1)(t_{cov}-t)}\right) & \text{, if } 0\le t<t_{cov}\\ \nu_2\left(S_{p_2}+\frac{h^L+\theta_d c_d}{\theta_d}\right)\frac{1-e^{-\delta_2(t_{vac}-t)}}{\delta_2}+\frac{\nu_2(h^L+\theta_d c_d)e^{\theta_d t}}{\theta_d(\theta_d-\delta_2)}\left(1-e^{(\theta_d-\delta_2)(t_{vac}-t)}\right) & \text{, if } t_{cov}\le t\le t_{vac}\\ \nu_3\left(S_{p_3}+\frac{h^L+\theta_d c_d}{\theta_d}\right)\frac{1-e^{-\delta_3(T-t)}}{\delta_3}+\frac{\nu_3(h^L+\theta_d c_d)e^{\theta_d t}}{\theta_d(\theta_d-\delta_3)}\left(1-e^{(\theta_d-\delta_3)(T-t)}\right) & \text{, if } t_{vac}\le t\le T\end{cases} \tag{21}$$

In the following analysis, suppose $\theta_d(u)\neq\delta_i$. For the case $\theta_d(u)=\delta_i$, all the results also hold when taking the limit in mathematics such that $\theta_d(u)=\delta_i$.

Corollary 1 : The service investment rate $f_{s_i}^{opt}\equiv 0$ for any $t\in[0,T]$ if the sales price $S_{p_i}\le S_{p_j}$

and $f_{s_i} > 0$ for any $t \in [0, T)$ if the sale price $S_{p_i} \geq S_{p_0}$, where

$$\begin{aligned}
S_{p_1} &= \frac{\gamma_1(c_d\theta_d + h^L)(1-e^{-\delta_1 t_{cov}})}{\theta_d(\delta_1-\theta_d)} - \lambda_1(t_{cov})e^{-\delta_1 t_{cov}}\frac{\delta_1}{\gamma_1(1-e^{-\delta_1 t_{cov}})} - c_d - \frac{h^L}{\theta_d}\\
S_{p_2} &= \frac{\gamma_2(c_d\theta_d + h^L)(1-e^{-\delta_2 T})}{\theta_d(\delta_2-\theta_d)} - \lambda_1(t_{vac})e^{-\delta_2 t_{vac}}\frac{\delta_2}{\gamma_2(1-e^{-\delta_2 t_{vac}})} - c_d - \frac{h^L}{\theta_d}\\
S_{p_3} &= -\frac{(c_d\theta_d + h^L)\delta_3}{\theta_d(\theta_d-\delta_3)} - c_d - \frac{h^L}{\theta_d}
\end{aligned} \tag{22}$$

and

$$S_{p_0} = \frac{h^L + \theta_d c_d}{\theta_d}\left(e^{\theta_d T} - 1\right) \tag{23}$$

The optimal control policies $\left(S_{p_1}^{opt},\ S_{p_2}^{opt},S_{p_3}^{opt},u^{opt},T^{opt}\right)$ and state trajectories $\left(F_s^{opt}(t), I^{opt}(t)\right)$ have to maximize the Hamiltonian Function at all the points, that is,

$$\begin{aligned}
&H^{opt}\left(F_s^{opt}(t), I^{opt}(t), S_{p_1}^{opt},\ S_{p_2}^{opt},S_{p_3}^{opt},u^{opt},T^{opt},t\right)\\
&\geq H^{opt}\left(F_s^{opt}(t), I^{opt}(t), S_{p_1},\ S_{p_2},S_{p_3},u,T,t\right)\ \forall\, t
\end{aligned} \tag{24}$$

It is to be noted that:

$$\frac{\partial\lambda_1(t)}{\partial t} = -\nu_3\left(S_{p_3} + c_d + h^L/\theta_d\right)e^{-\delta_3(T-t)} + \frac{\nu_3\left(h^L + \theta_d c_d\right)}{\theta_d\left(\theta_d - \delta_3\right)}\left(e^{\theta_d t}\theta_d - \delta_3 e^{(\theta_d-\delta_3)T+\delta_3 t}\right) < 0, \forall t \in [t_{vac}, T] \tag{25}$$

i.e. $\lambda_1(t)$ decreases in the final stage of the product lifespan. To maximize the Hamiltonian H with respect to $f_{s_i}(t)$, the first order condition is obtained by solving $\frac{\partial H}{d_{f_{s_i}} d_{f_{s_i}}(t)} = 0$. On simplification,

$$f_{s_1}(t) = \frac{\lambda_1(t)}{2\alpha_1}, 0 \leq t \leq t_{cov} \tag{26}$$

$$f_{s_2}(t) = \frac{\lambda_1(t)}{2\alpha_2}, t_{cov} \leq t \leq t_{vac} \tag{27}$$

$$f_{s_3}(t) = \frac{\lambda_1(t)}{2\alpha_3}, t_{vac} \leq t \leq T \tag{28}$$

Moreover, $\frac{\partial^2 H}{\partial f_{s_i}(t)^2} = -2\alpha_i < 0$ and $\frac{\partial^2 H}{\partial f_{s_i}(t) d f_{s_j}(t)} = 0\ \forall\ i \neq j$, therefore H is concave with respect to $f_{s_i}(t)$ and the optimal path of $f_{s_i}(t)$ depends on $\lambda_1(t)$.

Finally, Theorem 6.2 can be proved for the optimum strategy for the retailer.

Theorem 6.2 The dynamical investment rate of the retailer decreases continuously if always the retailer's uniform-price, production-sensitive, and decay rates are identical at each stage.

proof At the beginning of the declining stage, $\lambda_1\left(t_{vac}\right) = \frac{\nu_3}{\delta_3}\left(S_{p_3} + c_d +\ h/\theta_d\right)\left(1 - e^{-\delta_3(T+t)}\right) + \frac{\nu_3\left(h^L+\theta_d c_d\right)}{\theta_d(\theta_d-\delta_3)}\left(e^{\theta_d t} - e^{(t-\delta_3)T+\delta_3 t}\right)$, thus the solution exists if

$$S_{p_3} > \frac{\left(h^L + c_d\theta_d\right)\left(e^{\delta_3 t_{vac}}\left(\theta_d - \delta_3\left(1 - e^{T\theta_d}\right)\right) - e^{\delta_3 T}\left(\theta_d - \delta_3\left(1 - e^{\theta_d t_{vac}}\right)\right)\right)}{\left(e^{\delta_3 T} - e^{\delta_3 t_{vac}}\right)\theta_d\left(\theta_d - \delta_3\right)}\left(=\Gamma_1,\ \text{say}\ \right). \tag{29}$$

Substituting $S_{p_1} = S_{p_2} = S_{p_3}, \nu_1 = \nu_2 = \nu_3$, and $\delta_1 = \delta_2 = \delta_3$, one can obtain

$$\frac{\partial \lambda_1(t)}{\partial t_i} = -\nu_3 \left(S_{p_3} + c_d + h/\theta_d\right) e^{-\delta_3(T-t_i)} + \frac{\nu_3 \left(h + \theta_d c_d\right)}{\theta_d \left(\theta_d - \delta_3\right)} \left(e^{\theta_d t_i}\theta_d - \delta_3 e^{(\theta_d - \delta_3)T + \delta_3 t_i}\right) < 0,$$

where $t_1 \in [0, t_{cov}], t_2 \in [t_{cov}, t_{vac}]$, and $t_3 \in [t_{vac}, T]$. Therefore, the investment rate of the retailer decreases continuously if the retailer considers uniform price, promotion sensitivity, and decay rates identical at each stage.

Substituting Equations (26), (27) and (28) in Equation (2), one can find that the optimal path representing the promotional level as follows:

$$F_s^{opt}(t) = \begin{cases} \dfrac{\nu_1(S_{p_1} + \omega_1)\left(\frac{1-e^{-\delta_1 t}}{\delta_1} - e^{-\delta_1 t_{cov}}\omega_2\right)}{2\alpha_1\delta_1} + \dfrac{\nu_1\omega_1\left(\frac{e^{\theta_d t} - e^{-\delta_1 t}}{\theta_d + \delta_1} - e^{(\theta_d - \delta_1)t_{cov}}\omega_2\right)}{2\alpha_1\left(\theta_d - \delta_1\right)} \\ + \frac{\lambda_1(t_{cov})e^{-\delta_1 t_{cov}}\omega_2}{2\alpha_1} + F_{s_0} e^{-\delta_1 t} \\ \text{if } 0 \le t \le t_{cov} \\ \\ \dfrac{\nu_2(S_{p_2} + \omega_1)\left(\frac{1-e^{\delta_2(t_{cov}-t)}}{\delta_2} - e^{-\delta_2 t_{vac}}\omega_9\right)}{2\alpha_2\delta_2} + \dfrac{\nu_2\omega_1\left(\frac{e^{\theta_d t} - e^{(\theta_d+\delta_2)t_{cov}}e^{-\delta_2 t}}{\theta_d + \delta_2} - e^{(\theta_d - \delta_2)t_{vac}}\omega_9\right)}{2\alpha_2\left(\theta_d - \delta_2\right)} \\ + \frac{\lambda_1(t_{vac})e^{-\delta_2 t_{vac}}\omega_9}{2\alpha_2} + F_s(t_{cov})e^{-\delta_2(t-t_{cov})} \\ \text{if } t_{cov} \le t \le t_{vac} \\ \\ \dfrac{\nu_3(S_{p_3} + \omega_1)\left(\frac{1-e^{\delta_3(t_{vac}-t)}}{\delta_3} - e^{-\delta_3 T}\omega_{10}\right)}{2\alpha_3\delta_3} + \dfrac{\nu_3\omega_1\left(\frac{e^{\theta_d t} - e^{(\theta_d+\delta_3)t_{vac}}e^{-\delta_3 t}}{\theta_d + \delta_3} - e^{(\theta_d - \delta_3)T}\omega_{10}\right)}{2\alpha_3\left(\theta_d - \delta_3\right)} \\ + F_s(t_{vac})e^{-\delta_3(t-t_{vac})} \\ \text{if } t_{vac} \le t < T \end{cases} \tag{30}$$

[See all values in Appendix A]. Finally, substituting Equation (30) in Equation (6), the optimal path representing the inventory level in the entire replenishment cycle is obtained by the following equations.

$$
I^{opt}(t)=\begin{cases}
Q_0e^{-\theta_d t}-\dfrac{(a_1-\eta_1 S_{p_1})\left(1-e^{-\theta_d t}\right)}{\theta_d}-b_1\left(\dfrac{t}{\theta_d}-\dfrac{1-e^{-\theta_d t}}{\theta_d^2}\right)+\nu_1F_{s_0}\omega_3 \\
-\frac{\nu_1}{2\alpha_1}\left[\frac{\lambda_1(t_{cov})e^{-\delta_1 t_{cov}}\omega_{11}}{2\delta_1}+\frac{\nu_1}{\delta_1}\left(S_{p_1}+\omega_1\right)\left(\frac{1}{\delta_1}\left(\frac{1-e^{-\theta_d t}}{\theta_d}-\omega_3\right)-\frac{e^{-\delta_1 t_{cov}}\omega_{11}}{2\delta_1}\right)\right. \\
\left.+\frac{\nu_1\omega_1}{(\theta_d-\delta_1)}\left(\frac{1}{\theta_d+\delta_1}\left(\frac{e^{\theta_d t}-e^{-\theta_d t}}{2\theta_d}-\omega_3\right)-\frac{e^{(\theta_d-\delta_1)t_{cov}}\omega_{11}}{2\delta_1}\right)\right] \\
\text{if } 0\le t\le t_{cov} \\
\\
I\left(t_{cov}\right)e^{-\theta_d(t-t_{cov})}-\dfrac{(D_0-\eta_2 S_{p_2})\left(1-e^{-\theta_d(t-t_{cov})}\right)}{\theta_d}-\nu_2F_s\left(t_{cov}\right)e^{\delta_2 t_{cov}}\omega_5 \\
-\frac{\nu_2}{2\alpha_2}\left[\frac{\nu_2}{\delta_2}\left(S_{p_2}+\omega_1\right)\left(\frac{1}{\delta_2}\left(\frac{1-e^{\theta_d(t_{cov}-t)}}{\theta_d}-e^{\delta_2 t_{cov}}\omega_5\right)-\frac{e^{-\delta_2 t_{vac}}\omega_{12}}{2\delta_2}\right)\right. \\
\left.+\frac{\nu_2\omega_1}{(\theta_d-\delta_2)}\left(\frac{e^{\theta_d t}-e^{2\theta_d t_{cov}}e^{-\theta_d t}}{2\theta_d(\theta_d+\delta_2)}-\frac{e^{(\theta_d+\delta_2)t_{cov}}\omega_5}{\theta_d+\delta_2}-\frac{e^{(\theta_d-\delta_2)t_{vac}}\omega_{12}}{2\delta_2}\right)+\frac{\lambda_1 t_{vac}e^{-\delta_2 t_{vac}}\omega_{12}}{2\delta_2}\right] \\
\text{if } t_{cov}\le t\le t_{vac} \\
\\
I\left(t_{vac}\right)e^{-\theta_d(t-t_{vac})}+\dfrac{\eta_3 S_{p_3}\left(1-e^{-\theta_d(t-t_{vac})}\right)}{\theta_d}-\dfrac{a_3\left(e^{-b_3 t}-e^{(\theta_d-b_3)t_{vac}}e^{-\theta_d t}\right)}{\theta_d-b_3} \\
-\nu_3F_s\left(t_{vac}\right)+e^{\delta_3 t_{vac}}\omega_7-\frac{\nu_3}{2\alpha_3}\left[\frac{\nu_3\left(S_{p_3}+\omega_1\right)}{\delta_3}\left(\frac{1-e^{\theta_d(t_{vac}-t)}}{\theta_d\delta_3}-\frac{e^{\delta_3 t_{vac}}\omega_7}{\delta_3}-\frac{e^{-\delta_3 T}\omega_{13}}{2\delta_3}\right)\right. \\
\left.+\frac{\nu_3\omega_1}{(\theta_d-\delta_3)}\left(\frac{e^{\theta_d t}-e^{2\theta_d t_{vac}}e^{-\theta_d t}}{2\theta_d(\theta_d+\delta_3)}-\frac{e^{(\theta_d+\delta_3)t_{vac}}\omega_7}{(\theta_d+\delta_3)}-\frac{e^{(\theta_d-\delta_3)T}\omega_{13}}{2\delta_3}\right)+\frac{\lambda_1(\mu_3)e^{-\delta_3\mu_3}\omega_{13}}{2\delta_3}\right] \\
\text{if } t_{vac}\le t\le T
\end{cases} \tag{31}
$$

Using Equation (31), the replenishment quantity Q_0 can be obtained by using the condition $I(T)=0$. Therefore, one needs to find the following optimization problem to get optimal decision

$$
\operatorname{Max}\Pi\left(S_{p_1},\ S_{p_2}\ ,S_{p_3}\ ,u\ ,T\right) \tag{32}
$$

subject to

$$
\begin{cases}
a_1-\eta_1 S_{p_1}+\nu_1 S(0)\ge 0 \\
D_0-\eta_2 S_{p_2}+\nu_2 S\left(t_{cov}\right)\ge 0 \\
a_2e^{-b_2 t_{vac}}-\eta_2 S_{p_2}+\nu_3 S\left(t_{vac}\right)\ge 0 \\
I(T)=0,\ S(0)=S_0,\ S_{p_1}>c,\ S_{p_2}>c, \\
S_{p_3}>\operatorname{Max}\left\{c,\ \Gamma_1\right\},\ T\ge t_{vac}, 0\le f_{s_i}\le\sqrt{\frac{2}{\alpha_i}(U-u(t))}
\end{cases} \tag{33}
$$

where the constraints set Equation (33) ensure non-negative demand throughout the product lifespan.

Where the constraints set (33) ensure non-negative demand throughout the product lifespan. Due to the complexity and nonlinear characteristics of both the objective function and constraints, the analytical solution is difficult to find. Therefore, the Ants Colony Algorithm and Cuckoo Search Algorithm for solving the above optimization problem (Yu et al. [35]) is given in the following section.

6.2. Ant Colony Optimization

Ant colony optimization offers an innovative and efficient approach to simplifying complex measures while maintaining validity and reliability (Karl et al. [14]). Ants are a kind of social insects. After a lot of research, it has been found that ants can transmit information through a substance called exogenous hormone in the course of their movement. They leave this substance on the path. At the same time, ants can perceive the existence and intensity of this substance in the process of movement and use this substance to guide their movement direction. Therefore, the more ants walk along a certain path, the higher the intensity of exogenous hormones left behind, and the greater the probability that the latter will choose the path. It is through the exchange of information among ants that the purpose of searching for food is achieved Yu et al. [34]. Algorithm 1 represents the pseudocode of the applied Ant Colony Optimization.

6.3. Cuckoo Search Algorithm

The Cuckoo Search is a stochastic search evolutionary algorithm. The Cuckoo Search algorithm was first proposed by Srinivasan et al. [28]. It is inspired by cuckoo breeding behaviour. The cuckoo lays eggs in the nests of other species with the intention that the host will look after the cuckoos eggs as if they were the hosts own. Each egg in a nest signifies a solution. A cuckoos egg signifies a new solution. The objective is to use new and possibly superior solutions to replace an inferior solution in the nests. The probability that the host identifies the cuckoos eggs and abandons the nests are denoted by P_a. To generate a new solution, the cuckoo search approach exhibits Levy flight distribution.

Cuckoo Search Algorithm

The following rules are applied in cuckoo search algorithm (Srinivasan et al. [28]):

1. Each cuckoo lays one egg at a time, and a nest is chosen randomly for dumping it.
2. The best nests are the ones with high-quality eggs that would survive to carry over to the next generations.
3. There is a fixed number of host nests, and if the laid egg is discovered with a probability p_a ($p_a \in [0, 1]$), the host bird may either throw it away or leave the nest to construct a completely new nest.

Considering L_i and U_i as the lower and upper bounds for decision variable x_i, the initial solutions for the cuckoo search algorithm can be determined by:

$$x_i = L_i + \text{rand}(U_i - L_i). \quad (22)$$

Moreover, the new solution for cuckoo i, $x_i(t+1)$, is generated by performing Levy flight as shown in:

$$x_i(t+1) = x_i(t) + \alpha \odot \text{Levy}(\lambda), \quad (23)$$

where $\alpha > 0$ is the step size and $\odot$ represents entry-wise multiplication. The Levy flight provides a random walk in which the random steps are drawn from a Levy distribution with infinite variance and mean:

$$\text{Levy} \sim u = t^{-\lambda}, \quad 1 \leq \lambda \leq 3. \quad (24)$$

Algorithm 2, represents the pseudocode of the applied cuckoo search algorithm.

7. Numerical Validations

This section explores the characteristics of the proposed model and the solution procedure with two numerical examples. The sensitivity analysis performs a crucial role in finding out the changes of key parameters over the optimal solution and convergence rate of the Algorithm. The following parametric values are considered to illustrate the model as given in Table 1.

Table 1. Input data of numerical example

$n = 0.2$	$\epsilon = 0.05$	$I_{cov} = 0.1$
$a_1 = 40$	$a_2 = 100$	$b_1 = 5$
$b_2 = 0.086$	$\eta_1 = 0.1$	$\eta_2 = 0.5$
$\eta_3 = 0.5$	$\gamma_{vac} = 0.9$	$h = 1.8$
$\theta_d = 1.3$	$\alpha_1 = 0.7$	$\alpha_2 = 0.8$
$\alpha_3 = 0.9$	$\nu_1 = 1.5$	$\nu_2 = 1.6$
$\nu_3 = 1.7$	$K = 2.0$	$P_c = 2.1$
$Q_0 = 2.2$	$N = 150$	$c_d = 5$
$D_0 = 50$	$V_{cov} = 5000$	$\lambda = 0.2$
$h = 1.5$	$h_l = 1.2$	$l = 1$
$K = 20$	$K_l = 18$	$P_c = 25$
$P_1 = 20$	$c_d = 5$	$c_{dl} = 3$
$\xi = 0.03$		

The optimal investment rate in promotion and the corresponding promotional level are governed by the following relations:

$$\begin{aligned} f_1(t) &= 70.8628 - 31.4823e^{0.202554t} + 14.2673e^{0.25t} && \text{if } 0 \le t \le t_{cov} \\ f_2(t) &= 69.1923 - 15.3285e^{0.202554t} + 1.24483e^{0.3t} && \text{if } t_{cov} \le t \le t_{vac} \\ f_3(t) &= 44.6242 - 11.4964e^{0.202554t} + 1.71617e^{0.3t} && \text{if } t_{vac} \le t \le T \end{aligned} \tag{34}$$

$$F(t) = \begin{cases} 283.451 - 192.42e^{-0.25t} - 69.5652e^{0.202554t} + 28.5342e^{0.25t} & \text{if } 0 \le t \le t_{cov} \\ 230.641 - 116.646e^{-0.3t} - 30.5011e^{0.202554t} + 2.07465e^{0.3t} & \text{if } t_{cov} \le t \le t_{vac} \\ 148.747 + 265.706e^{-0.3t} - 22.8758e^{0.202554t} + 2.86022e^{0.3t} & \text{if } t_{vac} \le t \le T. \end{cases} \tag{35}$$

For the experiment of Ant Colony Optimization (ACO). We consider 50 ants set up. The pheromone renewal mechanism was the optimal one. The Volatilization Coefficient of pheromone was 0.2. α = 2 and β = 5 are parameters that tune the balance between exploiting pheromone trails (α) and exploring based on heuristic information (β) in the ACO algorithm with initial iteration = 100. and for the experiment of Cuckoo Search Algorithm. We consider the number of cuckoos to be 40 and the number of iterations to be 100.

By applying Ant Colony Optimization (ACO) and Cuckoo Search Algorithm (CSA), the optimal decision of the proposed model is obtained as shown in Table 2 and Figure 8

When comparing Ant Colony Optimization (ACO) and Cuckoo Search Algorithm (CSA) for optimizing pharmaceutical inventory profit, it's evident that ACO performs better, generating a profit of 248,845.36 compared to CSA's 246,708.59. While there are slight differences in selling prices and cycle times, ACO boasts a slightly lower cycle time of 7.00 versus CSA's 7.01. Importantly, the significance of

Table 2. Optimal Results of Proposed Algorithms using Dataset from Table 1

Algorithm	$Profit(A_\pi^*)$	S_{p1}^*	S_{p2}^*	S_{p3}^*	T^*	E_1^*	u^*
ACO	340861.89	95.60	149.54	103.04	5.03	1.43	91.94
CSA	337478.27	90.07	149.67	98.96	5.07	1.52	70.23

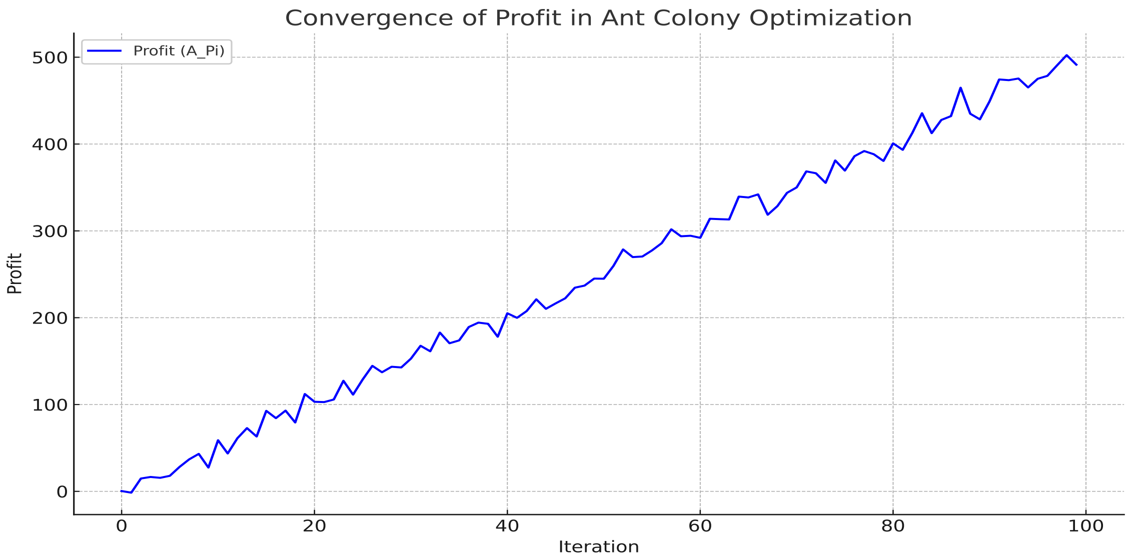

Figure 8. Convergence Graphs through Ant Colony Optimization for Inventory Profit from Table (1).

preservation technology is highlighted not only for maximizing profitability but also for reducing health risks associated with poorly preserved medications. This suggests that further investment in preservation technology could enhance the effectiveness of both algorithms. In summary, ACO emerges as the preferred choice for profit maximization, emphasizing the importance of selecting the right algorithm for inventory optimization strategies.

7.1. Statistical Analysis and Comparison of both Algorithm

The statistical test ANOVA was conducted to test the hypothesis in the research design. Based on the results presented in Tables 3 the comparison between Ant Colony Optimization and Cuckoo Search Algorithm reveals that not only was the Ant Colony Algorithm faster, but it also outperformed the Cuckoo Search Algorithm in terms of every solution and the best solution.

8. Sensitivity Analysis

The sensitivity analysis is conducted on the optimal policy concerning the price-sensitive parameters η_1, η_2, and η_3, as listed in Table 4. When the value of one parameter varies, all others remain constant as shown in Figures 9 and 10.

The price sensitivity of demand and total profit are closely intertwined. It is intuitive to understand that as the price sensitivity of products increases, profit decreases, impacting the investment opportunities for the retailer, either in reducing deterioration or promoting the product. The results presented in Table 4 corroborate this common intuition. However, the characteristics of the optimal solution undergo signif-

Table 3. ANOVA, Model Summary, Means, and Pairwise Comparisons for Factor (ACO Profit, CSA Profit)

Factor Information	**Factor (ACO Profit, CSA Profit)**				
Factor	**Levels**	**Values**		**2 (ACO Profit, CSA Profit)**	
Analysis of Variance					
Source	DF	Adj SS	Adj MS	F-Value	P-Value
Factor	1	1.47023E+11	1.47023E+11	1.32	0.257
Error	46	5.14130E+12	1.11767E+11		
Total	47	5.28832E+12			
Model Summary					
S	R-sq	R-sq(adj)	R-sq(pred)		
334316	2.78%	0.67%	0.00%		
Means					
Factor	**N**	**Mean**	**StDev**	**95% CI**	
ACO Profit	24	366266	471540	(228902, 503630)	
CSA Profit	24	255578	34412	(118213, 392942)	
Pooled StDev		334316			
Grouping Information		**Using the Tukey Method and 95% Confidence**			
Factor	N	Mean	Grouping		
ACO Profit	24	366266	A		
CSA Profit	24	255578	A		
Means that do not share a letter are significantly different.					
Tukey Simultaneous Tests for Differences of Means					
Difference of Levels	Difference of Means	SE of Difference	95% CI	T-Value	Adjusted P-Value
CSA Profit - ACO Profit	-110688	96509	(-304950, 83574)	-1.15	0.257
Individual confidence level = 95.00%					
Grouping Information		**Using the Fisher LSD Method and 95% Confidence**			
Factor	N	Mean	Grouping		
ACO Profit	24	366266	A		
CSA Profit	24	255578	A		
Fisher Individual Tests for Differences of Means					
Difference of Levels	Difference of Means	SE of Difference	95% CI	T-Value	Adjusted P-Value
CSA Profit - ACO Profit	-110688	96509	(-304950, 83574)	-1.15	0.257
Simultaneous confidence level = 95.00%					

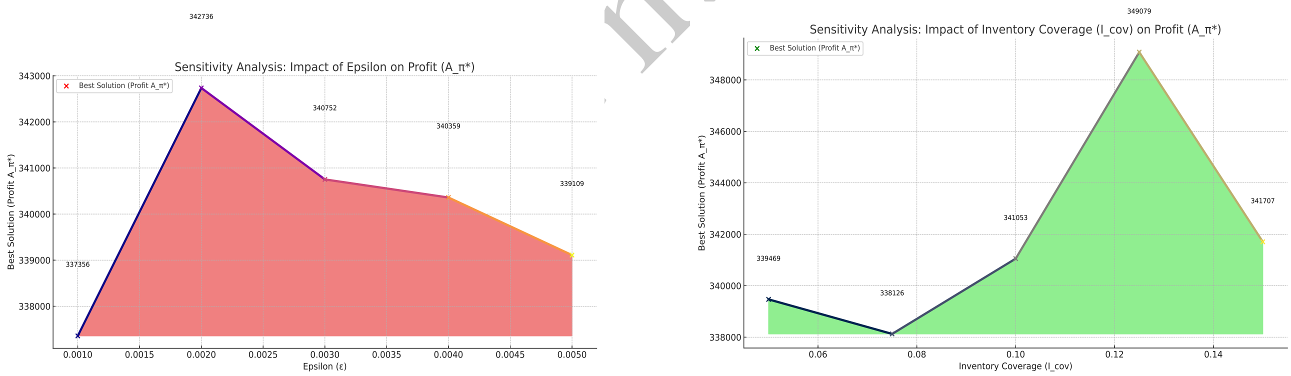

Figure 9. Sensitivity Analysis Graph for Parameters ϵ and I_{cov}.

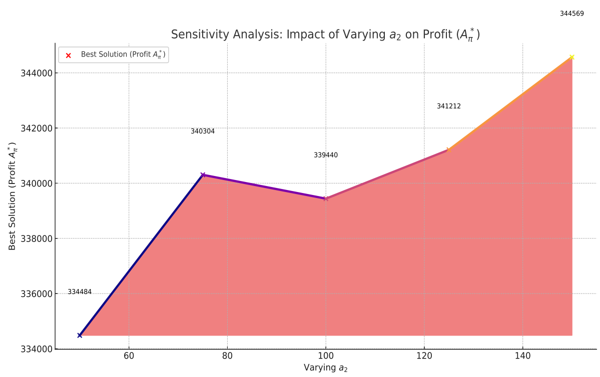

Figure 10. Sensitivity Analysis Graph for Parameter a_2.

icant changes. While increases in η_1 and η_2 lead to an increase in the retailer's replenishment cycle time, the opposite effect occurs for η_3. Therefore, the retailer should reduce the length of the replenishment cycle to maximize profit, especially in cases of high price sensitivity in the final stage. Importantly, optimal investment decisions and profit are influenced by the price sensitivity of products during the maturity stage, with the price sensitivity in the final stage exerting a greater influence than in the initial stage on determining the optimal preservation technology investment.

From Table 5, it is evident that the decision to apply market skimming, penetration, or a mixed pricing strategy should be based on the price sensitivity at different stages. Consequently, the retailer should monitor price sensitivity and decide on the pricing strategy accordingly to achieve optimal profit. Promotional investments are typically utilized to shift consumer focus away from price and towards product benefits. As consumer sensitivity to promotions increases, such investments become more profitable. Analysis of Table 4 reveals that as $\nu_i, i = 1, 2, 3$ increases, promotional investments also increase. With higher demand, the retailer can charge more, providing opportunities not only for increased promotional spending but also for investments in preservation technology. Implicitly, the profitability and replenishment cycle time of the retailer are significantly influenced by promotional sensitivity in the maturity stage. Additionally, as Q_0 increases, the retailer's profit increases, indicating the substantial impact of the retailer's reputation on profitability. Thus, investments in promotion not only enhance current cycle profits but also create opportunities for higher profits in the future.

From Table 4 and 5, it is evident that as the decay rate increases, so does the profit of the retailer. In other words, if the investment in promotion fails to effectively attract consumers, it could lead to a catastrophe in the retailer's profitability. Additionally, the retailer's profit is highly sensitive to the decay rate, particularly in the initial and growth stages compared to the decline stage. In situations where the decay rate is high, the retailer should consider reducing both the length of the replenishment cycle and the price of the product to maximize the impact of promotion. Moreover, a higher decay rate reduces opportunities for investment in preservation technology.

In comparison to existing literature on preservation technology, this study assumes that the retailer can reduce the rate of deterioration up to a certain threshold limit (refer to Tables 4 and 5). For instance, perishable items like milk or vegetables degrade over time, necessitating various preservation measures that require adjustable investments. The findings suggest that determining the threshold value, which signifies the extent to which deterioration can be mitigated, is crucial for guiding preservation technology investments. The results demonstrate that the retailer's profit is highly sensitive to this threshold value. A smaller deterioration rate necessitates a lesser investment in preservation technology by the retailer. The pricing of products is also influenced by the threshold value. It is logical to observe that as ξ increases, the retailer's profit rises while investment in preservation technology decreases. This observation aligns with the results of the sensitivity analysis conducted in the model. Therefore, it is essential for the retailer to assess the nature of the product and make preservation technology investment decisions accordingly.

From Tables 4 and 5, it is evident that the impact of holding cost and disposal cost on the retailer's profitability is not as significant as the unit cost of the product. It is natural to observe that as P_c and h increase, the retailer's profit decreases. This deduction is supported by the results presented above. However, the rate of decrement is highly sensitive to P_c, and as P_c increases, the rate of investment in promotion and preservation technology decreases. Furthermore, the unit cost of the product has a considerable impact on the price at the maturity stage. The replenishment time decreases across all cost parameters due to the higher deterioration rate. It is evident that as the promotion investment cost coefficients α_i $(i = 1, 2, 3)$ increase, profit, replenishment cycle time, and product prices decrease. If the promotion cost coefficients are relatively large, the higher promotional costs compel the retailer to invest less in preservation technology. In order to capitalize on the expanding demand, the retailer is motivated to set lower prices. Consequently, inefficient promotion activities always negatively impact the retailer's profitability. Notably, under the infection awareness-price-and trapezoidal-type demand, the retailer's effectiveness in promotion during the initial and growth stages is a critical factor in achieving higher profits.

The tables present optimal decision variable values and total profit for varying parameters in a model. Table 6 outlines the results for different cumulative frequency values ('l'). As 'l' increases, the optimal values of decision variables $S^*_{p_1}$, $S^*_{p_2}$, $S^*_{p_3}$, u^*, T^*, and E_R show notable variations, with the total profit fluctuating accordingly. For instance, with 'l' set to 2, the model achieves the highest total profit of 344,272.54. In contrast, Table 7 explores the impact of the learning rate (λ) on the decision variables and total profit. Different λ values yield varying results, with the best profit of 347,304.98 achieved when λ is 0.2. Overall, changes in both 'l' and λ affect the decision variables and profitability, illustrating the sensitivity of the model to these parameters. The application of fuzzy learning, as evidenced by the tables, highlights its significance in optimizing decision variables and maximizing total profit. By adjusting parameters such as cumulative frequency ('l') and learning rate (λ). This approach effectively captures the impact of different parameter settings, demonstrating its capability to handle uncertainties and improve model performance.

9. Managerial Insights from Industrial Cases: Pharmaceutical Products

The examination of industrial cases within the pharmaceutical sector offers crucial insights into managing deteriorated products. Due to the unique challenges associated with pharmaceutical items, where deterioration rates can fluctuate significantly, a static investment strategy may prove economically unfeasible. Adopting a dynamic investment approach, which adjusts based on the condition and lifespan of the products, tends to be more advantageous for pharmaceutical managers.

9.1. Key Insights

- Managing the rate of deterioration in pharmaceuticals is critical. By optimizing factors such as storage conditions and packaging, managers can extend the shelf life of drugs. This approach not only ensures prolonged effectiveness but also enhances economic benefits by minimizing wastage and maximizing of product utility
- Advanced preservation methods, including temperature control and specialized packaging, can greatly extend the lifespan of pharmaceutical products. However, their effectiveness diminishes once a product's deterioration crosses a critical threshold, such as when the active ingredients start to degrade rapidly.

Table 4. Sensitivity Analysis of Decision Variables for Overall Inventory Cost (A_π)

Inventory Parameters	Variability in Values	Selling Price $S^*_{p_1}$	Selling Price $S^*_{p_2}$	Selling Price $S^*_{p_3}$	Cycle T^* Length u^*	Preservation Technology	Environmental emission cost cost E_R	Best solution (Profit A^*_Π)
				ACO				
	0.10	99.80	148.20	113.10	05.00	70.48	2.04	330114.00
	0.15	99.04	149.70	111.10	05.01	83.27	3.05	339030.40
Varying n	0.20	98.89	149.80	109.00	05.01	87.77	3.22	343970.20
	0.25	98.05	149.96	108.40	05.02	97.12	3.38	342283.83
	0.30	97.42	149.98	095.80	05.04	99.37	3.66	347651.58
	0.001	98.71	149.70	112.90	05.08	96.82	0.54	337356.12
	0.002	91.92	149.80	104.70	05.07	90.89	1.48	342736.12
Varying ϵ	0.003	91.72	149.80	104.50	05.06	90.85	1.59	340751.86
	0.004	90.59	149.90	100.50	05.04	90.85	1.64	340359.22
	0.005	80.01	149.98	095.20	05.04	88.97	1.67	339109.38
	0.050	98.56	148.80	104.80	05.01	73.47	1.06	339469.32
	0.075	97.89	149.70	110.20	05.02	97.05	1.48	338126.00
Varying I_{cov}	0.100	96.69	149.75	112.60	05.03	82.65	3.34	341052.96
	0.125	095.72	148.50	122.90	05.05	95.91	1.03	349078.77
	0.150	90.65	147.80	130.90	05.07	75.37	3.32	341706.86
	20	97.72	149.30	096.2	05.01	71.78	0.73	338445.43
	30	95.42	149.90	097.2	05.03	76.84	1.25	339245.23
Varying a_1	40	91.64	148.40	101.9	05.03	82.90	2.04	339874.05
	50	91.54	148.30	110.1	05.07	91.29	3.05	339984.94
	60	90.14	148.10	111.2	05.08	94.39	3.25	341145.38
	050	94.10	149.6	103.6	99.25	05.08	1.11	334484.23
	075	99.86	149.8	108.3	85.88	05.03	3.71	340303.95
Varying a_2	100	98.74	147.9	106.1	95.60	05.00	0.38	339440.25
	125	96.22	149.9	092.3	73.38	05.04	2.70	341211.88
	150	92.12	149.5	116.0	80.96	05.03	0.23	344569.16
	2.50	90.89	148.80	113.70	91.11	05.01	0.41	340391.24
	3.75	97.96	147.85	113.30	89.48	05.03	1.84	339029.65
Varying b_1	5.00	98.74	147.60	106.10	95.60	05.00	0.38	339440.25
	6.25	99.37	146.50	105.20	83.36	05.01	3.66	342763.41
	7.50	99.39	149.70	098.9	99.91	05.04	1.79	340213.14
	0.0430	99.85	148.9	111.3	99.67	05.05	1.76	335236.72
	0.0645	99.57	148.7	096.0	99.29	05.00	2.81	336227.98
Varying b_2	0.0860	98.74	147.9	106.1	95.60	05.00	0.38	339440.25
	0.1075	95.02	148.7	104.6	91.33	05.02	1.56	335159.60
	0.1290	94.74	149.7	110.9	84.65	05.04	2.73	335712.99

Table 5. Sensitivity Analysis of Decision Variables for Overall Inventory Cost (A_π)

Inventory Parameters	Variability in Values	Selling Price $S^*_{p_1}$	Selling Price $S^*_{p_2}$	Selling Price $S^*_{p_3}$	Cycle T^* Length u^*	Preservation Technology	Environmental emission cost cost E_R	Best solution (Profit A^*_Π)
				ACO				
	0.050	94.14	149.2	098.6	95.80	05.05	03.04	334266.72
	0.075	94.19	148.6	113.6	99.23	05.00	02.50	336892.06
Varying η_1	0.100	98.74	147.9	106.1	95.60	05.00	00.38	339440.25
	0.125	96.62	147.7	118.1	00.27	05.02	00.13	334230.01
	0.150	93.89	147.9	098.5	97.25	05.00	00.06	334211.71
	0.250	96.38	149.8	118.2	74.45	05.06	1.881	337994.41
	0.375	93.33	149.0	113.3	84.86	05.03	3.099	338993.08
Varying η_2	0.500	98.74	147.9	106.1	95.60	05.00	0.384	339440.25
	0.625	91.62	148.8	107.5	73.69	05.03	0.653	340808.22
	0.750	94.22	149.0	113.9	87.37	05.00	3.172	345558.59
	0.250	98.29	147.0	147.0	94.47	05.01	2.140	339978.79
	0.375	96.46	148.2	115.9	96.31	05.04	3.074	339192.29
Varying η_3	0.500	98.74	147.9	106.1	95.60	05.00	0.384	339440.25
	0.625	93.82	149.7	106.2	90.58	05.07	0.181	337491.53
	0.750	94.41	149.1	101.3	87.61	05.03	3.786	337409.14
	0.450	90.36	149.0	112.7	83.12	05.02	1.691	173929.78
	0.675	90.80	149.2	112.9	86.42	05.00	1.279	258636.78
Varying γ_{vac}	0.900	98.74	147.9	106.1	95.60	05.00	0.384	339440.25
	1.125	90.16	149.7	108.0	84.21	05.02	2.721	425775.88
	1.350	96.57	149.5	090.6	72.45	05.02	2.328	507764.46
	0.650	93.82	148.6	095.3	82.20	05.00	0.647	340012.65
	0.975	92.85	148.4	111.6	92.23	05.00	3.169	340607.84
Varying θ_d	1.300	98.74	147.9	106.1	95.60	05.00	0.384	339440.25
	1.625	90.02	149.6	117.2	90.96	05.01	2.778	343140.07
	1.950	91.36	149.8	094.9	82.94	05.00	2.330	342135.21

Table 6. Optimal Decision Variable Values and Total Profit for Different Values of 'l'

Value of 'l'	$S^*_{p_1}$	$S^*_{p_2}$	$S^*_{p_3}$	u^*	T^*	E_R	Best Solution
1	97.25	149.69	112.33	90.70	5.00	3.13	347,304.98
2	94.99	148.94	102.08	79.30	5.00	3.37	344,272.54
3	97.49	146.31	117.30	93.18	5.00	2.13	340,350.93
4	90.32	149.46	090.15	74.62	5.02	0.58	342,591.81
5	93.44	147.51	107.44	71.51	5.02	1.44	340,652.96

Table 7. Optimal Decision Variables and Total Profit for Different Values of 'λ'

Value of 'λ'	$S^*_{p_1}$	$S^*_{p_2}$	$S^*_{p_3}$	u^*	T^*	E_R	Best Solution
0.1	99.05	149.43	110.83	87.93	5.02	2.28	345,273.33
0.2	97.25	149.69	112.33	90.70	5.00	3.13	347,304.98
0.3	95.61	148.92	109.32	84.25	5.00	1.45	345,127.04
0.4	90.31	149.88	110.30	99.75	5.03	0.58	344,872.07
0.5	98.23	149.32	107.46	90.72	5.01	2.40	345,024.26

- In the pharmaceutical industry, where product efficacy can decline over time, static investment strategies are often inadequate. A dynamic investment approach, which adjusts based on the specific conditions and life-cycle of each product, allows for optimal resource allocation. For instance, increasing investment in preservation technologies or storage solutions as a product nears its expiration date can be more effective than a one-time, fixed investment.
- When deterioration surpasses a certain level, additional investment like preservation investment may yield less returns. In such cases, managers can conserve resources by focusing on promoting faster sales or implementing dynamic preservation techniques. Strategically allocating investments according to a products condition and remaining shelf life can enhance return on investment and mitigate losses.

In summary, for pharmaceutical products, a dynamic investment strategy that considers variable deterioration rates and specific product needs enables more efficient resource allocation, reduced waste, and improved profitability. This approach involves ordering and investing strategically to reduce environmental emission losses. High environmental emissions necessitate reducing sales prices to expedite product turnover and minimize losses. Consequently, this shortens the replenishment cycle and naturally results in lower profits due to increased environmental emissions.

10. Conclusion

This paper analyzed a dynamic promotion rate based on several price ranges and studied an optimum investment to reduce the rate of deterioration by preservation technology considering their lifespan. Those products were used, and the demand pattern followed the trapezoidal type. Due to the dynamic investment rate and trapezoidal demand, the profit was maximized with Pontryagin's maximum principle, ACO and CSA together to calculate the selling price, promotion investment, preservation technology investment, and optimal order quantity simultaneously. Using the algorithm, the mathematical model obtained the optimum profit of the maximization problem. The Ant Colony Optimization (ACO) and Cuckoo Search Algorithm (CSA) were executed separately to compare their performance in solving the problem at hand, with findings showing that ACO performs better. Typically, such algorithms might get stuck in local optima; however, ACO leverages pheromone trails and positive feedback to enhance global search capabilities and avoid local optima. In contrast, CSA uses Lévy flights and parasitic egg-laying to explore the search space more broadly and probabilistically escape local optima. By examining these global optimization mechanisms separately, the authors highlight how ACO's structured exploration outperforms

CSA's random search in achieving more optimal global solutions. As the time-dependent promotional effort was used within the variable demand, the control theory was utilized to solve the proposed study. Utilizing the time-dependent investment for retailer's promotional effort, the study obtained the optimum dynamic investment based on time. Numerical studies showed the benefits of dynamic investment. Through the sensitivity analysis, the effectiveness of the key parameters are discussed.

Moreover, the retailer should decide a pricing strategy, i.e., penetration or skimming or their combination according to the product's price sensitivity at different stages. Third, investing in preservation technology is not always profitable for the retailer, and the retailer should make a decision on this issue based on the nature of the product. Finally, the retailer may determine the optimal replenishment time to gain the maximum profit with promotion, price dependent trapezoidal demand pattern.

10.1. Key Findings

- The study reveals that increased price sensitivity reduces profit and influences the optimal replenishment cycle, particularly during the final product stage. This effect extends to preservation technology investments, which are also shaped by fuzzy learning rates.
- Adapting pricing strategies based on price sensitivity at different product stages is crucial. Effective promotional investments, particularly during high-sensitivity periods, can shift consumer focus from price to product benefits, enhancing both immediate and long-term profits.
- The research highlights that higher decay rates can improve profitability if promotional efforts are successful, but they also limit preservation technology investments. Incorporating emission costs into these decisions further refines the retailer's strategy for managing profitability.
- The study identifies a threshold for deterioration reduction, guiding optimal preservation technology investments. These decisions are crucial for balancing costs, including those related to fuzzy learning rates and environmental emissions, to achieve maximum profit.
- The findings clarify that unit costs, including emission costs, have a significant impact on profit. Higher costs reduce the feasibility of investments in promotion and preservation technology, emphasizing the need for efficient cost management across all operational areas.

10.2. Limitations

The algorithm primarily focuses on investment strategies related to promotions, pricing, and preservation technology. However, it does not explicitly consider other factors influencing profitability, such as production costs, distribution channels, and regulatory requirements. Additionally, the algorithm assumes a trapezoidal-type demand pattern, which may not accurately represent all products or market segments. Despite its effectiveness, the proposed algorithm faces limitations in generalizability due to its reliance on specific demand patterns. Furthermore, the computational complexity and resource requirements might be prohibitive for smaller enterprises. The need for extensive parameter fine-tuning also demands significant domain-specific knowledge. Lastly, the assumption of accurately predictable dynamic investment rates and demand patterns may not hold in volatile markets.

10.3. Applicability of the Proposed Study

The proposed algorithm has the potential to bring numerous benefits to pharmaceutical companies. These include improved brand visibility through targeted marketing strategies, enhanced sales performance through optimized pricing and inventory management, cost optimization through identifying cost-saving opportunities, and risk mitigation through proactive measures to address potential risks. Ultimately, these benefits can lead to increased revenue, market share, profitability, and overall stability for pharmacists and pharmaceutical companies to optimize their investment strategies in the pharmaceutical sector.

- Pharmacists can benefit from this proposed strategy by gaining insights into the dynamic nature of promotional rates and pricing dynamics. This information allows them to strategically allocate resources towards promotional activities, enhance consumer behaviour, and manage costs effectively over time. Additionally, the algorithm guides the optimal utilization of preservation technology, helping pharmacists make informed inventory management and product storage decisions to reduce wastage and improve pharmacy operational efficiency.

- Pharmaceutical companies can also leverage the proposed strategy to refine pricing strategies, promotional campaigns, and investment decisions. Companies can enhance revenue generation, expand market share, and foster sustainable growth by utilizing advanced optimization techniques such as Pontryagin's maximum principle and Ant Colony Optimization.

11. Future Extension of the Proposed Study

One could explore several promising directions for extending this study. For instance, analyzing how dynamic investment in preservation technology affects optimal replenishment strategies could provide valuable insights. It would be intriguing to model the impact of price differentiation across different stages of the demand function, assuming uniform pricing at each stage. we can extend might include incorporating partial backlogging, time-dependent item deterioration, and fuzzy boundary points for each stage.

One can combine dynamic investment with an advanced payment strategy might yield a more robust model than algorithms alone. This combined approach could optimize selling prices, promotion investments, preservation technology investments, and order quantities simultaneously. The mathematical model could use Ant Colony Optimization (ACO) and Cuckoo Search Algorithm (CSA) to maximize profit. Moreover, incorporating control theory to address time-dependent promotional efforts within variable demand could be beneficial. The study could derive optimal dynamic investment strategies over time, and numerical analyses could demonstrate the benefits of dynamic investment. Sensitivity analyses could further clarify the effectiveness of key parameters, providing a comprehensive understanding of the model's performance.

Appendix A. List of additional notation used to simplify expression

$$\frac{h+\theta_d c_d}{\theta_d}=\omega_1,\ \frac{\left(e^{\delta_1 t}-e^{-\delta_1 t}\right)}{2\delta_1}=\omega_2,\ \frac{\left(e^{-\delta_1 t}-e^{-\theta_d t}\right)}{\theta_d-\delta_1}=\omega_3,\ \frac{e^{\delta_1 t}-e^{-\theta_d t}}{\theta_d+\delta_1}=\omega_4,$$
$$\frac{e^{-\delta_2 t}-e^{(\theta_d-\delta_2)t_{cov}}e^{-\theta_d t}}{\theta_d-\delta_2}=\omega_5,\ \frac{e^{\delta_2 t}-e^{(\theta_d+\delta_2)t_{cov}}e^{-\theta_d t}}{\theta_d+\delta_2}=\omega_6,\ \frac{e^{-\delta_3 t}-e^{(\theta_d-\delta_3)t_{vac}}e^{-\theta_d t}}{\theta_d-\delta_3}=\omega_7,$$
$$\frac{e^{\delta_3 t}-e^{(\theta_d+\delta_3)t_{vac}}e^{-\theta_d t}}{\theta_d+\delta_3}=\omega_8,\ \frac{\left(e^{\delta_2 t}-e^{2\delta_2 t_{cov}}e^{-\delta_2 t}\right)}{2\delta_2}=\omega_9,\ \frac{\left(e^{\delta_3 t}-e^{2\delta_3 t_{vac}}e^{-\delta_3 t}\right)}{2\delta_3}=\omega_{10},$$
$$\omega_4-\omega_3=\omega_{11},\ \left(\omega_6-e^{2\delta_2 t_{cov}}\omega_5\right)=\omega_{12},\ \left(\omega_8-e^{2\delta_3 t_{vac}}\omega_7\right)=\omega_{13}.$$

12. Declarations

12.1. Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

12.2. Ethical approval

This proposed study does not contain any research with human participants or animals performed by any of the authors.

12.3. Consent for publication

Not applicable

12.4. Availability of data and material

For numerical validation of different cases are obtained by optimizing respective functions through MAPLE 21 and Python software based on classical optimization method.

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