Vol. 35, No. 2 (2025) | DOI: 10.37190/ord250201

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Operations Research and Decisions

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OPERATIONS RESEARCH AND DECISIONS QUARTERLY

Solution of a bi-level linear programming problem with uncertain parameters and its application

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Abstract

In this paper, a bi-level linear programming problem characterized by interval uncertainty in the coefficients of both objectives and constraints is thoroughly examined. The Karush-Kuhn-Tucker (KKT) optimality conditions for interval nonlinear programming problems have been developed to address this challenge. Utilizing these conditions, the interval bi-level programming problem has been transformed into a deterministic nonlinear programming problem. Subsequently, a comprehensive methodology has been developed to solve the transformed problem. The proposed approach has been validated through numerous illustrative examples that demonstrate its successful execution. Furthermore, the developed methodology has been effectively applied to a practical problem in supply chain planning, showcasing its relevance and applicability in real-world scenarios.

Keywords: *Bi-level programming problem, Interval optimization problem, Interval analysis, KKT optimality conditions, Supply chain*

1. Introduction

Bi-level programming problem (*BLPP*) involves a couple of decision makers(DMs), the so-called leader or upper-level DM maker, and the follower or lower-level DM. There are two optimization problems in *BLPP*, where the constraint field of one problem is explicitly defined by the constraint field of the other. *BLPP* has been presented as a way to deal with hierarchical processes involving at least two levels of decision-making, and is receiving more attention in the literature today. A *BLPP* is formulated as

Received 23 May 2024, accepted 20 December 2024, published online 8 February 2025
ISSN 2391-6060 (Online)/

follows:

$$\min_{x,y} f_1(x,y), \text{ where } y \text{ solves} \tag{1}$$

$$\min_{y} f_2(x,y) \tag{2}$$

$$\text{subject to } (x,y) \in S, \tag{3}$$

where

- $x \in \mathbb{R}^{n_1}$ is an unknown vector associated with the upper level problem and is managed by the upper level DM;
- $y \in \mathbb{R}^{n_2}$ is a decision variable vector associated with the lower level and is managed by the lower level DM;
- $f_1, f_2 : \mathbb{R}^{n_1+n_2} \to \mathbb{R}$ are functions for objectives at the upper and lower levels, respectively;
- $S \subseteq \mathbb{R}^{n_1+n_2}$ is denoted constraint region for both problems.

BLPPs are non-convex and are exceptionally challenging to evaluate due to their structure. The *BLPP* has fascinated scholars as well as planners in the field of decision-making sciences over the past few decades (See [7, 10, 12, 13, 15, 24, 25, 33, 36, 42, 43]). Colson et al. (2007)[10] and Sinha et al. (2017)[39] provided surveys of the *BLPP*. Several researchers have attempted to study *BLPP* in recent years by submitting applications in various domains. Abdelaziz and Mejri (2018)[1] presented a multi-objective *BLPP* with an application to inventory considering the emergency and back-orders in the mathematical model. Mohanty et al. (2018)[27] studied the vendor-buyer joint production and inventory model with imperfect quality item. They have also considered the trade credit finance and variable setup cost in their model. Muneeb et al. (2020) [29] presented a *BLPP* approach to solve the vendor selection problem. Deb et al. (2020)[12] approximated a *BLPP* with population-based evolutionary algorithms.

Generally, coefficients of the objectives as well as constraints are assumed to be known precisely in the *BLPP* formulation. However, due to ambiguity in data collection, the coefficient values in some real-world problems are usually just estimated. Imprecise data encompasses uncertainty due to vagueness, randomness, and partial knowledge is categorized as fuzzy, random, and grey data. Fuzzy data manages ambiguity through degrees of membership, providing flexibility in representing subjective information. Random data involves probabilistic methods to account for inherent variability and chance events. Grey data handles partial knowledge scenarios, offering a middle ground between deterministic and stochastic models. Understanding these types of imprecise data enhances the robustness of models in uncertain environments [11].

Fuzzy set theory is often used to solve the problem of coefficient inaccuracy (Zheng et al. (2011)[41]), where fuzzy parameters are measured with predefined membership functions. However, defining membership functions of fuzzy parameters for DMs can be somewhat complicated in reality. To handle these uncertainty issues in mathematical programming problems, the interval analysis developed by Moore (1966) has been provided as an alternative approach. In this approach, we consider uncertain parameters as closed intervals, making it simple to evaluate both the lower and upper limits of the uncertain data.

Interval analysis has been successfully applied by many authors to solve optimization problems with interval parameters (See [5, 6, 9, 17–23, 26, 32, 38, 40]).

The following example will show that how can one present a real life optimization problem based on interval parameters bi-level programming problem.

Example 1. A logistics chain problem usually consists of several phases. All phases of the process in the chain are interconnected with respect to a decision made at one phase that has an impact on the following phase's performance. The decision goal and alternatives at each step of a logistics plan are dynamically impacted by previous stage decisions. Meanwhile, as one step seeks to maximize its goal, it may need to think about the goal of the following stage. The reaction of the following step will also influence its choice. This study will propose a bi-level decision-making model and its solution for the two stages since the supplier and the distributor are two important and essential stages in a logistics chain. Both the supplier and the distributor wish to enhance their profits while reducing their expenses. Here, the supplier makes the decision first. Then, the distributor will discover a strategy to optimize his or her goal for each potential option made by the source. The supplier is the leader in this circumstance, while the distributor is the follower. Each has its own goal (such as benefits or expenses) as well as limitations (such as time, location, and facilities).

However, in the decision-making process for logistics planning, ambiguity and imprecision are unavoidable. In reality, logistics managers frequently have a hazy understanding of the values of associated constraints and performance objectives when expressing an objective. They can only estimate inventory carrying costs and transportation costs for a certain set of items, for example. Furthermore, while evaluating different facilities, logistics managers may only provide values based on their own experience, and these values are sometimes assigned in ambiguous ways, such as these characteristics being designated closed intervals. Obviously, interval numbers are useful to define when developing a model for such a decision problem. Therefore, one can use the interval bi-level linear programming problem to solve the supply chain problem.

In the last few decades, the literature has also reported interval parameters based on linear *BLPP*s. Calvete and Gale (2012)[8] considered a linear *BLPP* with interval coefficients in both objective functions and discussed the best and worst solutions to the problem. Ren and Wang (2014)[34] developed a cutting plan to study the best and worst optimal solutions to the same problem as considered by Calvete and Gale (2012)[8]. Nehi and Hamidi (2015)[31] developed algorithms for calculating the worst and best optimal solutions of the general interval bi-level linear programming problem. Ren and Wang (2017)[35] presented a new preference-based index approach for the linear *BLPP* with interval parameters in both objectives and constraints. Recently, Nayak and Ojha (2020)[30] applied the interval numbers to solve BLPP, considering a fractional programming problem. Additionally, A bilevel interval-valued optimization problem is investigated, and it is reduced to a nonlinear nonsmooth program with necessary optimality conditions using upper convexificators and Abadie's constraint qualification [14]. They derive detailed results without assuming Lipschitz continuity or convexity, illustrated by examples [14]. An extensive survey of bilevel optimization under uncertainty is provided, reviewing classic stochastic and robust techniques, highlighting richer sources of uncertainty, and discussing applications in energy, interdiction games, security, management sciences, and networks [3]. There are few research works that exist in the literature to handle linear *BLPP* with uncertain environments. Among these papers, some are considered linear *BLPP*, consisting of objective functions with uncertain parameters, and all constraint functions are deterministic. Further, some papers have studied a general linear *BLPP* with interval parameters and obtained the search algorithm to find the best and worst solution to the problem. However, to the best of our knowledge and the aforementioned literature, there is no research paper for the solution of the general interval linear *BLPP* using KKT optimality conditions.

Therefore, this article investigates a general linear *BLPP* in which both objectives and all constraints are interval valued functions. This problem is rewritten in terms of the mean and radius of the interval valued functions using partial ordering. KKT optimality conditions for non-linear programming problems are developed using the partial order relations for a set of closed intervals. Using these KKT optimality conditions, the considered problem is transformed into a classical nonlinear programming problem. Furthermore, a method for solution procedure is established to study the optimal and feasible solution of the problem by solving the transformed problem. In order to validate the methodology, supply chain planning is considered and successfully applied the methodology under uncertain environments.

The paper is organized as follows: Section 2 defines some prerequisites in interval analysis and KKT optimality conditions for interval optimization problems, which are used for developing the results. In Section 3, a bi-level linear programming problem with interval parameters is defined and develop the solution procedure of the problem. Further, the developed solution procedure of the problem is applied in a real-life supply chain problem in Section 4.

2. Notations and Preliminaries

An English alphabet A denotes a closed interval, and a^L and a^R represent lower- and upper- bound, respectively, i.e., $A = [a^L, a^R]$ with $a^L \leq a^R$. If $a^L = a^R = a$, represents a real number. The set of closed interval on $\mathbb{R}$ is denoted by $\mathbb{I}(\mathbb{R})$, and $\mathbb{I}(\mathbb{R}) = \{[a^L, a^R] | \; a^L \leq a^R; \; a^L, a^R \in \mathbb{R}\}$. The mean (a^m) and half width (a^r) of an interval A are defined as $a^m = \frac{a^L + a^R}{2}$ and $a^r = \frac{a^R - a^L}{2}$. Using the mean and half-width of any closed interval, it can also be denoted as $A = \langle a^m, \; a^r \rangle$. In this paper, *mr* stands for mean-radius.

2.1. Partial ordering and interval valued function

It is necessary to define a partial order on a set of closed intervals due to the incomparability of two closed intervals exactly. There are several partial order relationships available in the literature ([4, 18, 28]). Two types of partial ordering relations for maximization ($\succeq_{\max}$) and minimization ($\preceq_{\min}$) problems are defined by Ishibuchi and Tanaka (1990)[18] as follows: For any two intervals $A = [a^L, a^R] = \langle a^m, a^r \rangle$ and $B = [b^L, b^R] = \langle b^m, b^r \rangle$,

$$A \succeq_{\max} B \Leftrightarrow a^m \geq b^m \;\text{ and }\; a^r \leq b^r$$
$$A \succ_{\max} B \Leftrightarrow A \succeq_{\max} B \;\text{ and }\; A \neq B.$$

$$A \preceq_{\min} B \Leftrightarrow a^m \leq b^m \;\text{ and }\; a^r \leq b^r$$
$$A \prec_{\min} B \Leftrightarrow A \preceq_{\min} B \;\text{ and }\; A \neq B.$$

An interval valued function has been defined as the extension of a real valued function onto an interval with one or more interval parameters in Moore (1966)[28] and Hansen and Walster (2003)[16]. Whereas, the interval valued function $F : X \to \mathbb{I}(\mathbb{R}),\ X \subseteq \mathbb{R}^n$ is defined by Ishibuchi and Tanaka (1990)[18] as follows.

$$F(x) = [f^L(x), f^R(x)] = \langle f^m(x), f^r(x)\rangle,$$

where $f^L,\ f^R,\ f^m,\ f^r : X \to \mathbb{R},\ \ f^L(x) \le f^R(x)\ \ \forall\, x \in X$; and $f^m(x) = \frac{f^L(x)+f^R(x)}{2}$ and $f^r(x) = \frac{f^R(x)-f^L(x)}{2}$.

Evaluating the best solution to interval bi-level programming problems requires defining optimality conditions for an interval optimization problem. Consequently, the following subsection shows the optimality condition for an interval optimization problem.

2.2. Optimality conditions for interval optimization problem

Consider the interval optimization problem

$$\min_{x \in X \subseteq \mathbb{R}^n} [f^L(x), f^R(x)]$$

Some important definitions of the solution of interval optimization problem are given as follows:

Definition 1. [4] A local minimum of the interval valued function $f(x)$ is a point $x^* \in X$, if this satisfies a $\delta > 0$ such that

$$[f^L(x^*), f^R(x^*)] \preceq_{\min} [f^L(x), f^R(x)],\ \forall\, x \in B(x^*, \delta) \cap X,$$

where $B(x^*, \delta)$ is an open ball whose center is at x^* and radius δ.

Definition 2. [4] A global minimum of the interval valued function $f(x)$ is a point $x^* \in X$, if this satisfies

$$[f^L(x^*), f^R(x^*)] \preceq_{\min} [f^L(x), f^R(x)],\ \forall\, x \in X.$$

Definition 3. [4] A local maximum of the interval valued function $f(x)$ is a point $x^* \in X$, if this satisfies a $\delta > 0$ such that

$$[f^L(x), f^R(x)] \preceq_{\max} [f^L(x^*), f^R(x^*)],\ \forall\, x \in B(x^*, \delta) \cap X.$$

Definition 4. [4] A global maximum of the interval valued function $f(x)$ is a point $x^* \in X$, if

$$[f^L(x), f^R(x)] \preceq_{\max} [f^L(x^*), f^R(x^*)],\ \forall\, x \in X.$$

Definition 5. [4] An interval valued function $f(x)$ defined on X, is said to be weakly differentiable at $x \in X$ if both $f^L(x)$ and $f^R(x)$ i.e., $f^m(x)$ and $f^r(x)$ are differentiable at x.

Definition 6. Let $f(x)$ be an interval valued function defined on a convex set $X \subseteq \mathbb{R}^n$. Then f is said to be $mr-$convex over X if for every $x_1, x_2 \in X$ and $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \preceq_{\min} \lambda f(x_1) + (1-\lambda) f(x_2).$$

Proposition 1. Let X be a convex set of $\mathbb{R}^n$ and f be an interval valued function. If f^m and f^r are convex, then $f(x)$ is $mr-$convex on X.

Consider an interval optimization problem with inequality constraints as follows.

$$(IOP)\ \min\ f(x) = [f^L(x), f^R(x)]$$
$$\text{subject to } g_i(x) = [g_i^L(x), g_i^R(x)] \preceq_{\min} [0,0],\ i = 1,2,\ldots,k,$$

where both $f(x)$ and $g_i(x),\ i = 1,2,\ldots,k$ are interval valued functions and weakly differentiable.

From the definition of partial ordering for minimization problem " $\preceq_{\min}$", the problem IOP can be rewritten as:

$$(IOP-I)\ \min\ f(x) = [f^L(x), f^R(x)] = \langle f^m(x), f^r(x)\rangle$$
$$\text{subject to } g_i^m(x) \le 0,\ \ g_i^r(x) = 0,\ i = 1,2,\ldots,k.$$

In order to derive KKT optimality conditions, we add non-negative slack variable s_i such that the constraint $G_i^m(x,s_i) = g_i^m(x) + s_i^2 = 0\ i = 1,2,\ldots,k$. Therefore, the IOP becomes

$$(IOP-II)\ \min\ f(x) = \langle f^m(x), f^r(x)\rangle$$
$$\text{subject to } G_i^m(x,s_i) = 0,\ i = 1,2,\ldots,k,$$
$$g_i^r(x) = 0, i = 1,2,\ldots,k.$$

Consider the Lagrange function corresponding to $IOP-II$ as follows:

$$\begin{aligned} L(x,\lambda_i,\mu_i,s_i) &= \langle f^m(x), f^r(x)\rangle + \sum_{i=1}^{k}\lambda_i G_i^m(x,s_i) + \sum_{i=1}^{k}\mu_i g_i^r(x) \\ &= \left\langle f^m(x) + \sum_{i=1}^{k}\lambda_i G_i^m(x,s_i) + \sum_{i=1}^{k}\mu_i g_i^r(x),\ f^r(x)\right\rangle \\ &= \langle L^m(x,\lambda_i,\mu_i,s_i), L^r(x,\lambda_i,\mu_i,s_i)\rangle, \end{aligned}$$

where $L^m(x,\lambda_i,s_i) = f^m(x) + \sum_{i=1}^{k}\lambda_i G_i^m(x,s_i) + \sum_{i=1}^{k}\mu_i g_i^r(x),\ L^r(x,\lambda_i,s_i) = f^r(x)$, and $(\lambda,\mu) = (\lambda_1,\lambda_2,\ldots,\lambda_k,\mu_1,\mu_2,\ldots,\mu_k)$ be the Lagrange multipliers. Now, if x be the local minimum of $L(x,\lambda_i,\mu_i,s_i)$ then from the necessary conditions for optimality of unconstrained optimization, we have

$$\nabla L^m(x,\lambda_i,\mu_i,s_i) = 0,\ \nabla L^r(x,\lambda_i,\mu_i,s_i) = 0$$

Now, $\nabla L^m(x,\lambda_i,\mu_i,s_i) = 0$ gives

$$\nabla f^m(x) + \sum_{i=1}^{k}\lambda_i \nabla G_i^m(x,s_i) + \sum_{i=1}^{k}\mu_i \nabla g_i^r(x) = 0, \tag{4}$$
$$\lambda_i g_i^m(x) = 0,\ \mu_i g_i^r(x) = 0, i = 1,2,\ldots,k, \tag{5}$$
$$g_i^m(x) \le 0,\ \ g_i^r(x) = 0,\ i = 1,2,\ldots,k, \tag{6}$$
$$\lambda_i \ge 0,\ i = 1,2,\ldots,k, \tag{7}$$

and $\nabla L^r(x,\lambda_i,\mu_i,s_i)=0$ gives

$$\nabla f^r(x)=0. \tag{8}$$

The condition $\nabla f^r(x)=0$ will be considerable only for $f^r(x)$ as a constant function; however, for linear and nonlinear function, it will not be considerable. Thus, we obtain the necessary optimality conditions for *IOP*. These conditions are known as KKT optimality conditions.

Theorem 1. (Necessary optimality conditions) Let x^* be a local $mr-$ minimum of the problem *IOP* at which $f(x)$ and $g_i(x),(i=1,2,\ldots,k)$ are weakly differentiable on X. Then there exist multipliers λ_i,μ_i $i=1,2,\ldots,k$ such that the following conditions hold

$$\left.\begin{array}{l} \nabla f^m(x)+\sum_{i=1}^{k}\lambda_i\nabla g_i^m(x)+\sum_{i=1}^{k}\mu_i\nabla g_i^r(x)=0,\\ \lambda_i g_i^m(x)=0,\ \mu_i g_i^r(x)=0, i=1,2,\ldots,k,\\ g_i^m(x)\le 0,\ \ g_i^r(x)=0,\ i=1,2,\ldots,k,\\ \lambda_i\ge 0,\ i=1,2,\ldots,k. \end{array}\right\} \tag{9}$$

Theorem 2. (Sufficient optimality conditions) Let $(x^*,\lambda_1,\ldots,\lambda_k,\mu_1,\ldots,\mu_k)$ satisfy the KKT conditions (9) and $f(x)$, $g_i(x),(i=1,2,\ldots,k)$ are $mr-$convex interval valued functions. Then x^* is the global $mr-$minimizer of *IOP*.

Example 2. Find the $mr-$ minimum point of the following interval optimization problem:

$$(IOP)\ \min\ f(x)=[x_1+x_2,\ x_1+x_2+2]$$
$$\text{subject to } [x_1^2+x_2^2-3,\ 3x_1^2+x_2^2-5]\preceq_{\min}[0,0]$$

Let $g(x_1,x_2)=[x_1^2+x_2^2-3,\ 3x_1^2+x_2^2-5]$, $g^m(x_1,x_2)=2x_1^2+x_2^2-4$, $g^r(x_1,x_2)=x_1^2-1$, and $f^m(x_1,x_2)=x_1+x_2+1, f^r(x_1,x_2)=1$, where f,g^m,g^r are continuously differentiable convex functions. The necessary optimality conditions for the problem *IOP* are:

$$\begin{array}{l} 1+\lambda(4x_1)+\mu(2x_1)=0,\ 1+\lambda(2x_2)=0,\\ \lambda(2x_1^2+x_2^2-4)=0,\ \mu(x_1^2-1)=0,\\ 2x_1^2+x_2^2\le 4,\ x_1^2=1,\\ \lambda\ge 0. \end{array}$$

By solving the above conditions, we obtain the optimal solution $x_1^*=-1, x_2^*=-\sqrt{2}, \lambda^*=\frac{1}{2\sqrt{2}}$, $\mu=\frac{1}{2}(1-\sqrt{2})$. Hence $(-1,-\sqrt{2})$ is the global $mr-$minimum solution of the *IOP* due to convexity of f, g^m, g^r.

2.3. Comparative study

Bhunia and Samanta (2014)[4] considered the following partial ordering relation to develop the KKT conditions for interval non-linear programming problem with minimization case.

Definition 7. For $A = [a^L, a^R] = \langle a^m, a^r \rangle$ and $B = [b^L, b^R] = \langle b^m, b^r \rangle$,

$$A \preceq_{\min} B \Leftrightarrow \begin{cases} a^m < b^m, & \text{if } a^m \neq b^m; \\ a^r \leq b^r, & \text{if } a^m = b^m \end{cases}$$

and $A \prec_{\min} B \Leftrightarrow A \preceq_{\min} B$ and $A \neq B$.

In the above definition of partial ordering, the idea is to lexicographically compare the mean components and (if there is equality in the mean components) the radius components of the two intervals. For the constraint $g(x_1,x_2) = [x_1^2 + x_2^2 - 4, x_1^2 + x_2^2] \preceq_{\min} [0,0]$ in Example 2, this means that it is equivalent to (case-1) $g^m(x_1,x_2) = x_1^2 + x_2^2 - 2 < 0$ or (case-2) $g^m(x_1,x_2) = x_1^2 + x_2^2 - 2 = 0$ and $g^r(x_1,x_2) = 2 \leq 0$, where the second case can obviously never occur. Hence, it is equivalent to $x_1^2 + x_2^2 - 2 < 0$, and not $x_1^2 + x_2^2 - 2 \leq 0$ as in the formulation of the necessary conditions due to Bhunia and Samanta (2014)[4]. This might make the feasible region not closed in general, which can be very problematic because the existence of optimal solutions might not be guaranteed then. They solved the Example 2, and obtained the solution of the problem as $(x_1,x_2) = (-1,-1)$, which is claimed to be the global minimum of the problem, even if it is not feasible due to constraint $x_1^2 + x_2^2 - 2 < 0$. However, we developed KKT conditions for interval non-linear optimization problems using partial ordering due to Ishibuchi and Tanaka (1990)[18], which provide a feasible optimal solution to the problem.

3. Linear bi-level programming problem

A bi-level programming problem in which both objectives and constraint are linear is said to be a linear bi-level programming problem, and it is defined as follows:

$$\begin{aligned} &\min_{x \in X} f_1(x,y) = cx + dy, \\ &\text{subject to } Ax + By \geq e, \\ &\min_{y \in Y} f_2(x,y) = ay, \\ &\text{subject to } Cx + Dy \geq \mathfrak{f}, \end{aligned} \tag{10}$$

where $X \subset \mathbb{R}^{n_1}$; $Y \subset \mathbb{R}^{n_2}$; $f_1, f_2 : X \times Y \to \mathbb{R}$; $c \in \mathbb{R}^{n_1}$; $d, a \in \mathbb{R}^{n_2}$; $e \in \mathbb{R}^{m_1}$; $\mathfrak{f} \in \mathbb{R}^{m_2}$; $A \in \mathbb{R}^{m_1 \times n_1}$; $B \in \mathbb{R}^{m_1 \times n_2}$; $C \in \mathbb{R}^{m_2 \times n_1}$; $D \in \mathbb{R}^{m_2 \times n_2}$. Some terminologies for the *BLPP* are given as follows:

Definition 8. ([2])

(a) Constraint region of the *BLPP*:
$S = \{(x,y) : x \in X, y \in Y, Ax + By \geq e, Cx + Dy \geq \mathfrak{f}\}$

(b) Projection of S onto the leader's decision space:
$S_1 = \{x \in X : \exists y \in Y, Ax + By \geq e, Cx + Dy \geq \mathfrak{f}\}$

(c) Feasible set for the follower for each fixed $x \in X$:
$S(x) = \{y \in Y : Dy \geq \mathfrak{f} - Cx\}$.

(d) Follower's rational reaction set for $x \in S_1$:
$M(x) = \{y \in Y : y \in argmin\{f_2(x,y^*) : y^* \in S(x)\}\}$.

(e) Inducible region:
$FR = \{(x,y) : x \in S_1, y \in M(x)\}$.

The set X and Y are generally considered as $X = \{x \in \mathbb{R}^{n_1} : x \geq 0\}, Y = \{y \in \mathbb{R}^{n_2} : y \geq 0\}$.

The problem (10) can be written as follows:

$$\begin{aligned}
&\min_{x \in X} \ f_1(x,y) = \sum_{i=1}^{n_1} c_i x_i + \sum_{j=1}^{n_2} d_j y_j, \\
&\text{subject to} \\
&\sum_{i=1}^{n_1} a_{ik} x_i + \sum_{j=1}^{n_2} b_{jk} y_j \geq e_k, k = 1,2,\ldots,m_1, \\
&\min_{y \in Y} \ f_2(x,y) = \sum_{j=1}^{n_2} a_j y_j, \\
&\text{subject to} \\
&\sum_{i=1}^{n_1} c_{il} x_i + \sum_{j=1}^{n_2} d_{jl} y_j \geq \mathfrak{f}_l, l = 1,2,\ldots,m_2.
\end{aligned} \tag{11}$$

3.1. Interval bi-level programming problem

The interval linear bi-level programming problem (*IBLPP*) involving interval coefficients in both objectives functions, as well as constraints, can be formulated as:

$$\begin{aligned}
&\min_{x \in X} \ f_1(x,y) = \textstyle\sum_{i=1}^{n_1} [c_i^L, c_i^R] x_i + \sum_{j=1}^{n_2} [d_j^L, d_j^R] y_j, \\
&\text{subject to} \\
&\textstyle\sum_{i=1}^{n_1} [a_{ik}^L, a_{ik}^R] x_i + \sum_{j=1}^{n_2} [b_{jk}^L, b_{jk}^R] y_j \geq [e_k^L, e_k^R], k = 1,2,\ldots,m_1, \\
&\min_{y \in Y} \ f_2(x,y) = \textstyle\sum_{j=1}^{n_2} [a_j^L, a_j^R] y_j, \\
&\text{subject to} \\
&\textstyle\sum_{i=1}^{n_1} [c_{il}^L, c_{il}^R] x_i + \sum_{j=1}^{n_2} [d_{jl}^L, d_{jl}^R] y_j \geq [\mathfrak{f}_l^L, \mathfrak{f}_l^R], l = 1,2,\ldots,m_2.
\end{aligned} \tag{12}$$

Here we define some terminologies for the *IBLPP* as follows:

Definition 9. (a) Constraint region of the *IBLPP*:

$$S_I = \Big\{(x,y) : x \in X, y \in Y, \sum_{i=1}^{n_1} [a_{ik}^L, a_{ik}^R] x_i + \sum_{j=1}^{n_2} [b_{jk}^L, b_{jk}^R] y_j \geq [e_k^L, e_k^R], k = 1,2,\ldots,m_1, \\ \sum_{i=1}^{n_1} [c_{il}^L, c_{il}^R] x_i + \sum_{j=1}^{n_2} [d_{jl}^L, d_{jl}^R] y_j \geq [\mathfrak{f}_l^L, \mathfrak{f}_l^R], l = 1,2,\ldots,m_2 \Big\}$$

(b) Projection of S_I onto the leader's decision space:

$$S_I^1 = \Big\{x \in X : \exists y \in Y, \sum_{i=1}^{n_1}[a_{ik}^L, a_{ik}^R]x_i + \sum_{j=1}^{n_2}[b_{jk}^L, b_{jk}^R]y_j \geq [e_k^L, e_k^R], k = 1,2,\ldots,m_1,$$
$$\sum_{i=1}^{n_1}[c_{il}^L, c_{il}^R]x_i + \sum_{j=1}^{n_2}[d_{jl}^L, d_{jl}^R]y_j \geq [\mathfrak{f}_l^L, \mathfrak{f}_l^R], l = 1,2,\ldots,m_2\Big\}$$

(c) Feasible set for the follower for each fixed $x \in X$:

$$S_I(x) = \Big\{y \in Y : \sum_{i=1}^{n_1}[c_{il}^L, c_{il}^R]x_i + \sum_{j=1}^{n_2}[d_{jl}^L, d_{jl}^R]y_j \geq [\mathfrak{f}_l^L, \mathfrak{f}_l^R], l = 1,2,\ldots,m_2\Big\}$$

(d) Follower's rational reaction set for $x \in S_I^1$:

$$M_I(x) = \{y \in Y : y \in argmin\{f_2(x,y^*) : y^* \in S_I(x)\}\}.$$

(e) Inducible region:

$$FR_I = \{(x,y) : x \in S_I^1, y \in M_I(x)\}.$$

The above problem 12 can be rewritten in terms of its mean and radius of the interval parameters as follows.

$$\begin{aligned}
&\min_{x\in X} f_1(x,y) = \sum_{i=1}^{n_1}\langle c_i^m, c_i^r\rangle x_i + \sum_{j=1}^{n_2}\langle d_j^m, d_j^r\rangle y_j,\\
&\text{subject to}\\
&\sum_{i=1}^{n_1}\langle a_{ik}^m, a_{ik}^r\rangle x_i + \sum_{j=1}^{n_2}\langle b_{jk}^m, b_{jk}^r\rangle y_j \succeq_{\min} \langle e_k^m, e_k^r\rangle, k = 1,2,\ldots,m_1,\\
&\min_{y\in Y} f_2(x,y) = \sum_{j=1}^{n_2}\langle a_j^m, a_j^r\rangle y_j,\\
&\text{subject to}\\
&\sum_{i=1}^{n_1}\langle c_{il}^m, c_{il}^r\rangle x_i + \sum_{j=1}^{n_2}\langle d_{jl}^m, d_{jl}^r\rangle y_j \succeq_{\min} \langle \mathfrak{f}_l^m, \mathfrak{f}_l^r\rangle, l = 1,2,\ldots,m_2.
\end{aligned} \tag{13}$$

Then by the definition of order relation $\succeq_{\min}$, the given constraints can be rewritten as:

$$\sum_{i=1}^{n_1}\langle a_{ik}^m, a_{ik}^r\rangle x_i + \sum_{j=1}^{n_2}\langle b_{jk}^m, b_{jk}^r\rangle y_j \succeq_{\min} \langle e_k^m, e_k^r\rangle, k = 1,2,\ldots,m_1,$$
$$\Rightarrow \begin{cases} \sum_{i=1}^{n_1} a_{ik}^m x_i + \sum_{j=1}^{n_2} b_{jk}^m y_j \geq e_k^m, & k = 1,2,\ldots,m_1; \\ \sum_{i=1}^{n_1} a_{ik}^r x_i + \sum_{j=1}^{n_2} b_{jk}^r y_j \geq e_k^r, & k = 1,2,...,m_1. \end{cases}$$

$$\sum_{i=1}^{n_1}\langle c_{il}^m, c_{il}^r\rangle x_i + \sum_{j=1}^{n_2}\langle d_{jl}^m, d_{jl}^r\rangle y_j \succeq_{\min} \langle \mathfrak{f}_l^m, \mathfrak{f}_l^r\rangle, l = 1,2,\ldots,m_2$$

$$\Rightarrow \begin{cases} \sum_{i=1}^{n_1} c_{il}^m x_i + \sum_{j=1}^{n_2} d_{jl}^m y_j \geq \mathfrak{f}_l^m, & l = 1,2,\ldots,m_2; \\ \sum_{i=1}^{n_1} c_{il}^r x_i + \sum_{j=1}^{n_2} d_{jl}^r y_j \geq \mathfrak{f}_l^r, & l = 1,2\ldots,m_2. \end{cases}$$

Then, the above problem 13 can be equivalently written as follows:

$$\begin{aligned} &\min_{x\in X} f_1(x,y) = \sum_{i=1}^{n_1}\langle c_i^m, c_i^r\rangle x_i + \sum_{j=1}^{n_2}\langle d_j^m, d_j^r\rangle y_j, \\ &\text{subject to} \\ &\sum_{i=1}^{n_1} a_{ik}^m x_i + \sum_{j=1}^{n_2} b_{jk}^m y_j \geq e_k^m, k = 1,2,\ldots,m_1, \\ &\textstyle\sum_{i=1}^{n_1} a_{ik}^r x_i + \sum_{j=1}^{n_2} b_{jk}^r y_j \geq e_k^r, k = 1,2,...,m_1. \\ &\min_{y\in Y} f_2(x,y) = \sum_{j=1}^{n_2}\langle a_j^m, a_j^r\rangle y_j, \\ &\text{subject to} \\ &\sum_{i=1}^{n_1} c_{il}^m x_i + \sum_{j=1}^{n_2} d_{jl}^m y_j \geq \mathfrak{f}_l^m, l = 1,2,\ldots,m_2, \\ &\sum_{i=1}^{n_1} c_{il}^r x_i + \sum_{j=1}^{n_2} d_{jl}^r y_j \geq \mathfrak{f}_l^r, l = 1,2\ldots,m_2. \end{aligned} \tag{14}$$

First, we consider lower level problem without considering upper level objective and constraints as follows:

$$\begin{aligned} &\min_{y\in Y} f_2(x,y) = \sum_{j=1}^{n_2}\langle a_j^m, a_j^r\rangle y_j, \\ &\text{subject to} \\ &\sum_{i=1}^{n_1} c_{il}^m x_i + \sum_{j=1}^{n_2} d_{jl}^m y_j \geq \mathfrak{f}_l^m, l = 1,2,\ldots,m_2, \\ &\sum_{i=1}^{n_1} c_{il}^r x_i + \sum_{j=1}^{n_2} d_{jl}^r y_j \geq \mathfrak{f}_l^r, l = 1,2\ldots,m_2. \end{aligned} \tag{15}$$

Necessary optimality conditions: Let y be a local $mr-$minimum of the problem IBLPP. Then there exist multipliers λ_l, μ_l, $l = 1,2,\ldots,m_2$ such that the following conditions are

$$\begin{aligned}
& a_j^m - \sum_{l=1}^{m_2} \lambda_l d_{jl}^m - \sum_{l=1}^{m_2} \mu_l d_{jl}^r = 0,\ j = 1,2,\ldots,n_2, \\
& \lambda_l (\sum_{i=1}^{n_1} c_{il}^m x_i + \sum_{j=1}^{n_2} d_{jl}^m y_j - \mathfrak{f}_l^m) = 0,\ l = 1,2,\ldots,m_2, \\
& \mu_l (\sum_{i=1}^{n_1} c_{il}^r x_i + \sum_{j=1}^{n_2} d_{jl}^r y_j - \mathfrak{f}_l^r) = 0,\ l = 1,2,\ldots,m_2, \\
& \sum_{i=1}^{n_1} c_{il}^m x_i + \sum_{j=1}^{n_2} d_{jl}^m y_j \geq \mathfrak{f}_l^m,\ l = 1,2,\ldots,m_2, \\
& \sum_{i=1}^{n_1} c_{il}^r x_i + \sum_{j=1}^{n_2} d_{jl}^r y_j \geq \mathfrak{f}_l^r,\ l = 1,2\ldots,m_2, \\
& \lambda_l \geq 0, \mu_l \geq 0,\ l = 1,2,\ldots,m_2.
\end{aligned} \tag{16}$$

The local $mr-$ minimum of the problem *IBLPP* corresponding to the y response of the lower level problem can be calculated by solving the following non-linear interval optimization problem

$$\begin{aligned}
& \min_{x \in X}\ f_1(x,y) = \sum_{i=1}^{n_1} \langle c_i^m, c_i^r \rangle x_i + \sum_{j=1}^{n_2} \langle d_j^m, d_j^r \rangle y_j, \\
& \text{subject to} \\
& \sum_{i=1}^{n_1} a_{ik}^m x_i + \sum_{j=1}^{n_2} b_{jk}^m y_j \geq e_k^m,\ k = 1,2,\ldots,m_1, \\
& \sum_{i=1}^{n_1} a_{ik}^r x_i + \sum_{j=1}^{n_2} b_{jk}^r y_j \geq e_k^r,\ k = 1,2,\ldots,m_1, \\
& a_j^m - \sum_{l=1}^{m_2} \lambda_l d_{jl}^m - \sum_{l=1}^{m_2} \mu_l d_{jl}^r = 0,\ j = 1,2,\ldots,n_2, \\
& \lambda_l (\sum_{i=1}^{n_1} c_{il}^m x_i + \sum_{j=1}^{n_2} d_{jl}^m y_j - \mathfrak{f}_l^m) = 0,\ l = 1,2,\ldots,m_2, \\
& \mu_l (\sum_{i=1}^{n_1} c_{il}^r x_i + \sum_{j=1}^{n_2} d_{jl}^r y_j - \mathfrak{f}_l^r) = 0,\ l = 1,2,\ldots,m_2, \\
& \sum_{i=1}^{n_1} c_{il}^m x_i + \sum_{j=1}^{n_2} d_{jl}^m y_j \geq \mathfrak{f}_l^m,\ l = 1,2,\ldots,m_2, \\
& \sum_{i=1}^{n_1} c_{il}^r x_i + \sum_{j=1}^{n_2} d_{jl}^r y_j \geq \mathfrak{f}_l^r,\ l = 1,2\ldots,m_2, \\
& \lambda_l \geq 0, \mu_l \geq 0,\ l = 1,2,\ldots,m_2.
\end{aligned} \tag{17}$$

Based on the partial ordering defined in Subsection 2.1, minimizing an interval involves minimizing both its mean and its radius. This approach ensures that the interval is centred at the lowest possible value and as narrow as possible, reducing uncertainty. Therefore, solving the interval minimization problem (17) is equivalent to solving the following bi-objective programming problem (18). The solution to the above problem can be calculated by the following bi-objective programming problem: In this bi-objective formulation, the two objectives—minimizing the mean of the interval and minimizing its radius—are addressed simultaneously, converting the interval minimization into a more tractable optimization problem.

This transformation allows the use of established bi-objective optimization techniques to find solutions that effectively balance both aspects of interval minimization, providing a more comprehensive approach to the problem (Ishibuchi & Tanaka, 1990 [18]).

$$
\begin{aligned}
&\min_{x\in X}\ \left\{f_1^m(x,y), f_1^r(x,y)\right\}\\
&\text{subject to}\\
&\sum_{i=1}^{n_1} a_{ik}^m x_i + \sum_{j=1}^{n_2} b_{jk}^m y_j \geq e_k^m,\ k=1,2,\ldots,m_1,\\
&\sum_{i=1}^{n_1} a_{ik}^r x_i + \sum_{j=1}^{n_2} b_{jk}^r y_j \geq e_k^r,\ k=1,2,\ldots,m_1,\\
&a_j^m - \sum_{l=1}^{m_2} \lambda_l d_{jl}^m - \sum_{l=1}^{m_2} \mu_l d_{jl}^r = 0,\ j=1,2,\ldots,n_2,\\
&\lambda_l(\sum_{i=1}^{n_1} c_{il}^m x_i + \sum_{j=1}^{n_2} d_{jl}^m y_j - \mathfrak{f}_l^m) = 0,\ l=1,2,\ldots,m_2,\\
&\mu_l(\sum_{i=1}^{n_1} c_{il}^r x_i + \sum_{j=1}^{n_2} d_{jl}^r y_j - \mathfrak{f}_l^r) = 0,\ l=1,2,\ldots,m_2,\\
&\sum_{i=1}^{n_1} c_{il}^m x_i + \sum_{j=1}^{n_2} d_{jl}^m y_j \geq \mathfrak{f}_l^m,\ l=1,2,\ldots,m_2,\\
&\sum_{i=1}^{n_1} c_{il}^r x_i + \sum_{j=1}^{n_2} d_{jl}^r y_j \geq \mathfrak{f}_l^r,\ l=1,2\ldots,m_2,\\
&\lambda_l \geq 0, \mu_l \geq 0,\ l=1,2,\ldots,m_2,
\end{aligned}
\tag{18}
$$

where $f_1^m(x,y) = \sum_{i=1}^{n_1} c_i^m x_i + \sum_{j=1}^{n_2} d_j^m y_j$ and $f_1^r(x,y) = \sum_{i=1}^{n_1} c_i^r x_i + \sum_{j=1}^{n_2} d_j^r y_j$. Using the weighted sum method, the solution to the above problem can be calculated by the following problem:

$$
\begin{aligned}
&\min_{x\in X}\ wf_1^m(x,y)+(1-w)f_1^r(x,y),\\
&\text{subject to}\\
&\sum_{i=1}^{n_1} a_{ik}^m x_i+\sum_{j=1}^{n_2} b_{jk}^m y_j \geq e_k^m,\ k=1,2,\ldots,m_1,\\
&\sum_{i=1}^{n_1} a_{ik}^r x_i+\sum_{j=1}^{n_2} b_{jk}^r y_j \geq e_k^r,\ k=1,2,\ldots,m_1,\\
&a_j^m-\sum_{l=1}^{m_2}\lambda_l d_{jl}^m-\sum_{l=1}^{m_2}\mu_l d_{jl}^r=0,\ j=1,2,\ldots,n_2,\\
&\lambda_l\Big(\sum_{i=1}^{n_1} c_{il}^m x_i+\sum_{j=1}^{n_2} d_{jl}^m y_j-\mathfrak{f}_l^m\Big)=0,\ l=1,2,\ldots,m_2,\\
&\mu_l\Big(\sum_{i=1}^{n_1} c_{il}^r x_i+\sum_{j=1}^{n_2} d_{jl}^r y_j-\mathfrak{f}_l^r\Big)=0,\ l=1,2,\ldots,m_2,\\
&\sum_{i=1}^{n_1} c_{il}^m x_i+\sum_{j=1}^{n_2} d_{jl}^m y_j \geq \mathfrak{f}_l^m,\ l=1,2,\ldots,m_2,\\
&\sum_{i=1}^{n_1} c_{il}^r x_i+\sum_{j=1}^{n_2} d_{jl}^r y_j \geq \mathfrak{f}_l^r,\ l=1,2\ldots,m_2.\\
&0\leq w\leq 1,\ \lambda_l\geq 0,\mu_l\geq 0,\ l=1,2,\ldots,m_2.
\end{aligned}
\tag{19}
$$

The whole developed solution methodology is summarized in the following step-by-step procedure.

Solution procedure for IBLPP: The following steps are needed to solve an *IBLPP* by developed methodology:

Step-1. Write the given *IBLPP* in terms of the mean and radius of the interval coefficients.

Step-2. Consider the lower level problem and find its KKT optimality conditions using (9) for the decision variables of the problem.

Step-3. Write the upper level problem with KKT conditions of lower level problem as constraints.

Step-4. Solve the problem obtained from Step 3 using any optimization software like Lingo, Mathematica, etc.

Step-5. The solution obtained in Step 4 is the local optimal solution of the given problem.

3.2. Numerical examples

Example 3. Consider a general *IBLPP* in which coefficients of both objectives and constraints are as follows:

$$\min_{x\geq 0} F_1 = [1,2]x + [-1,5]y,$$
subject to
$$[0.5,1]x + [1.9,2]y \geq [10,10.5]$$
$$\min_{y\geq 0} F_2 = [1,2]y,$$
subject to
$$[-2,-1]x + [1,2]y \geq [-6,-5],$$
$$[-3,-2]x + [0.5,1]y \geq [-21,20],$$
$$[-2,-1]x + [-3,-2]y \geq [-38,-37],$$
$$[0.5,1]x + [-3,-2]y \geq [-18,-17].$$

The mean and radius form of the given *IBLPP* is written as follows:

$$\min_{x\geq 0} F_1 = \langle 1.5,0.5\rangle x + \langle 2,3\rangle y,$$
subject to
$$\langle 0.75,0.25\rangle x + \langle 1.95,0.05\rangle y \geq \langle 10.25,0.25\rangle$$
$$\min_{y\geq 0} F_2 = \langle 1.5,0.5\rangle y,$$
subject to
$$\langle -1.5,0.5\rangle x + \langle 1.5,0.5\rangle y \geq \langle -5.5,0.5\rangle,$$
$$\langle -2.5,0.5\rangle x + \langle 0.75,0.25\rangle y \geq \langle -0.5,20.5\rangle,$$
$$\langle -1.5,0.5\rangle x + \langle -2.5,0.5\rangle y \geq \langle -37.5,0.5\rangle,$$
$$\langle 0.75,0.25\rangle x + \langle -2.5,0.5\rangle y \geq \langle -17.5,0.5\rangle.$$

First we consider lower level programming by ignoring upper level objectives as follows.

$$\min_{y\geq 0} F_2 = \langle 1.5,0.5\rangle y,$$
subject to
$$\langle -1.5,0.5\rangle x + \langle 1.5,0.5\rangle y \geq \langle -5.5,0.5\rangle,$$
$$\langle -2.5,0.5\rangle x + \langle 0.75,0.25\rangle y \geq \langle -0.5,20.5\rangle,$$
$$\langle -1.5,0.5\rangle x + \langle -2.5,0.5\rangle y \geq \langle -37.5,0.5\rangle,$$
$$\langle 0.75,0.25\rangle x + \langle -2.5,0.5\rangle y \geq \langle -17.5,0.5\rangle.$$

The conditions for the best response y of lower level problem is given as follows.

$$
\begin{aligned}
&1.5-1.5\lambda_1-0.75\lambda_2+2.5\lambda_3+2.5\lambda_4=0,\\
&\lambda_1(-1.5x+1.5y+5.5)=0,\\
&\lambda_2(-2.5x+0.75y+0.5)=0,\\
&\lambda_3(-1.5x-2.5y+37.5)=0,\\
&\lambda_4(0.75x-2.5y+17.5)=0,\\
&-1.5x+1.5y+5.5\geq 0,\\
&-2.5x+0.75y+0.5\geq 0,\\
&-1.5x-2.5y+37.5\geq 0,\\
&0.75x-2.5y+17.5\geq 0,\\
&x\geq 0, y\geq 0; \lambda_1,\lambda_2,\lambda_3\geq 0,\lambda_4\geq 0.
\end{aligned}
$$

Hence the local optimal solution of the given *IBLPP* is given by the optimal solution of the following non-linear programming problem:

$$
\begin{aligned}
&\min_{x,y}\ w(1.5x+2y)+(1-w)(0.5x+3y),\\
&\text{subject to}\\
&0.75x+1.95y\geq 10.25,\\
&1.5-1.5\lambda_1-0.75\lambda_2+2.5\lambda_3+2.5\lambda_4=0,\\
&\lambda_1(-1.5x+1.5y+5.5)=0,\\
&\lambda_2(-2.5x+0.75y+0.5)=0,\\
&\lambda_3(-1.5x-2.5y+37.5)=0,\\
&\lambda_4(0.75x-2.5y+17.5)=0,\\
&-1.5x+1.5y+5.5\geq 0,\\
&-2.5x+0.75y+0.5\geq 0,\\
&-1.5x-2.5y+37.5\geq 0,\\
&0.75x-2.5y+17.5\geq 0,\\
&x\geq 0, y\geq 0,\ 0\leq w\leq 1,\\
&\lambda_1\geq 0,\lambda_2\geq 0,\lambda_3\geq 0,\lambda_4\geq 0.
\end{aligned}
$$

The local optimal solution of the problem is $x^*=1.593103, y^*=4.643678, \lambda_1^*=0, \lambda_2^*=2, \lambda_3^*=0, \lambda_4^*=0$. Hence local optimal solution of the given *IBLPP* is $x^*=1.593103, y^*=4.643678$.

4. Application of *IBLP* to a supply chain planning with interval uncertain environment

Consider a manufacturing company with two plants P_1 and P_2, one distribution centre (DC), and two items A and B. The company's goal is to reduce its total cost, which includes both manufacturing and distribution costs. The supply chain model may be formally expressed as the following constraints using the notation in the table below.

y_{11} :	Product A production quantity at plant P_1 (ton)
y_{12} :	Product B production quantity at plant P_1 (ton)
y_{21} :	Product A production quantity at plant P_2 (ton)
y_{22} :	Product B production quantity at plant P_2 (ton)
x_1 :	Product A's inventory is held in DC (ton)
x_2 :	Product B's inventory is held in DC (ton)

The goal of the production department is to reduce the production cost, which is generally expressed as follows: (For detail see [37]):

$$\min Z_{PC} = 1.5x_1 + 2x_2 + 7y_{11} + 3y_{12} + 10y_{21} + 6y_{22} \tag{20}$$

$$\text{subject to } y_{11} + y_{12} + y_{21} + y_{22} \leq 500, \tag{21}$$

$$2y_{11} + y_{12} \leq 200, \tag{22}$$

$$y_{21} + y_{22} \leq 250, \tag{23}$$

$$y_{11} + y_{21} \leq x_1, \tag{24}$$

$$y_{12} + y_{22} \leq x_2, \tag{25}$$

where constraint (21) indicates that both plants share the same resources. Constraints (22-23) represent that individual plant circumstances may influence the availability of some resources. Constraints (24-25) indicates that the production levels obtained by both plants should not fall below the inventory requirements..

The objective of the distribution center is to minimize a distribution cost which may be formulated as follows:

$$\min Z_{DC} = 15x_1 + 13x_2 + 3y_{11} + 2y_{12} + 3.5y_{21} + 2.5y_{22} \tag{26}$$

$$\text{subject to } 3x_1 + 2x_2 \leq 500, \tag{27}$$

$$x_1 \geq 100, \tag{28}$$

$$x_2 \geq 100, \tag{29}$$

where constraint (27) restricted the overall capacity of the inventory level. Constraints (28-29) represent that the inventory levels should meet demands.

Now, we formulate the above two minimization problems as a $(BLPP)$, which is as follows:

(**First-level: Production model**)

$$\begin{aligned}
\min_{y_{11},y_{12},y_{21},y_{22}} \; Z_{PC} = & 1.5x_1 + 2x_2 + 7y_{11} + 3y_{12} + 10y_{21} + 6y_{22} \\
\text{subject to } & y_{11} + y_{12} + y_{21} + y_{22} \leq 500 \\
& 2y_{11} + y_{12} \leq 200, \\
& y_{21} + y_{22} \leq 250, \\
& y_{11} + y_{21} \leq x_1, \\
& y_{12} + y_{22} \leq x_2 \\
& y_{11} \geq 0, y_{12} \geq 0, y_{21} \geq 0, y_{22} \geq 0.
\end{aligned}$$

(**Second-level: Distribution model**)

$$\begin{aligned}
\min_{x_1,x_2} \; Z_{DC} = & 15x_1 + 13x_2 + 3y_{11} + 2y_{12} + 3.5y_{21} + 2.5y_{22} \\
\text{subject to } & 3x_1 + 2x_2 \leq 500, \\
& x_1 \geq 100, \\
& x_2 \geq 100, \\
& y_{11} + y_{21} \leq x_1, \\
& y_{12} + y_{22} \leq x_2 \\
& x_1 \geq 0, x_2 \geq 0.
\end{aligned}$$

There is typically some inconsistency in data in real-life scenarios. To deal with this vagueness in the problem, the interval numbers are considered as coefficients of the problem. The bi-level linear programming problem is reformulated as an interval bi-level linear programming problem as follows:

First-level: Production model

$$\begin{aligned}
\min_{y_{11},y_{12},y_{21},y_{22}} \; Z_{PC} = & [1,2]x_1 + [1.5,2.5]x_2 + [6.5,7.5]y_{11} + [2.5,3.5]y_{12} + [9.5,10.5]y_{21} + [5.5,6.5]y_{22} \\
\text{subject to } & y_{11} + y_{12} + y_{21} + y_{22} \leq [450,550], \\
& [1.5,2.5]y_{11} + [0.5,1.5]y_{12} \leq [150,250], \\
& [0.5,1.5]y_{21} + y_{22} \leq [200,300], \\
& y_{11} + y_{21} \leq x_1, \\
& y_{12} + y_{22} \leq x_2, \\
& y_{11} \geq 0, y_{12} \geq 0, y_{21} \geq 0, y_{22} \geq 0.
\end{aligned}$$

Second-level: Distribution model

$$\begin{aligned}\min_{x_1,x_2} Z_{DC} =& [14,16]x_1+[12,14]x_2+[2,4]y_{11}+[1,3]y_{12}+[3,4]y_{21}+[2,3]y_{22}\\ \text{subject to } & [2.5,3.5]x_1+[1.5,2.5]x_2 \leq [450,550],\\ & x_1 \geq 100,\\ & x_2 \geq 100,\\ & y_{11}+y_{21} \leq x_1,\\ & y_{12}+y_{22} \leq x_2,\\ & x_1 \geq 0, x_2 \geq 0.\end{aligned}$$

The mean and radius form of the above *IBLPP* is written as follows:

First-level: Production model

$$\begin{aligned}\min_{y_{11},y_{12},y_{21},y_{22}} Z_{PC} =& \langle 1.5,0.5\rangle x_1+\langle 2,0.5\rangle x_2+\langle 7,0.5\rangle y_{11}+\langle 3,0.5\rangle y_{12}+\langle 10,0.5\rangle y_{21}+\langle 6,0.5\rangle y_{22}\\ \text{subject to } & y_{11}+y_{12}+y_{21}+y_{22} \leq \langle 500,50\rangle,\\ & \langle 2,0.5\rangle y_{11}+\langle 1,0.5\rangle y_{12} \leq \langle 200,50\rangle,\\ & \langle 1,0.5\rangle y_{21}+y_{22} \leq \langle 250,50\rangle,\\ & y_{11}+y_{21} \leq x_1,\\ & y_{12}+y_{22} \leq x_2,\\ & y_{11} \geq 0, y_{12} \geq 0, y_{21} \geq 0, y_{22} \geq 0.\end{aligned}$$

Second-level: Distribution model

$$\begin{aligned}\min_{x_1,x_2} Z_{DC} = \quad & \langle 15,1\rangle x_1+\langle 13,1\rangle x_2+\langle 3,1\rangle y_{11}+\langle 2,1\rangle y_{12}+\langle 3.5,0.5\rangle y_{21}+\langle 2.5,0.5\rangle y_{22}\\ \text{subject to} \quad & \langle 3,0.5\rangle x_1+\langle 2,0.5\rangle x_2 \leq \langle 500,50\rangle,\\ & x_1 \geq 100,\\ & x_2 \geq 100,\\ & y_{11}+y_{21} \leq x_1,\\ & y_{12}+y_{22} \leq x_2,\\ & x_1 \geq 0, x_2 \geq 0.\end{aligned} \tag{30}$$

A nonlinear programming problem is formulated by applying the developed procedure as discussed in

Section 3. The *IBLPP* (30) can be solved by solving the following non-linear programming problem:

$$\begin{aligned}
\min\ Z =& w(1.5x_1+2x_2+7y_{11}+3y_{12}+10y_{21}+6y_{22})+(1-w)(x_1+x_2+y_{11}+y_{12}+0.5y_{21}+0.5y_{22})\\
\text{subject to } & y_{11}+y_{12}+y_{21}+y_{22}\leq 500,\\
& 2y_{11}+y_{12}\leq 200, y_{21}+y_{22}\leq 250,\\
& y_{11}+y_{21}\leq x_1, y_{12}+y_{22}\leq x_2,\\
& 15+3\lambda_1-\lambda_2-\lambda_4=0,\\
& 13+2\lambda_1-\lambda_3-\lambda_5=0,\\
& \lambda_1(3x_1+2x_2-500)=0,\\
& \lambda_2(x_1-100)=0,\ \lambda_3(x_2-100)=0\\
& \lambda_4(y_{11}+y_{21}-x_1)=0\\
& \lambda_5(y_{12}+y_{22}-x_2)=0\\
& 3x_1+2x_2\leq 500, y_{11}+y_{21}\leq x_1,\\
& y_{12}+y_{22}\leq x_2;\ x_1\geq 100, x_2\geq 100;\ 0\leq w\leq 1;\\
& y_{11}\geq 0, y_{12}\geq 0, y_{21}\geq 0, y_{22}\geq 0.
\end{aligned}$$

The solution of the above problem can be obtained using any optimization software such as Lingo, Mathematica, Matlab, etc. Here we solve the above problem by Lingo software and obtain the solution as $x_1=100, x_2=100, y_{11}=50, y_{12}=100, y_{21}=50, y_{22}=0, \lambda_1=0, \lambda_2=0, \lambda_3=1.217516, \lambda_4=15, \lambda_5=11.78248$.

4.1. Result and Discussion

In Section 4, we successfully applied the developed algorithm to solve a real-life supply chain problem with an uncertain environment and obtained the following decisions:

50 tons of product A is produced in the plant P_1
100 tons product B is produced in the plant P_1
50 tons of product A is produced in the plant P_2
There is no production of product B in the plant P_2.
100 tons of product A's inventory is held in DC
100 tons of product B's inventory is held in DC
The minimum production cost for the production department lies in the closed interval $[1300, 1700]$.
The minimum distribution cost for the distribution center lies in the closed interval $[2950, 3700]$.

5. Conclusion

This paper is considered a general bi-level linear programming problem with coefficients in both objectives and constraints are closed intervals. The problem is transformed into a deterministic non-linear programming problem using partial ordering in terms of means and radius of the interval parameters. Further, the solution of the original problem is studied by solving the transformed deterministic programming problem. The developed approach may be applied to solve various real life management problems with uncertainty which contains two level hierarchy models. In light of the developments in this paper, we may develop a solution method for multi-objective *BLPP*s with interval uncertainty.

Compliance with ethical standards

Data availability: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest: The authors declare that they have no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Funding: This research has no funding from any organization or individual.

Acknowledgement

The authors are grateful to two anonymous reviewers for their valuable comments and suggestions made on the previous draft of this manuscript.

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