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Sub-coalitional egalitarian and solidarity values

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Abstract

In our sub-coalition approach, players and their blocs, termed sub-coalitions, form a grand coalition following the queue bargaining model. So, sub-coalitional versions of marginal contribution values need to consider the marginal contributions of blocs of players. This article is a novel contribution in two aspects. The first aspect consists of introducing a new sub-coalitional value. This value is obtained by a modification of the solidarity value, introduced by Nowak and Radzik in 1994, and an egalitarian way to divide the blocs' contributions. The second aspect consists of applying the queue bargaining model to provide new formulations for the sub-coalitional egalitarian Shapley value introduced in 2017 and the new sub-coalitional value. Additionally, using a combinatorial approach, we prove that the solidarity value can be calculated using the queue model where players share the marginal contributions with their predecessors, which is the same idea as in procedure 4 proposed by Malawski in 2013.

Keywords: *cooperative games, values, sub-coalitional values, Shapley value, solidarity value*

1. Introduction

This paper focuses on a specific group of one-point solution concepts from cooperative games—the sub-coalitional values. The notion of sub-coalitional value was first introduced by Stach in 2017 [20]. In the general sub-coalition approach, individual players and their pre-constituted blocs, termed sub-coalitions, either form a grand coalition or join one at a time in a random sequence. In the sub-coalition approach considered here, the grand coalition is formed through the successive addition of blocs of players, randomly one by one, including blocs composed of single players. An intriguing issue is how the players or blocs distribute their marginal contribution among their members. The method of dividing the marginal contribution of a sub-coalition among its members, once applied, leads to various sub-coalitional values.

This work continues the research on sub-coalitional values initiated by Stach [20], where some sub-coalitional values were introduced and analyzed, considering some general properties of the values.

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Among these sub-coalitional values, one—referred to as the egalitarian SC-Shapley value – employs an equal division of the marginal contributions of the blocs of players. In this paper, we propose a formula for this value, as it was introduced in [20] in a descriptive manner.

Here, we also propose a new sub-coalitional value based on the solidarity value, introduced by Nowak and Radzik in 1994 [16], and an egalitarian division of marginal contributions of blocs of players. We provide a formula for this new sub-coalitional value as well. It is named the sub-coalitional egalitarian solidarity value. To align with the name of the new sub-coalitional value, we have renamed the egalitarian sub-coalitional value introduced in [20] to the sub-coalitional egalitarian Shapley value. The rationale for this is that the order of words in the new name accurately describes the order of nesting of the sets of values used, from the largest to the smallest.

The formulas we propose for the sub-coalitional egalitarian values are based on the queue bargaining model of the Shapley value [18]. The formation of a grand coalition is realized through the random successive addition of individual players—the queue bargaining model (see also [3]), for example. The notation we use to define the formulas is inspired by that used by Felsenthal and Machover [3].

The queue bargaining model of the solidarity value [16] is very similar to that of the Shapley value. In Section 2.1, the differences between the Shapley value and the solidarity value, as well as their bargaining models, are explained in detail. Broadly speaking, the difference lies in how the marginal contribution of a singular player to each coalition is considered in calculating the final value assigned to players. For a more thorough comparison of the sub-coalition egalitarian values considered in this paper with their baseline values, we also provide formulas for the latter using the queue model. Particularly, in the formula based on the queue model for the solidarity value, players distribute their marginal contribution with their predecessors in each queue, leading to the grand coalition (see also [13]). Malawski in [13] introduced some procedural values with formulas for their calculations. The formula for Malawski's "procedure 4" leads to the solidarity value and is the same as ours but written using a different notation. Additionally, using a combinatorial approach, we demonstrate that the solidarity value can be calculated according to the provided formula here, as to the best of the authors' knowledge, this has not been done yet.

The solidarity value and the sub-coalitional egalitarian values considered here belong to a group of values that show solidarity with weaker players. All these values do not satisfy the null player property, as can be seen in Examples 2 and 3 in Section 4. Thus, potential applications of the new sub-coalitional value could be in the social context where goods are public, and stronger players donate a portion of their winnings to weaker ones. Further considerations related to solidarity solutions can be found in [23].

The rest of this paper is structured as follows: in Section 2, we introduce some basic definitions and notations of cooperative games, values, and sub-coalitional values. Here, using the queue bargaining model, we provide a formula for calculating the solidarity value [16] and demonstrate that calculations by this formula yield the same results as the original formula. Section 3 introduces two new formulas. One, in Section 3.1, for the sub-coalitional egalitarian value, and the second one, in Section 3.2, for the new sub-coalitional value introduced in this paper. Section 4 compares all considered values in theoretical examples and terms of some known and desirable properties in cooperative games defined in Section 2. Finally, Section 5 concludes the paper with some discussions and suggestions for further development.

2. Preliminaries on cooperative games and values

Let $N = \{1, 2, \ldots, n\}$ denote an arbitrary set of players, where $n = |N|$ stands for the size (cardinality) of N. Each subset S of N is called a coalition, and N is called the grand coalition. The set of all coalitions is denoted by 2^N. For brevity, if a coalition is denoted by a capital letter, the corresponding lowercase letter denotes the cardinality of the coalition, e.g., $s = |S|$.

A cooperative game is a pair (N, v) formed by a finite set N and a real-valued function $v : 2^N \to R$, called the characteristic function, with a requirement such that $v(\emptyset) = 0$. For each coalition, $S \in 2^N$, $v(S)$ is called the worth of a coalition, and it is interpreted as the payoff that can be achieved if the members of S cooperate. Let G_N be a set of all cooperative games on N.

Many solution concepts have been proposed for cooperative games. Among these concepts is a group of one-point solutions called values. A value is a real-valued function that assigns to a cooperative game (N, v) a unique vector $f(N, v) = (f_1(N, v), f_2(N, v), \ldots, f_n(N, v))$. The component $f_i(N, v)$ is a payoff/value that player $i \in N$ can expect to obtain playing the cooperative game (N, v). Since N does not change in the rest of the paper, we will write $f_i(v)$ instead of $f_i(N, v)$ hereafter.

Let us present below some known desirable properties of values.

- Efficiency property. In a cooperative game (N, v), the main issue is the sharing of the worth $v(N)$ of the grand coalition among the players. A value f is therefore said to be efficient if $\sum_{i \in N} f(v) = v(N)$ for all $v \in G_N$.
- Individual rationality property. A value f is said to be individual rational if, for each player $i \in N$ obtains at least their worth, i.e., $f(\{i\}) \geq v(\{i\})$.
- Null player property. A value f fulfills the null player postulate if it allocates zero payoff to all individuals $i \in N$ whose net contribution to any coalition is zero. Formally, for all games $v \in G_N$ $f(v) = 0$ if i is a null player in (N, v), i.e., if marginal contribution of player $i \in N$ to coalition S is $[v(S) - v(S \backslash \{i\})]$, for all coalitions $S \subseteq N \backslash \{i\}$.
- Symmetry (anonymity) property. A value f fulfills the symmetry postulate if, it assigns the same payoff/value to the "symmetric" players. Formally, for all games $v \in G_N$ and for each $i \in N$ and each permutation $Q : N \to N$ the following holds $f(v) = f_{Q(i)}(Q(v))$, where $Q(v)(S)$ $= v(Q^{-1}(S))$.

2.1. Some known values in cooperative games

Next, we define three well-known values that we use in Section 3 to construct the sub-coalitional values. The definitions of the values below are given for each cooperative game (N, v) and each player $i \in N$.

The egalitarian value *e* (also called the egalitarian division or egalitarian solution) is an efficient and symmetric value that assigns to each player $i \in N$ equal part of $v(N)$, i.e., $e_i(v) = \dfrac{v(N)}{n}$.

The Shapley value (σ), introduced by Shapley in 1953 [18], is defined as follows:

$$\sigma_i(v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!(s-1)!}{n!} \big(v(S) - v(S \setminus \{i\})\big) \tag{1}$$

Considering the queue bargaining model of the Shapley value [18] or of Shapley and Shubik [19] (see [3, p. 182], for example, an alternative equivalent formula for the Shapley value exists. Before presenting this formula, let us introduce some notions.

Let N be a finite set of n players. A permutation (queue or sequential coalition) Q of N is any bijection from N to the set $\{1, 2, \ldots, n\}$. $Q(N)$ is a set of all permutations of N. In each singular permutation Q, the order players are listed reflects the order they joined the coalition. By Q_i we denote the place of i in Q. So, Q_i is a positive integer and $1 \leq Q_i \leq n$. Next, we define $h_iQ = \{j \in N : Q_j \leq Q_i\}$ as a head of i in Q. It means that $h_iQ \in 2^N$ is a coalition with i and those players in N placed ahead of i in Q.

Following the queue bargaining model, we can calculate the Shapley value [18] as follows:

$$\sigma_i(v) = \frac{1}{n!} \sum_{Q \in Q(N)} (v(h_iQ) - v(h_iQ \backslash \{i\})) \tag{2}$$

where the sum ranges over all $n!$ orders $Q \in Q(N)$.

The solidarity value (ψ), introduced and characterized axiomatically by Nowak and Radzik [16], is defined as follows:

$$\psi_i(v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!(s-1)!}{n!} A^v(S) \tag{3}$$

where $A^v(S)$ is the average marginal contribution of a member of a coalition S and it is defined as follows:

$$A^v(S) = \frac{1}{s} \sum_{i \in S} (v(S) - v(S \backslash \{i\})$$

for each coalition $S \in 2^N$.

Note that how players form the grand coalition in the bargaining model of the solidarity value is the same as that of the Shapley value [18]. This means, loosely speaking, that the formation of the grand coalition N starts with a single player, and then the coalition adds one player at a time until everyone has been admitted. Players join the coalition in a random order, and all $n!$ sequences are equally probable.

The difference between the two values lies in how the grand coalition's total win is divided among the players. In the Shapley value bargaining model, each player, upon their admission to the group, is promised the amount their adherence to the group contributes to the value of the coalition (as determined by the characteristic function), i.e., the so-called marginal contribution of a player. The Shapley value assigns each player their average marginal contribution across all coalitions to which they belong. Meanwhile, the solidarity value assigns each player the sum of the coalition members' average marginal contribution across all coalitions to which the player belongs. Thus, the foundation of the solidarity value is to be in solidarity with the weaker players. In other words, if a player become a member of coalition S, then he obtains $A^v(S) = \frac{1}{s} \sum_{i \in S} (v(S) - v(S \backslash \{i\})$—the average marginal contribution of a member of S.

Hereafter, we provide another formula for calculating the solidarity value, but first, let us introduce further notation. Consider the set of all permutations of N—$Q(N)$. For each $Q \in Q(N)$ and $i \in N$, let us define $t_iQ = \{j \in N : Q_j \geq Q_i\}$ as a tail of i in Q. It means that $t_iQ \in 2^N$ is a coalition with i and those players in N placed behind of i in Q.

If we follow the queue bargaining model of Shapley and Shubik [19] (see [3]) as well, we can calculate the solidarity value as follows:

$$\psi_i(v) = \frac{1}{n!} \sum_{Q \in Q(N)} \sum_{k \in t_i Q} \frac{v(h_k Q) - v(h_k Q \backslash \{k\})}{|h_k Q|} \tag{4}$$

where $\frac{v(h_k Q) - v(h_k Q \backslash \{k\})}{n!}$ is the probability that a given coalition $S = h_k Q$ will form, i.e., the probability that S is the head of a player in position k in a queue (see also [3]). Note that, in the Shapley value, the full worth of a marginal contribution $v(h_k Q) - v(h_k Q \backslash \{k\})$ is assigned to one player, k, whose joining constitutes coalition $h_k Q$, whereas, in the solidarity value, this amount is divided equally among all members of $h_k Q$. So, each member of $h_k Q$ obtains $\frac{v(h_k Q) - v(h_k Q \backslash \{k\})}{|h_k Q|}$ and the remaining players in Q obtain zero.

Note that Malawski, in 2013 [13], using a slightly different notation provided a general formula for procedural values, where changing a coefficient results in a different procedural value. Specifically, Procedure 4 leads to the calculation of the solidarity value according to formula (4) as well.

In the next section, using a combinatorial approach, we demonstrate that calculations of the solidarity value according to formulas (3) and (4) lead to the same results, as seen in Proposition 1. The rationale for presenting the demonstration in Section 2.2, rather than immediately, is that the considerations in the proof become clearer after the presentation of the solidarity value calculation in Example 1. Example 1 is provided there, in Section 2.2.

2.2. The Shapley and solidarity values in comparison

In this section, we calculate and compare the Shapley and solidarity values in Example 1. The calculations for both values are based on the queue bargaining models, i.e., using formulas (2) and (4). The reason for this is to facilitate a better understanding of the differences between them and their sub-coalitional egalitarian versions defined in Section 3.

Example 1. Let us consider a three-player cooperative game given by the following characteristic form: $v(\varnothing) = 0$, $v(\{1\}) = 1$, $v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = 4$, $v(\{1,3\}) = 3$, $v(\{2,3\}) = 2$, $v(\{1,2,3\}) = 5$.

Table 1. The Shapley value calculation according to the queue model in Example 1

Q order of players	Coalitions $S = h_i Q$ in Q, $i = 1,2,3$	Player i's marginal contributions to coalition S $i = 1$	$i = 2$	$i = 3$
(1, 2, 3)	{1}, {1, 2}, {1, 2, 3}	1	3	1
(1, 3, 2)	{1}, {1, 3}, {1, 2, 3}	1	2	2
(2, 1, 3)	{2}, {1, 2}, {1, 2, 3}	4	0	1
(2, 3, 1)	{2}, {2, 3}, {1, 2, 3}	3	0	2
(3, 1, 2)	{3}, {1, 3}, {1, 2, 3}	3	2	0
(3, 2, 1)	{3}, {2, 3}, {1, 2, 3}	3	2	0
	Total	15	9	6
	Shapley value	15/6 = 2.5	9/6 = 1.5	6/6 = 1

Table 1 shows the Shapley value calculations in Example 1. The first column provides the order in which players join in forming the grand coalition N. The last three columns provide, for each order and for each player, the amounts contributed to the value of the coalition by the players' joining the group.

Table 2 provides the solidarity value calculations according to the queue bargaining model in Example 1. We can find, for each player in each order of formation grand coalition, all sum components of the average marginal contribution of a coalition member (formula (4)).

Table 2. The solidarity value calculation according to the queue model in Example 1

Q order of players	Coalitions $S = h_i Q$ in Q, $i = 1, 2, 3$	Sum of fractions/summands of average marginal contributions $A^v(S)$ assigned to player i		
		$i = 1$	$i = 2$	$i = 3$
(1, 2, 3)	{1}, {1, 2}, {1, 2, 3}	1 + 3/2 + 1/3	3/2 + 1/3	1/3
(1, 3, 2)	{1}, {1, 3}, {1, 2, 3}	1 + 1 + 2/3	2/3	1 + 2/3
(2, 1, 3)	{2}, {1, 2}, {1, 2, 3}	2 + 1/3	0 + 2 + 1/3	1/3
(2, 3, 1)	{2}, {2, 3}, {1, 2, 3}	1	0 + 1 + 1	1 + 1
(3, 1, 2)	{3}, {1, 3}, {1, 2, 3}	3/2 + 2/3	2/3	0 + 3/2 + 2/3
(3, 2, 1)	{3}, {2, 3}, {1, 2, 3}	1	1 + 1	0 + 1 + 1
	Total	12	19/2	17/2
	Solidarity value	12/6 = 2	19/12 = 1.58(3)	17/12 = 1.41(6)

In Example 1, we can observe that the solidarity value is solidaric with weaker players. Player 1 is the strongest player, and Player 3 is the weakest one, according to both the characteristic function and the Shapley value (Table 1). By the definition of the solidarity value, players who contribute to a coalition S more than the average marginal contribution, in some sense, support the "weaker" partners in S, as shown in formula (3). As a result, Player 1 helps both Players 2 and 3, but offers more support to Player 3.

Before we demonstrate that the solidarity value can be calculate following formula (4), in Table 3, we present calculations of this value according to original formula (3) in Example 1.

Table 3. The solidarity value calculation according to original formula (3) in Example 1

Coalition S	$A^v(S)$	$v(S) - v(S\backslash\{i\})$			Coefficient	$A^v(S)$ assigned to a player and shown with summand that it consists of		
		$i = 1$	$i = 2$	$i = 3$	$(n-s)!(s-1)!$	$i = 1$	$i = 2$	$i = 3$
{1}	1	1			2	1		
{2}	0		0		2		0	
{3}	0			0	2			0
{1, 2}	7/2	4	3		1	(4 + 3)/2	(4 + 3)/2	
{1, 3}	5/2	3		2	1	(3 + 2)/2		(3 + 2)/2
{2, 3}	4/2		2	2	1		(2 + 2)/2	(2 + 2)/2
{1, 2, 3}	6/3	3	2	1	2	(3+2+1)/3	(3+2+1)/3	(3+2+1)/3
Total (considering multiplication by a coefficient)						12	19/2	17/2
Solidarity value						12/6	19/12	17/12

As we can observe, the payoff assigned to players by formulas (3) and (4) of the solidarity value in Example 1 is the same (Tables 2 and 3).

Now, let us present a demonstration that the solidarity value, calculated according to the queue bargaining model (formula (4)) yields the same results as those calculated by formula (3) in the general case.

Proposition 1. For each game (N, v) and player i in N, the value obtained by the queue bargaining model (formula (4)) is the solidarity value defined by formula (3).

Proof. Let fix a game (N,v). Let denote by $A_i^v(S) = \dfrac{v(S) - v(S\backslash\{i\})}{s}$ the player i's contribution to the average marginal contribution $A^v(S)$, $i \in S \subseteq N$. Observe, that by the definition, the average marginal contribution to coalition $S \subseteq N$, $A^v(S)$, is calculated as the sum of the marginal contribution of players divided by the cardinality of S. So, $A^v(S) = \sum_{i \in S} A_i^v(S)$. For example, for coalition $S = \{1,2,3\}$, $s = 3$, $A^v(S) = \dfrac{(v(S) - v(S\backslash\{1\}))}{3} + \dfrac{(v(S) - v(S\backslash\{2\}))}{3} + \dfrac{(v(S) - v(S\backslash\{3\}))}{3}$, what in Example 1 is equal to $A^v(N) = \dfrac{3}{3} + \dfrac{2}{3} + \dfrac{1}{3}$. The contribution of Player 1 to average marginal contribution to coalition N is $A_i^v(N) = \dfrac{v(N) - v(\{2,3\})}{3} = \dfrac{3}{3}$, $A_2^v(N) = \dfrac{2}{3}$, and $A_3^v(N) = \dfrac{1}{3}$, see Table 3. Hence, when calculating the solidarity value for a player $i \in N$ by formula (3), for every coalition $S \ni i$ and each of its members $k \in S$, $A_k^v(S)$ appears $(n-s)!(s-1)!$ times (as a component of $A^v(S)$), as the formula itself indicates.

Thus, in order to prove Proposition 1, it is sufficient to show that, using the queue bargaining model and Malawski's Procedure 4 [13], for player $i \in N$ and all coalitions $S \subseteq N$ with him, $i \in S$, the number of all parts assigned to player i, i.e., $A_k^v(S)$, where $S = h_kQ$, $k \in t_iQ$, over all orders of players $Q \in Q(N)$, is equal to $(n-s)!(s-1)!$

Below, step by step, we give all observations that helps to prove Proposition 1.

1. When we calculate the value using formula (4), we regard all possible permutations of players. For $|N| = n$, we have $n!$ permutations—queues of players.
2. For each permutation (order of players), we consider formation of n coalitions. We start with empty coalitions, then the first player arrives forming a coalition of size 1. Subsequently, the second player joins the first one, forming a coalition of size 2, and so on, until the last player joins the previous players, in the order, forming coalitions of size n, denoted as N.
3. The above implies that, if $n > 2$, then each non-empty coalition $S \subseteq N$ appears more than one time considering all possible queues of player—permutations. In particular, each coalition S (of size s) is considered $(n-s)!s!$ times over all $n!$ queues of players, which is not difficult to show. Namely, when we choose s players, these s players can form $s!$ queues/orders, then the remaining players can create $(n-s)!$ orders. For example, coalition $\{1,2\}$ is regarded $(n-s)!s! = (3-2)!2! = 2$ times, in Example 1. Specifically, $\{1,2\}$ appears, for the first time, considering permutation $(1,2,3)$ and, for the second time, considering permutation $(2,1,3)$, see second columns in Tables 1 and 2.
4. For each coalition $S = h_iQ$, formed by the addition of player $i \in N$, in formula (4), the marginal contribution $\dfrac{v(h_iQ) - v(h_kQ\backslash\{i\})}{|h_iQ|}$ is calculated only for one player—the last player in the queue /order that constituted coalition S—i. This contribution is nothing more than the contribution $A_i^v(S)$ $= \dfrac{v(S) - v(S\backslash\{i\})}{s}$ to the average marginal contribution $A^v(S)$. For example, in queue $(1,2,3)$ when Player 2 joins Player 1 coalition $\{1,2\}$ is formed and we calculate the average contribution of Player 2 only, which is equal to $(4-1)/2 = 3/2 = A_2^v(\{1,2\})$, see Table 2. Then, this value is assigned to all members of S. Hence, also Player 1 obtains this value $A_2^v(\{1,2\})$. In permutation $(2,1,3)$, when Player 1 joins Player 2, a coalition $\{1,2\}$ is formed and we calculate the average marginal contribution of Player 1 only, $(4-0)/2 = 4/2 = 2 = A_1^v(\{1,2\})$, see Table 2. Concluding,

fixing a coalition S of size s, this coalition appears $(n-s)!s!$ times in all permutations, and each time a marginal contribution is calculated for one player only. Then, this marginal contribution is divided by s and the result $(A_k^v(S))$ is assigned to all members of S. In all $(n-s)!s!$ cases, the marginal contribution of a particular member of S is considered only $(n-s)!s!/s = (n-s)!(s-1)!$ times, as the number $(n-s)!(s-1)!$ indicates the number of times that a chosen player is on the last position in the queue constituting coalition S. Thus, in total, each member of S obtains $(n-s)!(s-1)!\sum_{k\in S} A_k^v(S) = (n-s)!(s-1)!A^v(S)$.

Thus, starting from formula (4), we have

$$\psi_i(v) = \frac{1}{n!}\sum_{Q\in Q(N)}\sum_{k\in t_iQ}\frac{v(h_kQ)-v(h_kQ\backslash\{k\})}{|h_kQ|}$$

$$= \frac{1}{n!}\sum_{S\ni i}(n-s)!(s-1)!\sum_{k\in S\ni i}A_k^v(S) = \sum_{S\ni i}\frac{(n-s)!(s-1)!}{n!}A^v(S).$$

This ends the proof. The method of calculation bases on the queue bargaining model (formula (4)) leads to formula (3) of the solidarity value. $\square$

2.3. Sub-coalitional approach to values based on ordered partitions

There are two types of sub-coalitional values introduced in [20]. One group of sub-coalitional values is based on ordered partitions of grand coalition N and the second group on partitions of N. The sub-coalitional approach of the first group assumes that all ordered partitions of N are possible, and the grand coalition is formed by joining an individual or group of players. Below we provide the formal definition of the ordered partition of the grand coalition.

An ordered partition $\pi = (S_1, \ldots, S_k)$ of grand coalition N into k groups (i.e., blocs of players called by us sub-coalitions) where $2 \le k \le n$ is any permutation of k non-empty subsets $S_j \subset N$, $j \in \{1, 2, \ldots, k\}$ such that every member of N belongs to one and only one of these subsets. In other words, permutation $(S_1, \ldots, S_k)$, is an ordered partition of N if the following holds:

1. $S_j \neq \varnothing$ for all $1 \le j \le k$.
2. $S_1 \cup S_2 \cup \ldots \cup S_k = N$.
3. $S_i \cap S_n = \varnothing$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$.

Of course, the arrangement of blocs $S_j \in (2^N\backslash\varnothing)$ for all $1 \le j \le k$ in $(S_1, \ldots, S_k)$ is important. The number of ways to arrange a set of sub-coalitions $\{S_1, \ldots, S_k\}$ in a specific order is equal to $k!$

By $\Pi_k(N)$ we denote the set of all ordered partitions of N into k blocs and by $\Pi(N)$ the set of all possible ordered partitions of N. If $|N| = n$, then the number of all possible ordered partitions of N is equal to

$$|\Pi(N)| = \sum_{k=2}^{n}|\Pi_k(N)|$$

So, for $n = 2$, we have two different ordered partitions: $(\{1\},\{2\})$ and $(\{2\},\{1\})$.

For $n = 3$, there is twelve ordered partitions. In particular, $|\Pi_2(N)| = |\Pi_3(N)| = 6$ and $\Pi(N) = \{(\{1\}, \{2\}, \{3\}), (\{1\}, \{3\}, \{2\}), (\{2\}, \{1\}, \{3\}), (\{2\}, \{3\}, \{1\}), (\{3\}, \{1\}, \{2\}), (\{3\}, \{2\}, \{1\}), (\{1\}, \{2,3\}), (\{2,3\}, \{1\}), (\{2\}, \{1,3\}), (\{1,3\}, \{2\}), (\{3\}, \{1,2\}), (\{1,2\}, \{3\})\}$.

In general, the number of ways to partition a set of n elements into k non-empty subsets is given by so--called Stirling number of the second kind and expressed by the formula $S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n$ (see, for example, [9, p. 824–825]). Thus, $|\Pi_k(N)| = k!S(n,k)$. Because for $j = k$ and $n \geq 2$ $(k-j)^n = 0$, we have

$$|\Pi_k(N)| = \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}(k-j)^n \quad \text{and} \quad |\Pi(N)| = \sum_{k=2}^{n}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}(k-j)^n$$

3. The sub-coalitional egalitarian values

In this section, we revisit the procedure of the sub-coalitional egalitarian Shapley value introduced by Stach (2017) [20]. We then provide a formula for this value, which represents a novel contribution of this paper (see Section 3.1).

Section 3.2 introduces a new value as a modification/extension of the solidarity value introduced by Nowak and Radzik [16]. Specifically, we outline a three-step procedure to calculate this value and subsequently propose a formula for what we have named the sub-coalitional egalitarian solidarity value.

Let us introduce the notation necessary in Sections 3.1 and 3.2 to define both sub-coalitional values already mentioned. Let us fix a game (N, v), an ordered partition of N, $\pi = (S_1, \ldots, S_k) \in \Pi(N)$ where $2 \leq k \leq n$, and a player $i \in N$. We denote the place of a bloc S in π by $p_S\pi$. Let us denote the place of the bloc with player i in π by πi. Thus, $p_{S_{\pi i}}\pi = \pi i$. For not create so many levels in describing the place of bloc with player i, we shortly denote the bloc with i by iS. A coalition $h_S\pi$ is called a *head* of bloc S in an ordered partition $\pi = (S_1, \ldots, S_k)$, where $2 \leq k \leq n$, if $h_S\pi = \{j \in N : j \in \bigcup_{l=1}^{p_S\pi} S_l$, $\pi = (S_1, \ldots, S_k)$, $2 \leq k \leq n\}$. This coalition consists of members of bloc S and members of those blocs in π placed ahead of bloc S in π. A tail of a bloc S in an ordered partition $\pi = (S_1, \ldots S_k)$, where $2 \leq k \leq n$, is defined as follows $t_S\pi = \{j \in N : j \in \bigcup_{l=p_S\pi}^{k} S_l$, $\pi = (S_1, \ldots, S_k)$, $2 \leq k \leq n\}$. The $t_S\pi$ coalition consists of those blocs in π placed behind S.

3.1. The sub-coalitional egalitarian Shapley value

The following three-step algorithm calculates the sub-coalitional egalitarian Shapley value.

Step 0. Select a player $i \in N$.

Step 1. For each ordered partition of N—$\pi = (S_1, \ldots, S_k)$—that consists of k blocs of players $(k = 2, \ldots, n)$ fix the bloc with player i in π by i–S.

Step 2. Calculate the marginal contribution of a bloc selected in step 1 (bloc iS): $v(h_{iS}\pi)-v(h_{iS}\pi\backslash iS)$. Then divide equally the marginal contribution of bloc iS among its members. Then, assign to player i the following amount $\dfrac{v(h_{iS}\pi)-v(h_{iS}\pi\backslash iS)}{|iS|}$.

Step 3. Sum up all values assigned to player i over all ordered partitions and then divide the result by $|\Pi(N)|$ to get the sub-coalitional egalitarian Shapley value of player i.

Proposition 2. The sub-coalitional egalitarian Shapley value (σ^e) for each cooperative game (N,v) and a player $i\in N$ is given as follows:

$$\sigma_i^e(v)=\sum_{\pi\in\Pi(N)}\left(\frac{v(h_{iS}\pi)-v(h_{iS}\pi\backslash iS)}{|\Pi(N)|\cdot|iS|}\right) \tag{5}$$

Proof. From the three-step algorithm given above, we immediately obtain (5), which is what is needed to be proven. □

3.2. The sub-coalitional egalitarian solidarity value

The following three-step algorithm calculates the sub-coalitional egalitarian solidarity value.

Step 0. Select a player $i\in N$.

Step 1. For each ordered partition of N—$\pi=(S_1,\dots,S_k)$—that consists of k blocs of players ($k=2,\dots,n$), fix the bloc with player i in π–iS.

Step 2. Starting from bloc iS to S_k, i.e., for each bloc T in tail of iS ($T\in t_{iS}\pi$), calculate the marginal contribution of a bloc selected, $v(h_T\pi)-v(h_T\pi\backslash T)$, and distribute this value equally among the blocs placed ahead of bloc T in π, i.e., among blocs in head of $T(h_T\pi)$. So, the bloc with player i obtains $\dfrac{v(h_T\pi)-v(h_T\pi\backslash T)}{p_T\pi}$ as iS belongs to this group of blocs. Then, the amount assigned to iS is distributed equally among its members. Total all values assigned to player i over all blocs in $t_{iS}\pi$: $\displaystyle\sum_{T\in t_{iS}\pi}\frac{v(h_T\pi)-v(h_T\pi\backslash T)}{p_T\pi\cdot|iS|}$.

Step 3. Sum up all values assigned to player i over all ordered partitions and the amount obtained then divide by $|\Pi(N)|$. The value obtained is the sub-coalitional solidarity egalitarian value of player i.

Proposition 3. The sub-coalitional egalitarian solidarity value (ψ^e) for each cooperative game (N,v) and a player $i\in N$ is given as follows:

$$\psi_i^e(v)=\sum_{\pi\in\Pi(N)}\sum_{T\in t_{iS}\pi}\frac{v(h_T\pi)-v(h_T\pi\backslash T)}{|\Pi(N)|\cdot p_T\pi\cdot|iS|} \tag{6}$$

Proof. From the three-step algorithm given above, we immediately obtain (6), which is what needs to be proven. □

4. Comparison of sub-coalitional egalitarian values

Let us start with the comparison of both sub-coalitional values considered in Section 3, i.e., the sub-coalitional egalitarian Shapley value and the sub-coalitional egalitarian solidarity value in Example 1. Tables 4 and 5 present the calculations for Example 1.

Table 4. The sub-coalitional egalitarian Shapley value calculation in Example 1

π ordered partition	Part of contribution of iS to $h_{iS}\pi$ assigned to player i		
	$i=1$	$i=2$	$i=3$
({1}, {2}, {3})	1	3	1
({1}, {3}, {2})	1	2	2
({2}, {1}, {3})	4	0	1
({2}, {3}, {1})	3	0	2
({3}, {1}, {2})	3	2	0
({3}, {2}, {1})	3	2	0
({1}, {2, 3})	1	2	2
({2, 3}, {1})	3	1	1
({2}, {1, 3})	5/2	0	5/2
({1, 3}, {2})	3/2	2	3/2
({3}, {1, 2})	5/2	5/2	0
({1, 2}, {3})	2	2	1
Total	55/2	37/2	28/2
σ^e	$55/24 = 2.29167$	$37/24 = 1.54167$	$28/24 = 1.16667$

Table 5 provides the sub-coalitional egalitarian solidarity value calculations according to the queue bargaining model in Example 1. In particular, we can find, for each player in each ordered partition of grand coalition, all sum components of average marginal contributions assigned to each player (see also formula (6)).

Table 5. The sub-coalitional egalitarian solidarity value calculated in Example 1

π ordered partition	Sum of fractions/summands of average marginal contributions $A^v(S)$ assigned to player i		
	$i=1$	$i=2$	$i=3$
({1}, {2}, {3})	$1+3/2+1/3$	$3/2+1/3$	1/3
({1}, {3}, {2})	$1+1+2/3$	2/3	$1+2/3$
({2}, {1}, {3})	$2+1/3$	$0+2+1/3$	1/3
({2}, {3}, {1})	1	$0+1+1$	$1+1$
({3}, {1}, {2})	$3/2+2/3$	2/3	$0+3/2+2/3$
({3}, {2}, {1})	1	$1+1$	$0+1+1$
({1}, {2, 3})	$1+2$	1	1
({2, 3}, {1})	3/2	$1+3/4$	$1+3/4$
({2}, {1, 3})	5/4	$0+5/2$	5/4
({1, 3}, {2})	$3/2+1/2$	1	$3/2+1/2$
({3}, {1, 2})	5/4	5/4	$0+5/2$
({1, 2}, {3})	$2+1/4$	$2+1/4$	1/2
Total	23.25	19.25	17.50
ψ^e	1.9375	1.6042	1.4583

Both sub-coalitional egalitarian values are more solidaric with weaker players than the original values (Tables 1, 2, 4, and 5). This result was expected, as in both cases an egalitarian division was used to distribute the sub-coalition's contributions.

Two other examples of games are considered in [20]. We recall these examples (Examples 2 and 3) to add observations about the new sub-coalitional value introduced in this paper—ψ^e.

Example 2. Let us consider a three-player cooperative game given by the following characteristic form: $v(\varnothing) = 0$, $v(\{1\}) = v(\{1,3\}) = 1$, $v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0$, $v(\{1,2\}) = v(\{1,2,3\}) = 4$.
In Example 2, Player 3 is a null player. The sub-coalitional egalitarian Shapley value is equal to $\sigma^e = (53/24,\ 35/24,\ 8/24)$, while the sub-coalitional egalitarian solidarity value is equal $\psi^e = (245/144,\ 197/144,\ 67/72)$, see Table 6, where these results are compared with the Shapley and solidarity values.

Table 6. The sub-coalitional and classical values in Example 2

Value	Player payoff		
	1	2	3
The Shapley value	2.500	1.500	0
The solidarity value	1.778	1.361	0.861
The sub-coalitional egalitarian Shapley value	2.208	1.458	0.333
The sub-coalitional egalitarian solidarity value	1.701	1.368	0.931

Example 3. Let us consider a three-player cooperative game given by the following characteristic form: $v(\varnothing) = 0$, $v(\{1\}) = v(\{1,3\}) = 1$, $v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0$, $v(\{1,2\}) = v(\{1,2,3\}) = 1$.

Player 1 is a necessary member to make a coalition win 1. Players 2 and 3 contribute nothing to any coalition, so their position in each coalition is symmetric and null. The sub-coalitional egalitarian Shapley value is equal to $\sigma^e = (10/12,\ 1/12,\ 1/12)$, while the solidarity value is equal $\psi^e = (20/36,\ 8/36,\ 8/38)$ (see Table 7). The table also includes calculations of the Shapley and solidarity values to facilitate a comparison of all values considered in this paper.

Table 7. The sub-coalitional and classical values in Example 3

Value	Player payoff		
	1	2	3
The Shapley value	1	0	0
The solidarity value	0.611	0.194	0.194
The sub-coalitional egalitarian Shapley value	0.833	0.083	0.083
The sub-coalitional egalitarian solidarity value	0.556	0.222	0.222

Both sub-coalitional egalitarian values considered in this paper (i.e., σ^e and ψ^e) satisfy the symmetry and efficiency properties by definition, but do not satisfy the null player property, as the solidarity value, see Examples 2 and 3 and Tables 6 and 7. Moreover, in comparison with the Shapley value, the individual rationality property is violated for both sub-coalitional values, σ^e and ψ^e , and for the solidarity values as well, see Example 3 and Table 7.

5. Conclusions

One of the novel contributions of this paper is the proposal of a new sub-coalitional value with a formula for its calculation. Specifically, the paper introduces the sub-coalitional egalitarian solidarity value ψ^e

as a modification of the solidarity value proposed by Nowak and Radzik [16], and offers an application of the egalitarian value for the division of marginal contributions of blocs of players (Section 3.2 and formula (6)).

We also present a new formula for the sub-coalitional egalitarian Shapley value following the queue bargaining model proposed by Shapley and Shubik [19] (Section 3.1 and formula (5)).

Another novel contribution is a demonstration, using the combinatorial approach, that formula (4) leads to the calculation of the solidarity value (formula (3)).

An interesting issue that could be explored more deeply in further research are the advantages and disadvantages of the two proposed sub-coalitional values (ψ^e and σ^e) in real-life applications compared to each other and other solidarity measures. Now, based on the results obtained in this paper, we can suggest some directions for further developments in this regard.

Stach and Bertini [23], in a paper devoted to some solidarity measures for sharing (public) goods among members with different participation quotas in a binary decision-making process, compare them considering some properties in simple games. Among these measures, the solidarity value [16] was regarded as well. It would be interesting to extend this comparison to the new sub-coalitional values ψ^e and σ^e and observe which of the considered properties in [23] are satisfied and which are lost compared to the original values ψ and σ. It could help to better understand the advantages and disadvantages of the proposed values in real-life applications. Certainly, an advantage of the proposed values in real-life applications is that these new values picture all possible formation of grand coalition (N) giving the same chance to all possible divisions of N, it means permutations of blocs. So consequently, as also observed in all three examples of this paper, the new sub-coalitional values are more solidary towards weaker players than their original values. Thus, the potential applications of the two sub-coalitional values, ψ^e and σ^e, could be in the social context where stronger agents donate a portion of their winnings to weaker ones.

σ^e loses the dummy player property satisfied by σ. ψ and ψ^e do not fulfil the dummy player postulate as well. The portion of win assigned to a dummy/weak player could give an idea about the "cost" of keeping him in blocs/sub-coalitions. Both sub-coalitional values maintain the symmetry and efficiency property which do not eliminate a lot of possible real-life applications of these values.

It would also be worth expressing these new sub-coalitional values using the null player free winning coalitions, as it was made for some solidarity values in simple games [22]. This also could help with the acquisition of knowledge on these sub-coalitional values.

In this paper, we concentrate on the formulas and not practical applications. There is still a lot to be explored in this regard. For example, the disadvantage of the proposed values in real applications is their calculation in games with many players. According to the sub-coalition approach considered here, for example, for six players there are 4,682 possibilities to form a grand coalition (Section 2.3). So, the elaboration of efficient algorithms would help to facilitate the calculation.

The research presented in this paper could be further developed. Below we list some other potential ideas:

- Comparing new proposed sub-coalitional value taking into account some known properties of values or *a priori* unions of coalitions [2, 17].

- The formulas for the sub-coalitions proposed by Stach [20]. In this paper, we proposed the formulas only for the σ^e value and the new proposed value ψ^e.
- Modification of further values witha a sub-coalitional approach, e.g., the public good index of Holler [11, 12].
- Extending a sub-coalitional approach to games modeling voting rules with abstention could be a good idea [4–6], for example.
- Another very interesting aspect to examine is the sub-coalitional approach to the so-called probabilistic power indices. In particular, take into account that the power of any sub-coalition could significantly depend on the size of such a sub-coalition and its internal cohesion (measured by the probability that a member of the sub-coalition follows the leader) [7, 8, 14].
- As was just mentioned by Stach [20] and still not done, it would be interesting to study the relationship between the sub-coalitional values and the interaction index introduced by Grabisch and Roubens in 1999 [10].
- Regarding the efficient computation of the proposed sub-coalitional values, an approach like the one introduced by Staudacher et al. [27], i.e., applying the dynamic programming, could be considered.
- Referring to applications of sub-coalitional values, one application could be assessing the power of firms in complex shareholding structures, i.e., to measure indirect control power, as indicated, for example in [1, 15, 21, 24–26].

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References

[1] Bertini, C., Mercik, J., and Stach, I. Indirect control and power. *Operations Research and Decisions 26*, 2 (2016), 7–30.

[2] Calvo, E., and Gutiérrez, E. The Shapley-solidarity value for games with a coalition structure. *International Game Theory Review 15*, 1 (2013), 1350002.

[3] Felsenthal, D. S., and Machover, M. L. *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes*. Edward Elgar, Cheltenham, 1998.

[4] Freixas, J. Probabilistic power indices for voting rules with abstention. *Mathematical Social Sciences 64*, 1 (2012), 89–99.

[5] Freixas, J. The Banzhaf value for cooperative and simple multichoice games. *Group Decision and Negotiation 29*, 1 (2020), 61–74.

[6] Freixas, J., and Pons, M. An appropriate way to extend the Banzhaf index for multiple levels of approval. *Group Decision and Negotiation 30*, 2 (2021), 447–462.

[7] Gehrlein, W. V., and Fishburn, P. C. Division of power in legislatures with two cohesive subgroups. *Social Choice Welfare 3*, 2 (1986), 119–124.

[8] Gehrlein, W. V., Ord, J. K., and Fishburn, P. C. The limiting distribution of a measure of the voting power of subgroups. *Communications in Statistics - Simulation and Computation 15*, 2 (1986), 571–577.

[9] Goldberg, K., Newman, M., and Haynsworth E. Stirling numbers of the second kind. In *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1972), M. Abramowitz and I. A. Stegun, Eds., U.S. Government Printing Office, pp. 824–825.

[10] Grabisch, M., and Roubens, M. An axiomatic approach to the concept of interaction among players in cooperative games. *International Journal of Game Theory 28*, 4 (1999), 547–565.

[11] Holler, M. J. Forming coalitions and measuring voting power. *Political Studies 30*, 2 (1982), 262–271.

[12] Holler, M. J. The public good index: A brief introduction. In *Finnish-German Yearbook of Political Economy*, vol. 1, (München, 2018), T. Airaksinen, M. J. Holler and H. Nurmi, Eds., (2018), Accedo Verlag Holler, pp. 31–39.

[13] Malawski, M. "Procedural" values for cooperative games. *International Journal of Game Theory 42*, 1 (2013), 305–324.

[14] Mercik, J. M. Index of power for cabinet. *Homo Oeconomicus 17* (2000), 125–136.

[15] Mercik, J., and Stach, I. On Measurement of Control in Corporate Structures. In *Transactions on Computational Collective Intelligence XXXI* (Berlin, 2018), N. T. Nguyen, R. Kowalczyk, J. Mercik and A. Motylska-Kuźma, Eds., vol. 11290 of *Lecture Notes in Computer Science*, Springer, pp. 64–79.

[16] Nowak, A. S., and Radzik, T. A solidarity value for *n*-person transferable utility games. *International Journal of Game Theory 23*, 1 (1994), 43–48.

[17] Owen, G. Values of games with a priori unions. In *Mathematical Economics and Game Theory: Essays in Honor of Oskar Morgenstern* (Heidelberg, 1977), R. Henn and O. Moeschlin, Eds., vol. 141 of *Lecture Notes in Economics and Mathematical Systems* Springer, pp. 76–88.

[18] Shapley, L. S. A value for *n*-person games. In *Contributions to the Theory of Games II* (Princeton, 1953), H. W. Kuhn and A. W. Tucker, Eds., vol. 28 of *Annals of Mathematics Studies* Princeton University Press, pp. 307–317.

[19] Shapley, L. S., and Shubik, M. A method for evaluating the distribution of power in a committee system. *American Political Science Review 48*, 3 (1954), 787–792.

[20] Stach, I. Sub-coalitional approach to values. In *Transactions on Computational Collective Intelligence XXVII* (Cham, 2017), N. T. Nguyen and J. M. Kowalczyk and J. Mercik, Eds., vol. 10480 of *Lecture Notes in Computer Science*, Springer, pp. 74–86.

[21] Stach, I. Indirect control of corporations: analysis and simulations. *Decision Making in Manufacturing and Services 11*, 1-2 (2017), 31–51.

[22] Stach, I., and Bertini, C. Reformulation of some indices using null player free winning coalitions. In *Transactions on Computational Collective Intelligence XXXVI* (Berlin, 2021), N. T. Nguyen, R. Kowalczyk, A. Motylska-Kuźma, and J. Mercik, Eds., vol. 13010 of *Lecture Notes in Computer Science*, Springer, pp. 108–115.

[23] Stach, I., and Bertini, C. Solidarity measures. In *Transactions on Computational Collective Intelligence XXXVII* (Berlin, 2022), N. T. Nguyen, R. Kowalczyk, J. Mercik, and A. Motylska-Kuźma, Eds., vol. 13750 of *Lecture Notes in Computer Science*, Springer pp. 89–105.

[24] Stach, I., and Mercik, J. Measurement of control power in corporate networks. *Operations Research and Decisions 31*, 1 (2021), 97–121.

[25] Stach, I., Mercik, J., and Bertini, C. Some propositions of approaches for measuring indirect control power of firms and mutual connections in corporate shareholding structures. In *Transactions on Computational Collective Intelligence XXXV* (Berlin, 2020), N. T. Nguyen, R. Kowalczyk, J. Mercik and A. Motylska-Kuźma Eds., vol. 12330 of *Lecture Notes in Computer Science*, Springer, pp. 116–132.

[26] Stach, I., Mercik, J., Bertini, C., Gładysz, B., and Staudacher, J. Aggregated power indices for measuring indirect control in complex corporate networks with float shareholders. *Entropy 25*, 3 (2023), 429.

[27] Staudacher, J., Kóczy, L. A., Stach I. Filipp, J., Kramer, M., Noffke, T., Olsson, L., Pichler, J., and Singer, T. Computing power indices for weighted voting games via dynamic programming. *Operations Research and Decisions 31*, 2 (2021), 123–145.