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A generalized composition approach in network data envelopment analysis for complex structures. An application of higher education institutions in Poland

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Abstract

We present in this paper the pitfalls of the most established approach in network data envelopment analysis for units with a parallel internal structure. We show that these pitfalls are the cause of deficiencies of prevalent models employed for general series structures. To overcome these issues, we build a general composition approach that can be applied to units with any type of structure. Our approach relies on multi-objective programming and, unlike existing methods in the literature, we identify the divisional efficiency scores in a min-max and max-min sense simultaneously. This allows us to identify unique and unbiased efficiency scores that are not affected by the different magnitude of scores that the divisions can attain. Comparisons with other approaches, under various structures and assumptions, highlight the advantages of the proposed approach. We further employ this new approach to evaluate the teaching and research efficiency of the top 19 public higher education institutions in Poland with data drawn from the period 2020–2021. The proposed assessment framework departs from the employment of standard metrics such as number of publications and journal rankings, commonly used to evaluate the quantity and quality of research outcomes, and relies on other proxies, such as field-weighted citation impact factor and volume of research grants, that may provide more reliable results.

Keywords: *network data envelopment analysis, composition approach, mathematical programming, multi-objective programming, complex structures, higher education institutions*

1. Introduction

Data envelopment analysis (DEA) is a well-established non-parametric technique to assess the performance of homogeneous decision-making units (DMUs) based on multiple inputs and outputs. The two pioneering models CCR [7], under constant returns to scale (CRS) assumption, and BCC [3], under

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variable returns to scale (VRS) assumption, have become standards in the literature of performance measurement with a wide spectrum of applications, e.g., education [39], courts [37], IT projects [32], paper mills [1], telecommunication companies [46], banks [27], etc.

The conventional DEA models treat the DMUs as a black box, i.e., every unit is conceived as one--stage process which transforms a set of external inputs into a set of final outputs. In such cases, the unit's internal structure is ignored, and the performance analysis can provide managerial insights only to the unit as a whole. Nevertheless, there are cases where the internal structure of the unit is known, and it plays a crucial role in the efficiency assessment and policy making. Network data envelopment analysis (NDEA) is an extension of standard DEA which takes into consideration the internal structure of the DMUs. Specifically, in NDEA, every DMU is conceived as a network of sub-processes (divisions), interconnected with intermediate measures (links), which represent the DMU's production process more accurately. Therefore, in addition to the evaluation of the DMU as a whole (overall efficiency), NDEA assesses the performance of all sub-processes, i.e., the efficiency is a multi-dimensional vector. The evaluation of the overall and divisional efficiencies allows the decision-makers to identify more accurately the sources of inefficiencies within the production process. Thus, it provides more precise managerial insights compared to standard DEA.

The NDEA methods proposed in the literature can be grouped based on the optimization criterion they employ and the structure that they are applied on, i.e., series, parallel and complex which are a mix of series and parallel networks [18, 25, 28]. The majority of the studies focus on series structures. For instance, a series two-stage process is assumed by Tsolas [44] for credit risk evaluation, by Michali et al. [38] for the noise-pollution assessment of European railways and by Tsolas [45] who evaluated the efficiency of electric trolley bus routes. The dominant methodological approaches in Network DEA are the composition and the decomposition ones. In the decomposition approach, the objective of the optimization is the maximization of the overall efficiency, while the stage efficiencies are calculated ex-post from the optimal solution (top-down). On the contrary, in the composition approach, priority is given to the calculation of the stage efficiencies and the overall efficiency derives from the aggregation of the stage efficiency scores (bottom-up). Within the context of the decomposition approach, Liang et al. [35] and Kao and Hwang [26] proposed the multiplicative aggregation of the stage efficiencies, while Chen et al. [8] proposed the additive one. Despotis et al. [12] introduced the composition approach for two-stage series structures. A multi-objective program (MOP) was proposed where the stage efficiencies are treated as distinct objective functions and maximized simultaneously. As the conflicting nature of the intermediate measures leads to efficiency trade-offs among the divisions, the notion of the Pareto front in the divisional efficiencies space was introduced. Despotis et al. [14] extended the composition approach in general series structures of several types by employing the min-max method (Chebyshev scalarizing function) to solve the proposed MOP. Similarly, Despotis et al. [13] based on the notion of weak-link in supply chains adopted a max-min method to locate the most inefficient (weak-link) division in two-stage series structures. A direct comparison of the aforementioned decomposition and composition methods is provided by Koronakos et al. [30]. All these methods are primarily developed for two-stage series structures. Extensions of the above-mentioned decomposition methods are proposed by Koronakos et al. [31] for DMUs with more than two-stages series structures and in Kao [21, 24] and Cook et al. [9] for general structures. Alternative modeling approaches for complex structures are provided by Tone and Tsutsui [43] who developed a slack-based measure approach as well as by Yu and Lin [48] who employed a directional distance function to evaluate railway performance. Nevertheless, to the best of the authors' knowledge, little attention has been paid to parallel structures. The work of Kao [22], which relies on the system's slack inefficiencies, and the work of Kao [23], which is a more generalized approach, were the first ones dealing with parallel structures. Thereafter, a limited number of papers focused on parallel structures and most of them (e.g. Du et al. [16]) rely on the formulations of Kao [22, 24].

Focusing on the approaches that rely on radial measures, we select the approach of Kao [23], as the dominant and most representative approach for parallel structures and we present the pitfalls that permeate it. Then, we develop a new bottom-up approach for the evaluation of DMUs with parallel structures. We show that the new approach overcomes the deficiencies of Kao [23] and therefore, it provides reliable results. In addition, we extend the proposed modeling approach for series structures, and we show that this method is an extension of the min-max and max-min approach introduced for series structures. We further explain why the relational model [21, 24] cannot provide reliable results for general series, parallel and complex structures. Finally, the applicability of the new approach is illustrated in a complex structure through a dataset borrowed from the literature and a case study related to the evaluation of teaching and research efficiency of higher education institutions.

The paper unfolds as follows. Section 2 describes the approach of Kao [23] for parallel structures, the deficiencies that characterize it, and introduces a new unbiased approach to measure the efficiency of units with an internal parallel structure. Section 3 extends the proposed approach in general series structures and compares it with other prevalent approaches for series structures. A direct comparison with the relational model of Kao [21, 24] shows that the peculiarities that permeate the relational model are attributed to the way that the efficiencies of units with parallel structure are estimated [23]. Section 4 illustrates the applicability of the proposed approach in a complex structure with data drawn from the literature and Section 5 presents an application of higher education institutions in Poland with data drawn from the period 2020–2021.

2. Parallel structures

A DMU with a parallel internal structure consists of two or more sub-processes for which there are not any inter-dependencies, i.e., there are not any flows (links) among them. Thus, each sub-process utilizes some of the DMU's inputs to produce a vector of outputs that constitute part of the DMU's final outputs. Assume that the DMU k utilizes a vector of inputs X_k to produce a vector of outputs Y_k and that X_k^p, Y_k^p are the input and output vectors that the $p = 1, 2, \ldots, q$ process of DMU k consumes and produces, respectively. Figure 1 depicts a general representation of the internal structure for the DMU k which is composed of q parallel sub-processes.

Regarding the factors that each sub-process consumes/produces and their relationship with the input/output vectors of the DMU, three different assumptions[1] can be identified:

[1]The first two assumptions are analogous to the classification provided in Kao [25], i.e., multi-component (Assumption I) and multi-function (Assumption II) systems.

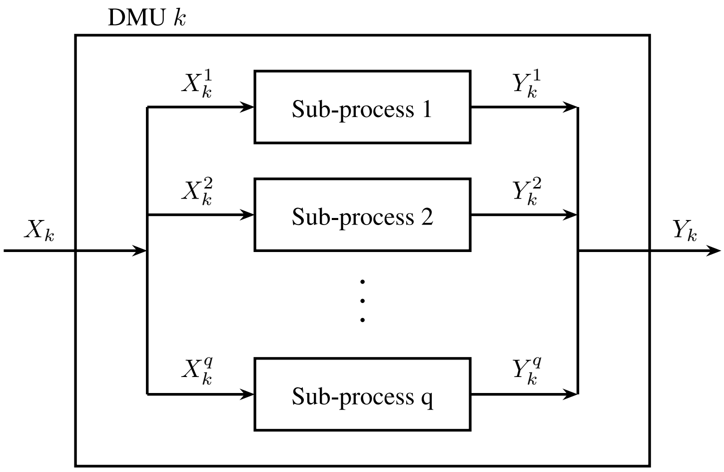

Figure 1. General representation of a DMU with multi-divisional parallel internal structure

I. Each sub-process consumes/produces a portion of each DMU's input/output. That is, all sub-processes consume the same input factors to produce the same output factors. Only the levels of inputs and outputs differ among the sub-processes. In this case, the sub-processes are homogeneous and the DMU's input/output vectors equal to the sum of the input/output vectors of all sub-processes, i.e., $X_k = \sum_{p=1}^{q} X_k^p$ and $Y_k = \sum_{p=1}^{q} Y_k^p$.

II. Each sub-process consumes/produces different inputs/outputs from the rest sub-processes. In this setting, the sub-processes are independent in the sense that each one of them utilizes exclusively some of the DMUs' inputs to entirely produce some of the DMUs' outputs. Therefore, the sub-processes are heterogeneous and the input/output vectors of the sub-processes constitute the components of the DMUs' total input/output vectors, i.e., $X_k = (X_k^1, X_k^2, \ldots, X_k^q)$ and $Y_k = (Y_k^1, Y_k^2, \ldots, Y_k^q)$.

III. Sub-processes may entirely utilize/produce some of the DMUs' input/output factors as well as a portion of some other factors.

Kao [23], assuming that all sub-processes consume the same type of inputs to produce the same types of outputs (Assumption I), developed a NDEA approach for DMUs with a parallel internal structure that follows the decomposition paradigm. That is, the overall efficiency of the system is calculated first through the optimization process and the divisional efficiencies derive ex-post from the optimal multipliers. Assume that there is a set of n DMUs ($k = 1, 2, \ldots, n$) and each DMU utilizes a vector of m inputs $X_k = (x_{1k}, x_{2k}, \ldots, x_{mk})$ to produce a vector of s outputs $Y_k = (y_{1k}, y_{2k}, \ldots, y_{sk})$. Then, the input and output vectors for the sub-process $p = 1, 2, \ldots, q$ are respectively $X_k^p = (x_{1k}^p, x_{2k}^p, \ldots, x_{mk}^p)$ and $Y_k^p = (y_{1k}^p, y_{2k}^p, \ldots, y_{mk}^p)$. The overall efficiency of the system is defined as the ratio of the aggregate value of the total outputs to the aggregate value of the total inputs, i.e., $e_k^0 = \dfrac{uY_k}{vX_k}$ where u and v are the multipliers associated with the outputs and the inputs, respectively. The divisional efficiency scores are $e_k^p = \dfrac{uY_k^p}{vX_k^p}, p = 1, 2, \ldots, q$. Consequently, Kao [23] proposed model (1) to estimate the overall efficiency of the system, for unit k.

$$\begin{aligned}
&\max uY_k \\
&\text{s.t.} \\
&vX_k = 1 \\
&uY_j^p - vX_j^p \leq 0,\ p = 1, 2, \ldots, q;\ j = 1, 2, \ldots, n \\
&uY_j - vX_j \leq 0,\ j = 1, 2, \ldots, n \\
&u, v \geq \varepsilon
\end{aligned} \tag{1}$$

Let u^*, v^* be an optimal solution in model (1). Then, the overall efficiency of the system is derived from the optimal value of the objective function in model (1), i.e., $E_k^0 = \frac{u^* Y_k}{v^* X_k} = u^* Y_k$. Accordingly, the efficiencies of the sub-processes derive as offspring from the optimal solution, according to equation (2).

$$E_k^p = \frac{u^* Y_k^p}{v^* X_k^p},\ p = 1, 2, \ldots, q \tag{2}$$

To link the overall efficiency with the divisional efficiency scores, a weighted average function is assumed. The weight attached to each division represents its size and it is calculated as the ratio of the aggregate value of the inputs the division utilizes to the total aggregate value of the inputs of the whole process, i.e., $w_k^p = \frac{v^* X_k^p}{\sum_{p=1}^{q} v^* X_k^p}$, $p = 1, 2, \ldots, q$. Thus, the overall efficiency can be expressed as a function of the divisional efficiency scores based on equation (3).

$$E_k^0 = \sum_{p=1}^{q} w_k^p E_k^p = \frac{u^* Y_k}{v^* X_k} \tag{3}$$

Notably, the approach of Kao [23] relies on the same concepts with the additive efficiency decomposition introduced for two-stage series structures by Chen [8]. Specifically, both approaches aim first to maximize the overall efficiency of the unit and then to decompose it to the divisions of the unit. In addition, both approaches assume that the overall efficiency derives from the weighted average of the divisional efficiency scores, whereas the weights of the divisions reflect the size of each division and are functions of the decision variables of the optimization process. Therefore, in this paper, we will refer to the approach of Kao [23] as the additive efficiency decomposition method for parallel structures.

2.1. Pitfalls of the additive efficiency decomposition method for parallel structures

Several pitfalls from the additive efficiency decomposition approach in parallel structures [23] derive, that have not been discussed in the literature. First, the size of each stage, as expressed by the weight associated with it, differs across the evaluated DMUs. That is, the contribution of each division towards the estimation of the overall efficiency, is viewed from the DMU perspective which does not allow for a common ground to compare the DMUs. In addition, the size of each division may not be unique from the perspective of the same DMU. Specifically, when multiple optimal solutions exist, the weights attached to the stages, as functions of the decision variables, may not be unique. Therefore, the size of each stage not only differs across different DMUs, but it may also have different values for the same DMU.

Another critical point is the uniqueness of the divisional efficiency scores. Model (1) maximizes the overall efficiency of the system, and the divisional efficiency scores are calculated ex-post from the optimal solution of model (1). Therefore, when multiple optimal solutions exist, the uniqueness of the divisional efficiency scores is not secured. As noted in Kao [24] *"Efficiency decomposition enables decision-makers to identify the stages that cause the inefficiency of the system, and to effectively improve the performance of the system"*. However, in such cases, the division with the lowest efficiency is randomly selected as the divisional efficiency scores depend solely on the solver employed to solve model (1).

Sotiros et al. [41] introduced the dominance property in NDEA for series structures. Specifically, they argued that in any assessment method, there should not exist any feasible solution that provides stage efficiency scores that dominate the estimated ones, i.e., feasible solutions that provide stage efficiency scores at least as high as the assessed ones and higher for at least one stage. They further found that violation of the dominance property in series structures can lead to peculiarities identified and criticized when standard DEA models are applied to DMUs with network structures. In this paper, we extend the work of Sotiros et al. [41] for parallel structures and we find out that the model of Kao [23] violates the dominance property and therefore, it provides misleading results. To illustrate this issue, we provide in Table 1 a synthetic dataset consisting of five DMUs that have two parallel sub-processes and in each sub-process two inputs are utilized to produce two outputs. To be in line with the assumption of model (1), we assume that the sub-processes are homogeneous, i.e., for every unit both sub-processes consume the same type of inputs to produce the same type of outputs (Assumption I). Therefore, the input and the output vectors for DMU k are respectively $X_k = (x^1_{1k} + x^2_{1k}, x^1_{2k} + x^2_{2k})$ and $Y_k = (y^1_{1k} + y^2_{1k}, y^1_{2k} + y^2_{2k})$. Columns 2–5 illustrate the inputs and the outputs of all units for the first sub-process whereas the last columns (6–9) present the inputs and outputs of all units for the second division.

Table 1. Synthetic dataset generated from a uniform distribution in the interval $[10, 100]$

DMU	x^1_1	x^1_2	y^1_1	y^1_2	x^2_1	x^2_2	y^2_1	y^2_2
1	96	91	54	98	51	90	79	34
2	98	87	13	32	42	36	86	32
3	43	17	49	64	87	57	53	22
4	29	29	52	82	91	18	78	49
5	64	18	99	32	68	87	37	91

In Table 2, we report the efficiency scores of the two sub-processes (columns 2 and 3) as well as the overall one (column 4) when model (1) is applied to the data of Table 1. Columns 5, and 6 show feasible pairs of divisional efficiency scores. These efficiency scores are derived by employing the leader-follower approach introduced by Liang et al. [35]. This approach was developed for two-stage series structures, but it can be extended to parallel structures as illustrated in models (4) and (5). In model (4) pre-emptive priority is given to the first sub-process and its maximum efficiency (E^{1+}_k) is calculated for each unit. In model (5), the efficiency of the second division is maximized (E^{2-}_k) given that the maximum efficiency of the first sub-process is maintained.

Table 2. Results from model (1) and the leader-follower approach

	Results from Kao [23] model (1)			Results from leader follower approach; models (4)–(5)		
DMUs	E^1	E^2	E^0	E^{1+}	E^{2-}	E^0
1	0.3214	0.7616	0.4741	0.3764	0.1475	0.2689
2	0.0876	1	0.3588	0.1266	0.3036	0.1787
3	0.8119	0.3287	0.4747	1	0.2721	0.4710
4	1	0.7396	0.8491	1	0.7396	0.8491
5	1	0.3073	0.5658	1	0.3295	0.5183

Notably, in model (4) the lower bound for the weights is ε, whereas in model (5) it is $\varepsilon v X_k^1$. This derives by applying a new Charnes–Cooper transformation[2] [6] in model (4) while taking into account the imposed lower bound (ε) on the decision variables. From the optimal solution of model (5), the overall efficiency of the system, as defined in Kao [23], can be calculated, i.e., $E_k^0 = \dfrac{u^* Y_k}{v^* X_k}$, which is provided in column 7.

$$
\begin{aligned}
& E_k^{1+} = \max u Y_k^1 \\
& \text{s.t.} \\
& v X_k^1 = 1 \\
& u Y_j^p - v X_j^p \leq 0, \; p = 1, 2; \; j = 1, 2, \ldots, n \\
& u Y_j - v X_j \leq 0, \; j = 1, 2, \ldots, n \\
& u, v \geq \varepsilon
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
& E_k^{2-} = \max u Y_k^2 \\
& \text{s.t.} \\
& v X_k^2 = 1 \\
& u Y_j^p - v X_j^p \leq 0, \; p = 1, 2; \; j = 1, 2, \ldots, n \\
& u Y_j - v X_j \leq 0, \; j = 1, 2, \ldots, n \\
& u Y_k^1 - E_k^{1+} v X_k^1 = 0 \\
& u, v \geq \varepsilon v X_k^1
\end{aligned} \tag{5}
$$

Regarding DMU 5, it can be observed that both approaches render the first sub-process efficient. However, model (5) provides a higher efficiency score for the second division than model (1). Therefore, model (1) does not comply with the dominance property, and its results can be misleading. Notably, the divisional efficiency scores derived from models (4) and (5) provide a lower overall efficiency score than model (1). This is in line with the findings of Guo et al. [17] and Sotiros et al. [41] for the additive model [8].

Homogeneity of the sub-processes is another issue that should be considered. Notably, the model of Kao [23] has been developed under the assumption that all sub-processes utilize and produce the same

[2] This transformation leads to new variables. However, for the sake of simplicity, in model (5) we follow the same notation as in model (4).

types of inputs and outputs. However, it is commonly employed in cases where some inputs and outputs are related to all sub-processes whereas the rest of them are sub-process specific (see, e.g., the case study in [23]). This is technically achieved by setting at zero level the elements of the X_k^p and Y_k^p vectors that the $p = 1, 2, \ldots, q$ sub-process is not related with and thus, satisfying the equations

$$X_k = \sum_{p=1}^{q} X_k^p \quad \text{and} \quad Y_k = \sum_{p=1}^{q} Y_k^p$$

Therefore, in such cases, the sub-processes are assumed artificially homogeneous, while they are not. Another issue in such cases is the requirement to set lower bounds on all decision variables $(u, v \geq \varepsilon)$ to define all divisional efficiency scores. If such lower bounds are omitted (and all decision variables are set to be greater than or equal to zero), the optimal weights for all the inputs and/or all the outputs of a sub-process may be at zero level, and therefore, the efficiency of the division to be zero or even not defined. Nevertheless, such results do not have any managerial interpretation and they are not in line with the performance measurement theory.

Kao [25] classified the DMUs in multi-component and multi-function systems. Nevertheless, in all cases, model (1) is employed. Notably, in multi-function systems each sub-process acts independently from each other, i.e., there is not any conflict among the sub-processes. Therefore, any assessment model, in order to comply with the dominance property [41], should produce the independent efficiency scores of the divisions. However, when the model of Kao [23] is used under such an assumption, the divisional efficiency scores that are derived are lower than the independent ones. This is an issue that not only questions the validity of the results for parallel structures but also for more complex structures when the relational model [24] is employed (further discussion about this issue is provided in Section 3). To illustrate this issue, we apply the independent approach (standard CCR model) and model (1) on the data of Table 1 under the assumption that each sub-process is using entirely different inputs to produce entirely different outputs from the rest of the sub-processes (Assumption II). To apply model (1) in this case, as in Kao [25], the input and the output vectors for the first sub-process are assumed to be $X_k^1 = (x_{1k}^1, x_{2k}^1, 0, 0)$ and $Y_k^1 = (y_{1k}^1, y_{2k}^1, 0, 0)$, respectively. Accordingly, the input and the output vectors for the second sub-process are $X_k^2 = (0, 0, x_{1k}^2, x_{2k}^2)$ and $Y_k^2 = (0, 0, y_{1k}^2, y_{2k}^2)$, respectively. Therefore, the input and the output vectors for DMU k are $X_k = (x_{1k}^1, x_{2k}^1, x_{1k}^2, x_{2k}^2)$ and $Y_k = (y_{1k}^1, y_{2k}^1, y_{1k}^2, y_{2k}^2)$. In Table 3, we present the results from these two approaches. Columns 2 and 3 report the efficiency scores of the sub-processes when model (1) is applied. The divisional efficiency scores from the independent approach are presented in columns 4 and 5.

Table 3. Results from model (1) and the independent approach when the sub-processes are independent

DMUs	E^1	E^2	$E^{1\,\text{Ind}}$	$E^{2\,\text{Ind}}$
1	0.3559	0.8148	0.3764	0.8148
2	0.1080	1	0.1266	1
3	1	0.3617	1	0.3638
4	1	1	1	1
5	1	1	1	1

The results coincide only for DMUs 4 and 5, which are efficient in both divisions. However, for the rest three units, model (1) provides lower divisional efficiency scores. Notably, as was mentioned before, every sub-process utilizes/produces different inputs/outputs from the rest and, therefore, they act independently from each other. Consequently, the independent efficiency scores can be simultaneously achieved for the two divisions. However, model (1) does not provide these scores. Thus, the approach of Kao [23, 25] does not comply with the dominance property under Assumption II.

Table 4. Synthetic dataset generated from uniform distribution in the interval $[10, 100]$

DMUs	x_1^1	x_2^1	y_1^1	y_2^1	x_1^2	x_2^2	y_1^2	y_2^2
1	52	65	19	70	12	18	55	53
2	56	86	31	72	47	64	62	57
3	66	89	86	34	71	79	69	34
4	33	31	99	83	22	72	17	27
5	31	71	56	55	67	65	50	88

One of the main principles in NDEA and generally in DEA, is that the efficiency scores, as they reflect the distance of the DMU from the efficient frontier, are non-negative. However, it is noted that the model of Kao [23] for parallel structures under VRS assumption can provide negative efficiency scores for the sub-processes [40]. To highlight this issue, we apply the VRS model presented in Kao [23] on the dataset reported in Table 4. The dataset consists of five DMUs which are composed of two parallel sub-processes. To be in line with the development of the VRS model (6) discussed in Kao [23], it is assumed that all sub-processes are homogeneous, i.e., they consume/produce the same inputs/outputs (Assumption I). Columns 2–5 illustrate the inputs and the outputs for the first sub-processes while the last columns (6–9) present the inputs and outputs of the second sub-processes.

$$
\begin{aligned}
&\max uY_k - \sum_{p=1}^{q} u^p \\
&\text{s.t.} \\
&vX_k = 1 \\
&uY_j^p - u^p - vX_j^p \leq 0,\ p = 1, 2, \ldots, q;\ j = 1, 2, \ldots, n \\
&uY_j - \sum_{p=1}^{q} u^p - vX_j \leq 0,\ j = 1, 2, \ldots, n \\
&u, v \geq \varepsilon \\
&u^p \in \mathfrak{R},\ p = 1, 2, \ldots, q
\end{aligned}
\tag{6}
$$

The results are reported in Table 5. The efficiency scores of the first sub-processes for DMUs 3 and 5 are negative. This implies that the model (1) cannot provide reliable efficiency scores under the VRS assumption.

To conclude, even though the model of Kao [23] is considered the most established model in NDEA for parallel structures, it is permeated by several peculiarities questioning the validity of the results it provides.

Table 5. Results from the VRS model of Kao [23] on the dataset in Table 3

DMUs	$E^{1\text{ VRS}}$	$E^{2\text{ VRS}}$	$E^{0\text{ VRS}}$
1	0.6252	1	0.6962
2	0.5752	0.2614	0.432
3	-0.3302	1	0.3592
4	1	0.5038	0.7889
5	-0.3549	1	0.5714

2.2. A composition approach for parallel structures

The approaches that follow the composition paradigm are based on MOP, but they employ different scalarizing functions depending on the different purposes they intend to achieve. The min-max method [14] aims to locate a point on the Pareto front that minimizes the maximum deviation from the ideal point. Contrarily, the max-min method [13] locates a point on the Pareto front that maximizes the minimum deviation from the nadir point. Ideally, an analyst would prefer to locate a point as far as possible from the worst situation (nadir point) and as close as possible to the best status (ideal point). However, the min-max and the max-min methods do not necessarily provide the same solution.

In this section, we introduce the composition (bottom-up) approach for parallel structures which, for the sake of simplicity, is illustrated for units with two sub-processes. This modeling approach overcomes the pitfalls of the additive efficiency decomposition approach for the efficiency assessment of DMUs with a parallel structure. In addition, we introduce a normalization technique in the divisional efficiencies space which allows us to locate a point in the Pareto front as close as possible to the ideal point and as far as possible from the nadir point. Therefore, the assessed scores are not affected by the different magnitude of scores that the divisions can attain and can be conceived as fair.

Assume a set of n DMUs where each DMU consists of two sub-processes ($p = 1, 2$) and both sub-processes utilize the same type of inputs to produce the same type of outputs (Assumption I in Section 2). To assess the efficiency of the first division, under the constants-returns-to-scale assumption, the fractional model (7) can be employed. Analogously, the fractional model (8) can be utilized to estimate the efficiency of the second division. Notice that the second set of constraints in model (7) derives from the second division whereas the first set of constraints in model (8) derives from the first division. The third set of constraints in both models is related to the overall efficiency of the DMU.

$$
\begin{aligned}
&E_k^1 = \max \frac{uY_k^1}{vX_k^1} \\
&\text{s.t.} \\
&\frac{uY_j^1}{vX_j^1} \le 1,\ j = 1, 2, \ldots, n \\
&\frac{uY_j^2}{vX_j^2} \le 1,\ j = 1, 2, \ldots, n \\
&\frac{uY_j}{vX_j} \le 1,\ j = 1, 2, \ldots, n \\
&u, v \ge 0
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
& E_k^2 = \max \frac{uY_k^2}{vX_k^2} \\
& \text{s.t.} \\
& \frac{uY_j^1}{vX_j^1} \le 1,\ j = 1,\ 2,\ \ldots,\ n \\
& \frac{uY_j^2}{vX_j^2} \le 1,\ j = 1,\ 2,\ \ldots,\ n \\
& \frac{uY_j}{vX_j} \le 1,\ j = 1,\ 2,\ \ldots,\ n \\
& u, v \ge 0
\end{aligned} \tag{8}
$$

Models (7) and (8) have the same set of constraints and consequently the same feasible region. Thus, they can be expressed as a bi-objective program.

$$
\begin{aligned}
& \max \left\{ E_k^1 = \frac{uY_k^1}{vX_k^1}, E_k^2 = \frac{uY_k^2}{vX_k^2} \right\} \\
& \text{s.t.} \\
& \frac{uY_j^1}{vX_j^1} \le 1,\ j = 1,\ 2,\ \ldots,\ n \\
& \frac{uY_j^2}{vX_j^2} \le 1,\ j = 1,\ 2,\ \ldots,\ n \\
& \frac{uY_j}{vX_j} \le 1,\ j = 1,\ 2,\ \ldots,\ n \\
& u, v \ge 0
\end{aligned} \tag{9}
$$

Model (9) is a bi-objective program that maximizes simultaneously the efficiencies of both divisions (E_k^1, E_k^2). By employing the lexicographic method to solve the bi-objective program (9) the efficiency scores of models (7) and (8) can be obtained. Specifically, by assigning pre-emptive priority to the first division, i.e., $\operatorname{lex\,max} \left\{ E_k^1 = \frac{uY_k^1}{vX_k^1}, E_k^2 = \frac{uY_k^2}{vX_k^2} \right\}$, the maximum efficiency of the first division E_k^{1+} and the efficiency of the second one E_k^{2-}, given that the first sub-process maintains its maximum efficiency, are obtained. Notice that this method is similar to the leader-follower method introduced by Liang et al. [35] for the two-stage series structures. For instance, if the first sub-process is the leader, E_k^{1+} is derived by model (7) whereas E_k^{2-} derives by model (8) after introducing the additional constraint $\frac{uY_k^1}{vX_k^1} = E_k^{1+}$ (see also models (4) and (5)). Analogously, if priority is given to the second division, i.e., $\operatorname{lex\,max} \left\{ E_k^2 = \frac{uY_k^2}{vX_k^2}, E_k^1 = \frac{uY_k^1}{vX_k^1} \right\}$ then, the following pair of efficiency scores is obtained $\left(E_k^{1-}, E_k^{2+}\right)$. In terms of multi-objective programming, the point $I\left(E_k^{1+}, E_k^{2+}\right)$ represents the ideal point in the divisional efficiencies space and $N\left(E_k^{1-}, E_k^{2-}\right)$ the nadir point. However, due to the conflicting nature of the two objective functions, the ideal point is not necessarily attainable. Notice, that the conflict between the two objectives derives from the assignment of common multipliers (weights) to the same type of factors that the two divisions utilize or produce. Consequently, similarly to the case of the series structures, the notion of Pareto front in the divisional efficiency space for the parallel structures is introduced.

Generally, the choice of the optimization criterion, that drives the efficiency assessment, plays a crucial role in the estimation of the efficiency scores and it depends on the managerial policies and preferences of the decision-maker(s). For instance, if the relative preference of the divisions $\left\{w_k^1, w_k^2 \mid w_k^1 + w_k^2 = 1\right\}$ is known then, model (9) can be solved by employing the weighted additive function $w_k^1 E_k^1 + w_k^2 E_k^2$ as a scalarizing function to transform the bi-objective program (9) to a single-objective optimization model. Notice, that in this case the weights (w_k^1, w_k^2) are constant and a priori defined by the decision-maker(s). However, this preference information may not be easy to extract or may even not exist. An alternative way to solve the bi-objective model (9) is the approach of global criterion where the objective is to locate a feasible point that optimizes the distance between the feasible region and a reference point, without any a priori preference information. The reference point is an integral part of the problem and can be conceived either as the nadir point or the ideal point. In the first case, the optimization criterion is to locate a feasible point as far as possible from the nadir point whereas in the second one, to find a point in the feasible region as close as possible to the ideal point. However, as already discussed, these two criteria may provide different solutions. Ideally, the decision-maker(s) would prefer to find a solution in the objective functions space that satisfies both criteria, i.e., a point that is as far as possible from the nadir point and as close as possible to the ideal one.

In this paper, initially we employ the lexicographic approach to solve the bi-objective model (9) and to obtain the coordinates of the two aforementioned reference points, i.e., the ideal point $I\left(E_k^{1+}, E_k^{2+}\right)$ and the nadir point $N\left(E_k^{1-}, E_k^{2-}\right)$. Then, contrarily to the existing approaches in NDEA developed for series structures, we normalize the objective functions space based on the distances between the coordinates of the two reference points, i.e., $\left(E_k^{1+} - E_k^{1-}\right)$ and $\left(E_k^{2+} - E_k^{2-}\right)$. Specifically, the efficiency of the k DMU in the $p = 1, 2$ sub-process is expressed though the normalization function

$$\hat{E}_k^p = \frac{E_k^p - E_k^{p-}}{E_k^{p+} - E_k^{p-}}, \; k = 1, 2, \ldots, n; \; p = 1, 2 \tag{10}$$

Thus, model (9) can be rewritten in the following form:

$$\begin{aligned}
&\max \left\{ \hat{E}_k^1 = \frac{\dfrac{uY_k^1}{vX_k^1} - E_k^{1-}}{E_k^{1+} - E_k^{1-}}, \hat{E}_k^2 = \frac{\dfrac{uY_k^2}{vX_k^2} - E_k^{2-}}{E_k^{2+} - E_k^{2-}} \right\} \\
&\text{s.t.} \\
&\frac{uY_j^1}{vX_j^1} \leq 1, \; j = 1, 2, \ldots, n \\
&\frac{uY_j^2}{vX_j^2} \leq 1, \; j = 1, 2, \ldots, n \\
&\frac{uY_j}{vX_j} \leq 1, \; j = 1, 2, \ldots, n \\
&u, v \geq 0
\end{aligned} \tag{11}$$

Notice that $\hat{E}_k^p$ is bounded in the interval $[0, 1]$ and reflects the distance of the efficiency of the pth sub-process from the point E_k^{p-} as a percentage of the distance $E_k^{p+} - E_k^{p-}$. $\hat{E}_k^p$ reaches its highest value (equal to one) when the pth sub-process achieves the maximum possible efficiency score $\left(E_k^{p+}\right)$. Similarly, the value of $\hat{E}_k^p$ drops to zero when E_k^p attains its lower level E_k^{p-}. Notably, different normalization techniques can be employed. However, the current normalization transforms the objective functions' space into a hypercube and omits any endogenous discrepancies among the objective functions [5].

To solve the bi-objective model (11) we employ the Tchebycheff distance achievement function, and we provide two different formulations, i.e., models (12)–(13). Model (12) relies on the max-min formulation which actually aims to maximize the weighted minimum deviation from the nadir point $N\left(E_k^{1-}, E_k^{2-}\right)$. On the contrary, model (13) seeks to minimize the maximum weighted deviation from the ideal point $I\left(E_k^{1+}, E_k^{2+}\right)$. Even though these two models rely on different formulations, they identify the same divisional efficiency scores, i.e., the same point on the Pareto front. Indeed, the first two constraints of model (13) are equivalent to the first two constraints of model (12). In addition, given that the employed normalization transforms the objective functions' space into a hypercube, the search rays in a max-min (model (12)) and min-max (model (13)) sense coincide. Therefore, these two models are equivalent. Notably, even by applying the Charnes–Cooper transformation [6] to linearize one of the first two constraints, either in model (12) or in model (13), both models remain non-linear. However, they can be easily solved by bisection search [10].

$$
\begin{aligned}
& \max \delta \\
& \text{s.t.} \\
& \frac{\dfrac{uY_k^1}{vX_k^1} - E_k^{1-}}{E_k^{1+} - E_k^{1-}} \geq \delta \\
& \frac{\dfrac{uY_k^2}{vX_k^2} - E_k^{2-}}{E_k^{2+} - E_k^{2-}} \geq \delta \\
& \frac{uY_j^1}{vX_j^1} \leq 1, \ j = 1, 2, \ldots, n \\
& \frac{uY_j^2}{vX_j^2} \leq 1, \ j = 1, 2, \ldots, n \\
& \frac{uY_j}{vX_j} \leq 1, \ j = 1, 2, \ldots, n \\
& u, v \geq 0
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
& \min d \\
& \text{s.t.} \\
& 1 - \frac{\dfrac{uY_k^1}{vX_k^1} - E_k^{1-}}{E_k^{1+} - E_k^{1-}} \leq d \\
& 1 - \frac{\dfrac{uY_k^2}{vX_k^2} - E_k^{2-}}{E_k^{2+} - E_k^{2-}} \leq d \\
& \frac{uY_j^1}{vX_j^1} \leq 1, \ j = 1, 2, \ldots, n
\end{aligned}
\tag{13}
$$

$$\frac{uY_j^2}{vX_j^2} \leq 1, \; j = 1, 2, \ldots, n$$
$$\frac{uY_j}{vX_j} \leq 1, \; j = 1, 2, \ldots, n$$
$$u, v \geq 0$$

Models (12) and (13) may provide a weakly Pareto optimal solution in the objective functions' space. Nevertheless, a Pareto optimal solution can be secured by model (14). Model (14) employs, in a second phase, the L_1 norm on the set of optimal solutions of models (12)–(13) in order to identify further potential increments on the divisional efficiency scores while maintaining the optimal value of the objective function (δ^*) in models (12)–(13). Model (14) is non-linear. By applying the Charnes–Cooper transformation [6] on the fraction representing the efficiency of the first sub-process, the first constraint will turn into a linear one but the second constraint will remain a quadratic one. Specifically, in the second constraint, the term $-\left(E_k^{2+} - E_k^{2-}\right) vX_k^2 S_2$ is of second order as both v and S_2 are variables of the optimization process. However, the weights (v) attached to S_2 can be substituted with the optimal solution (v^*) from model (12) or (13) to derive a new solution, $(v^{**}, u^{**}, S_1^{**}, S_2^{**})$. If $S_1^{**} = S_2^{**} = 0$, then the divisional efficiency scores from models (12)–(13) are Pareto optimal. Otherwise, model (14) is resolved again by substituting v (multiplied with S_2) with v^{**}. This iterative process will continue until there are no differences on the divisional efficiency scores among two successive iterations, in order to secure Pareto optimality of the divisional efficiency scores (cf. Despotis [14]).

Once the efficiency scores of the divisions are calculated, the overall efficiency of the system can derive from their aggregation. The most common aggregation schemes are the multiplicative aggregation and the arithmetic mean. The definition of overall efficiency mainly depends on the decision-maker and there is no consensus in the literature on which aggregation method is better. Nevertheless, the decision could rely on the level of compensation the decision-maker is willing to allow. For instance, the multiplicative aggregation can be viewed as a non-compensatory method that penalizes the overall efficiency when one of the divisions has a low-efficiency score. On the other hand, the arithmetic mean can be viewed as a compensatory method which allows for compensations among the divisional efficiency scores. If the decision-maker desires to define the overall efficiency as the ratio of the total virtual outputs over the total virtual inputs, then another mathematical program should be solved in which the overall efficiency of the system is maximized, while the divisional efficiency scores are fixed to the optimal levels identified by model (14).

$$\begin{aligned}
&\max S_1 + S_2 \\
&\text{s.t.} \\
&\frac{\dfrac{uY_k^1}{vX_k^1} - E_k^{1-}}{E_k^{1+} - E_k^{1-}} - S_1 = \delta^* \\
&\frac{\dfrac{uY_k^2}{vX_k^2} - E_k^{2-}}{E_k^{2+} - E_k^{2-}} - S_2 = \delta^*
\end{aligned} \tag{14}$$

$$\frac{uY_j^1}{vX_j^1} \leq 1, \ j = 1, 2, \ldots, n$$
$$\frac{uY_j^2}{vX_j^2} \leq 1, \ j = 1, 2, \ldots, n$$
$$\frac{uY_j}{vX_j} \leq 1, \ j = 1, 2, \ldots, n$$
$$u, v \geq 0$$

Notably, this approach has been developed in the case where the sub-processes are homogeneous (Assumption I in Section 2). Nevertheless, it can be straightforwardly extended to cases where the sub-processes are independent in the sense that they use/produce entirely only some of the inputs/outputs of the system (Assumption II in Section 2) or in cases where only some of the inputs/outputs of the DMU are sub-process specific (Assumption III in Section 2). In the case of independent sub-processes, by construction, this approach ensures that the divisional efficiency scores will be the independent ones. In addition, under VRS assumption, it is ensured that the divisional efficiency scores will always be positive. Furthermore, as this approach relies on the composition paradigm, it identifies unique and unbiased divisional efficiency scores. Therefore, it overcomes the issues deriving from the existence of multiple weights and efficiency scores. Furthermore, the methods that follow the composition paradigm inherently comply with the dominance property [41]. Consequently, the divisional efficiency scores derived from the new approach are always Pareto optimal. Finally, in this approach, it is not required to set lower bounds (ε) on the weights of the inputs/outputs of the system in order the divisional efficiency scores to be defined at optimality. This is an inherent property of the proposed approach as it pushes radially the divisional efficiency scores from the nadir to the ideal point. Nevertheless, it is not free of disadvantages. First, it does not ensure that at least one DMU will be overall efficient and cannot provide projections on the efficient frontier. These are common issues that permeate all NDEA models but, they contradict the benchmarking character of DEA [15]. However, as the proposed modeling approach relies on MOP theory, these issues are perfectly explicable. In addition, our approach is computationally more demanding as the identification of the nadir point is required, and two non-linear programs have to be solved. To conclude, even though this approach has a higher computational load, it overcomes the aforementioned deficiencies of the additive efficiency decomposition method [23] for parallel structures, and it can be extended to any other type of structure. In the next sections we extend the proposed approach to series and complex structures.

3. Extension to series structures

The extension of the proposed approach to series structures is straightforward. In fact, it is an extension of Despotis et al. [14] and Despotis et al. [13] that bridges the min-max and max-min formulations. For the sake of brevity, we illustrate the new approach for a unit (DMU k) with a general two-stage series structure as depicted in Figure 2. The first stage utilizes a set of external inputs $\left(X_k^1\right)$ to produce the intermediate measures (Z_k) and some outputs $\left(Y_k^1\right)$. Then, the second stage consumes the intermediate measures (Z_k) along with another set of external inputs $\left(X_k^2\right)$ to produce additional outputs $\left(Y_k^2\right)$.

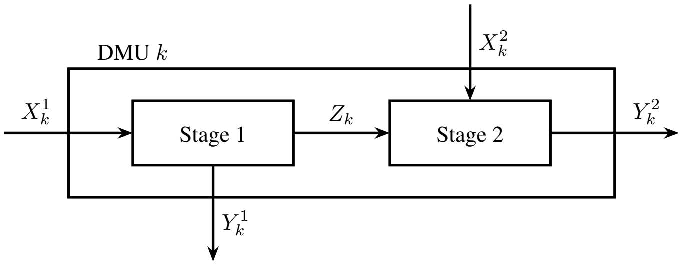

Figure 2. General two-stage series structure

The coordinates of the ideal $I\left(E_k^{1+}, E_k^{2+}\right)$ and nadir $N\left(E_k^{1-}, E_k^{2-}\right)$ point can be obtained by solving with the lexicographic approach the bi-objective model (15), in which v^1, v^2, w, u^1 and u^2 are the weights associated with X_k^1, X_k^2, Z_k, Y_k^1 and Y_k^2, respectively.

$$\begin{aligned}
&\max\left\{E_k^1=\frac{u^1Y_k^1+wZ_k}{v^1X_k^1}, E_k^2=\frac{u^2Y_k^2}{v^2X_k^2+wZ_k}\right\}\\
&\text{s.t.}\\
&\frac{u^1Y_j^1+wZ_j}{v^1X_j^1}\le 1,\ j=1,2,\ldots,n\\
&\frac{u^2Y_j^2}{v^2X_j^2+wZ_j}\le 1,\ j=1,2,\ldots,n\\
&\frac{u^1Y_j^1+u^2Y_j^2}{v^1X_j^1+v^2X_j^2}\le 1,\ j=1,2,\ldots,n\\
&v^1,v^2,w,u^1,u^2\ge 0
\end{aligned}\tag{15}$$

Once the coordinates of the reference points are estimated, the min-max or the max-min formulation can be employed as they both yield the same results when the proposed normalization technique is applied. In model (16) we present the max-min formulation.

$$\begin{aligned}
&\max\delta\\
&\text{s.t.}\\
&\frac{\dfrac{u^1Y_k^1+wZ_k}{v^1X_k^1}-E_k^{1-}}{E_k^{1+}-E_k^{1-}}\ge\delta\\
&\frac{\dfrac{u^2Y_k^2}{v^2X_k^2+wZ_k}-E_k^{2-}}{E_k^{2+}-E_k^{2-}}\ge\delta\\
&\frac{u^1Y_j^1+wZ_j}{v^1X_j^1}\le 1,\ j=1,2,\ldots,n\\
&\frac{u^2Y_j^2}{v^2X_j^2+wZ_j}\le 1,\ j=1,2,\ldots,n\\
&\frac{u^1Y_j^1+u^2Y_j^2}{v^1X_j^1+v^2X_j^2}\le 1,\ j=1,2,\ldots,n\\
&v^1,v^2,w,u^1,u^2\ge 0
\end{aligned}\tag{16}$$

Model (16) provides a weakly Pareto optimal solution. To secure that the efficiency scores are Pareto optimal, model (17) is solved, in which δ^* derives from the optimal solution of model (16). Notably, models (16) and (17) are non-linear. However, they can be solved by the techniques discussed in Section 2.

$$\begin{aligned}
&\max S_1 + S_2 \\
&\text{s.t.} \\
&\frac{\dfrac{u^1Y_k^1 + wZ_k}{v^1X_k^1} - E_k^{1-}}{E_k^{1+} - E_k^{1-}} - S_1 = \delta^* \\
&\frac{\dfrac{u^2Y_k^2}{v^2X_k^2 + wZ_k} - E_k^{2-}}{E_k^{2+} - E_k^{2-}} - S_2 = \delta^* \\
&\frac{u^1Y_j^1 + wZ_j}{v^1X_j^1} \le 1, \ j = 1, 2, \ldots, n \\
&\frac{u^2Y_j^2}{v^2X_j^2 + wZ_j} \le 1, \ j = 1, 2, \ldots, n \\
&\frac{u^1Y_j^1 + u^2Y_j^2}{v^1X_j^1 + v^2X_j^2} \le 1, \ j = 1, 2, \ldots, n \\
&v^1, v^2, w, u^1, u^2 \ge 0 \\
&\delta^* \ge S_k \ge 0, \ k = 1, 2
\end{aligned} \tag{17}$$

We compare the proposed approach with the min-max approach [14], max-min approach [13], and the relational model of Kao [24]. For the comparison we use a synthetic dataset provided in Despotis et al. [14], illustrated in Table 6. The results for all units are reported in Table 7[3] and in Figure 3 we provide a graphical comparison among these models for DMU 3.

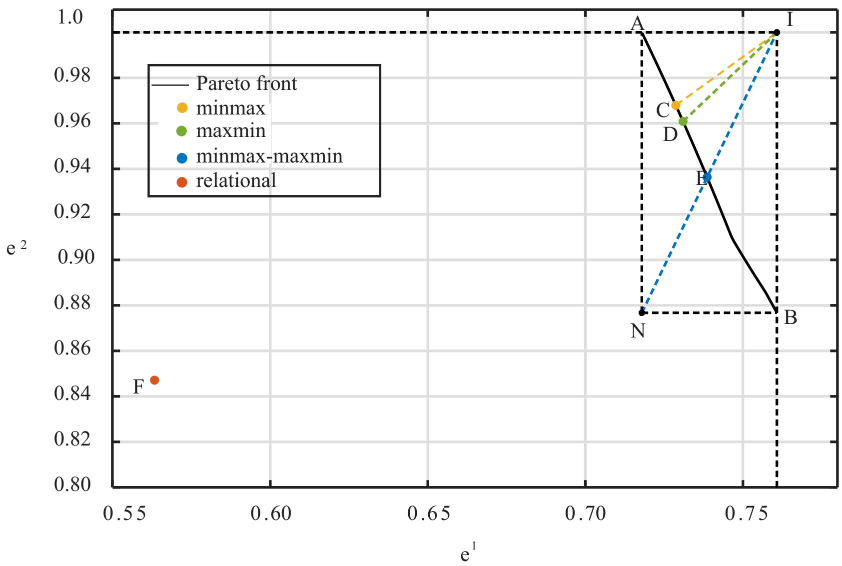

Figure 3. Comparison of the minmax-maxmin with other prevalent approaches for DMU 3 of Table 6

[3]The overall efficiency scores of the aforementioned approaches are omitted due to size limitations.

Table 6. Synthetic dataset for general two-stage series structure

DMU	x_1^1	x_2^1	x_3^1	y_1^1	y_2^1	z_1	z_2	x_1^2	x_2^2	y_1^2	y_2^2
1	22.3	13.2	54.6	21.8	44.6	110.1	66.1	18	31	13.3	12.5
2	68.3	8.3	15.8	19.8	12	75.4	116.4	19.6	25.8	2.4	18.2
3	52	19.2	31.2	47.3	47.4	94.3	59.9	11.5	22.5	2.3	36
4	31.8	12	40.3	10.5	35.8	66.4	127.2	16.8	37.1	3	19.5
5	95.3	12	29	15	22.5	108.9	52.3	14	27.2	15.8	16.7
6	52.8	6.1	22.6	69.6	27	102.4	78.8	14.8	44.9	12.6	20.4
7	50.5	9.3	48.7	52.2	49.8	124.6	120.6	5.9	38.5	9.5	20.5
8	80.1	17.4	58.4	37.7	14.6	64.5	131.2	10.3	65.6	16.7	39.9
9	53.9	14	36.9	60.9	24.1	129.8	122.1	11.9	49.5	16.8	15
10	20.9	9.5	48.8	12.2	68.7	66.4	132.5	10.1	54.5	10	28.3
11	82.5	7.1	16.8	47.7	60.7	71.9	138.9	5.6	19.1	19.7	33.6
12	27	10.6	25.6	47.3	63.3	51.9	84.4	11	39.6	12.2	43.7
13	49.6	10.7	20.6	15.3	32.6	125.5	97.3	17.7	38.9	18.9	44.7
14	55.7	19.4	46.6	79	60.3	91.5	117.3	11.8	26.4	7.5	38.7
15	55.1	18.2	52.5	12.2	24.9	90.1	61	17	33.5	17.2	43.9
16	66.3	8	34.9	57	30.7	131.1	63.7	10.7	52.5	11.2	15.5
17	93.3	6.3	43.5	38.6	32.1	53.5	133.9	13.4	45	19.7	15.4
18	10.8	11.9	31.5	34.9	23.6	118.7	89.4	11	67.3	8	20.5
19	98.5	6.8	21.3	28	28.9	75.4	133	16.1	26.7	9.5	20.3
20	27.8	17.1	24.9	30.6	14.3	81	52.2	16.9	35.3	17.4	15.4
21	42	7.2	59.7	29.2	39.4	98.4	147.5	14.8	42.3	10.7	44.5
22	98.7	8.5	51	27.3	69.3	132.8	60.6	19.8	61.9	19.9	33.3
23	53.5	15.6	25.7	31.3	34.6	93.5	121.6	19.7	56.5	13	47.6
24	25.1	16.7	56.8	62.1	74.8	81.6	145.6	11.7	17.4	7.6	29.9
25	96.3	15.3	45.1	25.7	56.8	120.5	133.6	19.7	16.9	14.9	38.5
26	97.9	6.8	53.1	45.7	49.6	103.8	89.8	17.7	56.3	4.9	12.5
27	37.4	15	15.5	53.1	22	63.1	128.2	5.5	57.7	5.8	11.8
28	70	12.8	21.5	28.3	44.3	126.1	97.2	11.4	56.7	4.9	47.2
29	24	5.8	33.8	73.7	76.2	91.2	82.6	19.4	42.4	7.8	10.3
30	48.6	18.6	55.9	15.4	57.6	126	73.8	17.1	76.2	13.2	25.6

In Figure 3, the horizontal and the vertical axes represent the efficiency scores of the first and the second stages, respectively. The curve AB denotes the Pareto front whereas the points I and N denote the ideal and the nadir points, respectively. Point C is estimated from the min-max method [14] and derives from the Pareto front intersection with a ray from the ideal point I with direction $(-1,-1)$. In the max-min method [13], the nadir point is assumed to be the origin of the axes. Therefore, the estimated point D derives from the intersection of the Pareto front with a ray from the origin of the axes with direction $\left(E_k^{1+},\, E_k^{2+}\right)$. Notably, these two methods locate a different point on the Pareto front and they generate different divisional efficiency scores. On the contrary, point E as estimated from the proposed minmax-maxmin method, is simultaneously as close as possible to the ideal point (I) and as far as possible from the true nadir point (N). Geometrically, point E is located at the intersection of the Pareto front with the ray that passes through the two reference points (ideal and nadir). All of these methods, as they rely on the composition approach, identify a pair of stage efficiency scores that lie on the Pareto front AB. The relational model of Kao [24] estimates point F, which lies below the Pareto front, i.e., it provides dominated stage efficiency scores (even from the nadir point). This is in line with the finding of Sotiros et al. [41] who identified that the relational model does not comply with the dominance property and can derive misleading results.

Table 7. Results from of the minmax-maxmin approach and other prevalent approaches on the dataset in Table 6

DMU	Minmax		Maxmin		Minmax-Maxmin		Kao	
	E^1	E^2	E^1	E^2	E^1	E^2	E^1	E^2
1	0.6751	0.6739	0.6750	0.6740	0.7011	0.6305	0.6883	0.6161
2	0.9868	0.4619	0.9792	0.4652	0.9780	0.4658	0.9566	0.4751
3	0.7287	0.9679	0.7310	0.9608	0.7386	0.9363	0.5633	0.8471
4	0.8280	0.3610	0.7789	0.3719	0.7084	0.3902	0.3855	0.3074
5	0.5488	0.8574	0.5719	0.8272	0.5735	0.8250	0.3874	1
6	1	0.6708	1	0.6708	1	0.6708	1	0.5689
7	0.8529	0.6033	0.8467	0.6069	0.8452	0.6077	0.8020	0.6316
8	0.4727	0.9028	0.5012	0.8934	0.4202	0.9243	0.5420	0.8522
9	0.8298	0.7198	0.8270	0.7218	0.8241	0.7234	0.7223	0.7499
10	1	0.6316	1	0.6316	1	0.6316	1	0.5211
11	1	1	1	1	1	1	1	1
12	1	1	1	1	1	1	1	1
13	0.9715	0.8405	0.9677	0.8409	0.8835	0.8533	1	0.8183
14	0.6686	0.9336	0.6686	0.9336	0.6686	0.9336	0.6156	0.7921
15	0.4119	0.9904	0.4132	0.9804	0.4165	0.9492	0.3625	0.8613
16	0.7815	0.5662	0.7525	0.5874	0.8035	0.5514	0.6032	0.4999
17	0.9344	0.9344	0.9344	0.9344	0.9322	0.9363	0.9397	0.9300
18	1	0.5189	1	0.5189	1	0.5189	1	0.3499
19	0.9858	0.4954	0.9757	0.4972	0.9557	0.5011	0.8793	0.5096
20	0.7067	0.9390	0.7109	0.9261	0.7301	0.8666	0.6679	1
21	0.9975	0.7846	0.9969	0.7846	0.9899	0.7856	1	0.7842
22	0.7993	0.8403	0.8031	0.8374	0.8252	0.8096	0.6349	0.9517
23	0.8178	0.7184	0.8171	0.7186	0.8094	0.7219	0.8094	0.7190
24	0.9623	0.8658	0.9597	0.8671	0.9516	0.8763	0.9331	0.7905
25	0.6613	1	0.6613	1	0.6613	1	0.5130	1
26	0.9343	0.2323	0.9189	0.2361	0.8819	0.2382	0.5552	0.2078
27	1	0.3791	1	0.3791	1	0.3791	1	0.2740
28	0.9030	0.9030	0.9030	0.9030	0.8768	0.9186	0.7393	0.9714
29	1	0.3762	1	0.3762	1	0.3762	1	0.2642
30	0.5512	0.6079	0.5544	0.6047	0.5707	0.5877	0.4753	0.6728

Notably, the violation of the dominance property in the relational model for a general two-stage series structure is attributed to the way that the efficiency scores for DMUs with a parallel internal structure are calculated. Kao [24] argued that: *"Moreover, by utilizing dummy processes, a network system can be represented by a series structure where each stage in the series is of a parallel structure composed of a set of processes"* and he illustrated that on a complex structure with three sub-processes. In the case of the general two-stage series structure depicted in Figure 2, the equivalent structure is illustrated in Figure 4, in which the square nodes represent the original stages, and the circle nodes reflect dummy processes that facilitate the flow of netputs.

In Kao [24], the efficiencies of the sub-systems in Figure 4 are estimated based on the method of Kao [23]. However, the parallel sub-processes in each sub-system are not homogeneous (Assumption II in Section 2). Indeed, in sub-system I, node (1) utilizes X_k^1 and produces Z_k and Y_k^1, whereas the dummy node (3) utilizes and produces X_k^2. Similarly, in the sub-system II, node (2) consumes Z_k and X_k^2 to produce Y_k^2, whereas the dummy node (4) consumes and produces Y_k^2. Therefore, the relational model for general two-stage series structures suffers from all the peculiarities described in Section 2.1

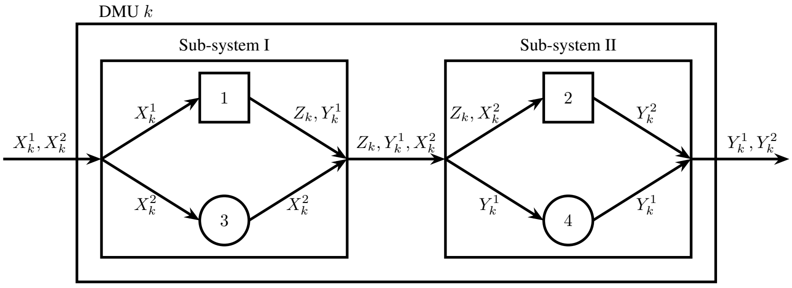

Figure 4. Transformed two-stage general structure

and cannot derive reliable results. This is happening for all network structures except multistage series structures in which the external inputs enter only to the first stage and the final outputs exit only from the last stage.

4. Illustration to complex structures

To illustrate the applicability of the proposed modeling approach to complex structures, we use the dataset provided in Kao [21]. In this dataset, each DMU is composed of three sub-processes which form a complex structure as depicted in Figure 5. Specifically, DMU k utilizes two external inputs (x_{1k}, x_{2k}) to produce three final outputs $\left(y_{1k}^{1o}, y_{2k}^{2o}, y_{3k}\right)$. Sub-process 1 consumes some amount $\left(x_{1k}^1, x_{2k}^1\right)$ of the external inputs to produce a single output $\left(y_{1k}^1\right)$. A portion of this output $\left(y_{1k}^{1o}\right)$ exits the system whereas the remaining amount $\left(y_{1k}^{1I} = y_{1k}^1 - y_{1k}^{1o}\right)$ is used as input to sub-process 3. Similarly, sub-process 2 consumes some amount $\left(x_{1k}^2, x_{2k}^2\right)$ of the external inputs to produce a single output $\left(y_{2k}^2\right)$. A fragment of this output $\left(y_{2k}^{2o}\right)$ exits the system and the remaining output $\left(y_{2k}^{2I} = y_{2k}^2 - y_{2k}^{2o}\right)$ is used as input to sub-process 3. Finally, sub-process 3 utilizes a portion $\left(x_{1k}^3, x_{2k}^3\right)$ of the external inputs along with y_{1k}^{1I} and y_{2k}^{2I} to produce a single output (y_{3k}). The dataset is presented in Table 8.

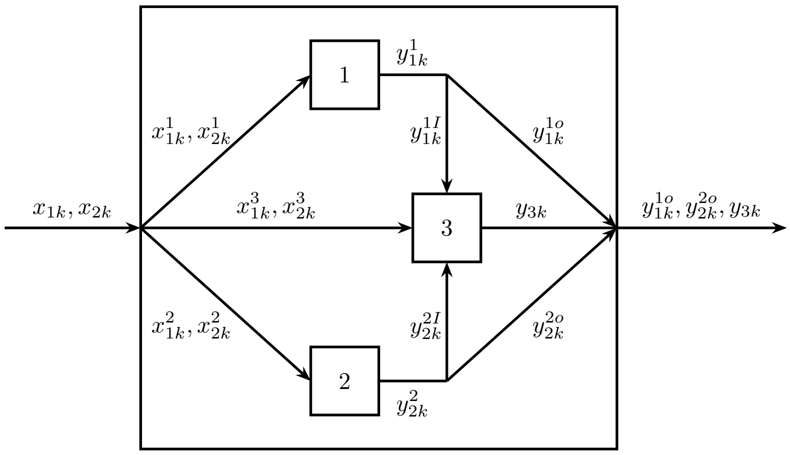

Figure 5. A complex structure
with three sub-processes from Kao [21]

Regarding the inputs, we can observe that the sub-processes 1 and 2 are homogeneous as they utilize the same type of inputs. However, sub-process 3 utilizes additional inputs $\left(y_1^{1I}, y_2^{2I}\right)$. Concerning the outputs, all sub-processes are heterogeneous as each sub-process produces different outputs from the rest sub-processes. Therefore, all sub-processes are heterogeneous (Assumption III) and the results provided by the relational model of Kao [23] are misleading.

Table 8. Dataset provided by Kao [21]

DMU	Process 1			Process 2			Process 3					System				
	Inputs		Output	Inputs		Output	Inputs				Output	Inputs		Output		
	x_1^1	x_2^1	y_1^1	x_1^2	x_2^2	y_2^2	x_1^3	x_2^3	y_1^{1I}	y_2^{2I}	y_3	x_1	x_2	y_1^{1o}	y_2^{2o}	y_3
A	3	5	4	4	3	3	4	6	2	1	1	11	14	2	2	1
B	2	3	2	2	1	2	3	3	1	1	1	7	7	1	1	1
C	3	4	2	5	3	2	3	7	1	1	2	11	14	1	1	2
D	4	6	3	5	5	4	5	3	1	1	1	14	14	2	3	1
E	5	6	4	5	4	4	4	5	1	2	3	14	15	3	2	3

To derive unbiased efficiency scores, we employ the proposed efficiency assessment method. The coordinates of the ideal point can be calculated by optimizing separately each objective function in model (18) after applying the corresponding Charnes–Cooper transformation [6].

$$\begin{aligned}
&\max E_k^1 = \frac{u^1 y_{1k}^1}{v^1 x_{1k}^1 + v^2 x_{2k}^1}\\
&\max E_k^2 = \frac{u^2 y_{2k}^2}{v^1 x_{1k}^2 + v^2 x_{2k}^2}\\
&\max E_k^3 = \frac{u^3 y_{3k}}{v^1 x_{1k}^3 + v^2 x_{2k}^3 + u^1 y_{1k}^{1I} + u^2 y_{2k}^{2I}}\\
&\text{s.t.}\\
&\frac{u^1 y_{1j}^1}{v^1 x_{1j}^1 + v^2 x_{2j}^1} \le 1,\ j = 1, 2, \ldots, n\\
&\frac{u^2 y_{2j}^2}{v^1 x_{1j}^2 + v^2 x_{2j}^2} \le 1,\ j = 1, 2, \ldots, n\\
&\frac{u^3 y_{3j}}{v^1 x_{1j}^3 + v^2 x_{2j}^3 + u^1 y_{1j}^{1I} + u^2 y_{2j}^{2I}} \le 1,\ j = 1, 2, \ldots, n\\
&v^1, v^2, u^1, u^2, u^3 \ge 0
\end{aligned} \tag{18}$$

Regarding the estimation of the nadir point, it is worth noting that model (18) has three objective functions and therefore, a more extensive search is required. This issue derives from the fact that different priorities on the objectives will provide different results. Therefore, the coordinates of the nadir point can be defined by the minimum efficiency scores derived by solving iteratively model (18) with the lexicographic method while different priority is given to the objective functions in each iteration. For instance, the first coordinate of the nadir point (E_k^{1-}) can be defined as $E_k^{1-} = \min\left\{\overline{E_k^1}, \overline{\overline{E_k^1}}\right\}$ where $\overline{E_k^1}$ and $\overline{\overline{E_k^1}}$ are the efficiency scores of the first sub-process when the $\operatorname{lex\,max}\left\{E_k^2, E_k^3, E_k^1\right\}$ and $\operatorname{lex\,max}\left\{E_k^3, E_k^2, E_k^1\right\}$ approaches are employed, respectively.

In model (19) the coordinates of the reference points (ideal and nadir) are used to derive the divisional efficiency scores in the 1st phase of our approach. Once the optimal value (δ^*) in the objective function is calculated, model (20) can be employed to identify Pareto optimal divisional efficiency scores. Table 9 reports the coordinates of the reference points and in Table 10 the efficiency scores from Kao [21] and model (20) are presented. For the estimation of the overall efficiency scores from model (20), the multiplicative aggregation is employed, i.e., $E^0 = E^1 \times E^2 \times E^3$. It is worth noting that the efficiency scores from model (20) are identical to those from model (19), i.e., the efficiency scores from model (19) are Pareto optimal.

$$
\begin{aligned}
&\max \delta \\
&\text{s.t.} \\
&\frac{\dfrac{u^1 y_{1k}^1}{v^1 x_{1k}^1 + v^2 x_{2k}^1} - E_k^{1-}}{E_k^{1+} - E_k^{1-}} \geq \delta \\
&\frac{\dfrac{u^2 y_{2k}^2}{v^1 x_{1k}^2 + v^2 x_{2k}^2} - E_k^{2-}}{E_k^{2+} - E_k^{2-}} \geq \delta \\
&\frac{\dfrac{u^3 y_{3k}}{v^1 x_{1k}^3 + v^2 x_{2k}^3 + u^1 y_{1k}^{1I} + u^2 y_{2k}^{2I}} - E_k^{3-}}{E_k^{3+} - E_k^{3-}} \geq \delta \\
&\frac{u^1 y_{1j}^1}{v^1 x_{1j}^1 + v^2 x_{2j}^1} \leq 1,\ j = 1, 2, \ldots, n \\
&\frac{u^2 y_{2j}^2}{v^1 x_{1j}^2 + v^2 x_{2j}^2} \leq 1,\ j = 1, 2, \ldots, n \\
&\frac{u^3 y_{3j}}{v^1 x_{1j}^3 + v^2 x_{2j}^3 + u^1 y_{1j}^{1I} + u^2 y_{2j}^{2I}} \leq 1,\ j = 1, 2, \ldots, n \\
&v^1, v^2, u^1, u^2, u^3 \geq 0
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
&\max S_1 + S_2 + S_3 \\
&\text{s.t.} \\
&\frac{\dfrac{u^1 y_{1k}^1}{v^1 x_{1k}^1 + v^2 x_{2k}^1} - E_k^{1-}}{E_k^{1+} - E_k^{1-}} - S_1 = \delta^* \\
&\frac{\dfrac{u^2 y_{2k}^2}{v^1 x_{1k}^2 + v^2 x_{2k}^2} - E_k^{2-}}{E_k^{2+} - E_k^{2-}} - S_2 = \delta^* \\
&\frac{\dfrac{u^3 y_{3k}}{v^1 x_{1k}^3 + v^2 x_{2k}^3 + u^1 y_{1k}^{1I} + u^2 y_{2k}^{2I}} - E_k^{3-}}{E_k^{3+} - E_k^{3-}} - S_3 = \delta^* \\
&\frac{u^1 y_{1j}^1}{v^1 x_{1j}^1 + v^2 x_{2j}^1} \leq 1,\ j = 1, 2, \ldots, n \\
&\frac{u^2 y_{2j}^2}{v^1 x_{1j}^2 + v^2 x_{2j}^2} \leq 1,\ j = 1, 2, \ldots, n \\
&\frac{u^3 y_{3j}}{v^1 x_{1j}^3 + v^2 x_{2j}^3 + u^1 y_{1j}^{1I} + u^2 y_{2j}^{2I}} \leq 1,\ j = 1, 2, \ldots, n \\
&v^1, v^2, u^1, u^2, u^3 \geq 0 \\
&\delta^* \geq S_k \geq 0,\ k = 1, 2, 3
\end{aligned} \tag{20}
$$

Table 9. Coordinates of the ideal and nadir points of the divisions according to the dataset in Table 8

DMU	Coordinates of ideal point			Coordinates of nadir point		
	E^{1+}	E^{2+}	E^{3+}	E^{1-}	E^{2-}	E^{3-}
A	1	0.75	0.4	0	0.75	0.3462
B	0.8333	1	0.5714	0	1	0.5088
C	0.625	0.4	1	0	0.3333	0.5524
D	0.625	0.8	0.5714	0	0.4	0.3333
E	0.8333	0.8	1	0.6	0.5	1

Table 10. Efficiency scores from Kao [21] and model (20)

DMU	Efficiency scores from Kao [21]				Efficiency scores from model (20)			
	E^1	E^2	E^3	E^0	E^1	E^2	E^3	E^0
A	1	0.75	0.3462	0.5227	0.4673	0.75	0.3713	0.1301
B	0.8333	1	0.5088	0.5952	0.3849	1	0.5377	0.2070
C	0.5	0.4	0.9474	0.5682	0.5074	0.3982	0.9158	0.1850
D	0.5625	0.8	0.3333	0.4821	0.258	0.5651	0.4316	0.0629
E	0.8333	0.5	1	0.8	0.7285	0.6652	1	0.4846

5. Teaching and research efficiency of higher education institutions in Poland

Universities, as higher education institutes (HEIs), play a pivotal role in providing students with the knowledge and skills essential for success in their careers. Offering a diverse range of academic programs, spanning from medicine and engineering to business and arts, they promote knowledge dissemination across various disciplines. Apart from education, they lead in knowledge advancement and research, which drive innovation, address complex challenges, and enhance the quality of life globally.

Worldwide several countries have implemented governmental policies to monitor the level of education and research that the national HEIs provide. Indicative examples are the Research Excellence Framework (REF) in the UK and the Excellence in Research for Australia (ERA). In Poland, the quality of education is ensured and enhanced by an independent institution, the Polish Accreditation Committee[4] (PKA), established in 2002. The PKA is responsible for evaluating the educational programs and providing opinions on applications for granting the right to conduct studies. The assessment of each HEI mainly relies on a qualitative analysis according to its previous evaluations, a self-report of the evaluated HEI, and an on-site visit by a team of experts. The quality of HEIs' scientific activity is periodically assessed by the Commission for the Evaluation of Science[5], which has been incorporated into the Ministry of Science since 2004. The evaluation model of scientific activity is quantitative and relies on three criteria 1) publications and patents, 2) income from grants, research and development projects, and commercialization, and 3) societal impact. The results of these evaluations do not only affect the prestige of the institutions, but they also determine the recognition of degrees as well as the level of governmental

[4] https://pka.edu.pl/en/home-page/

[5] https://www.gov.pl/web/nauka/komisja-ewaluacji-nauki

funding. Polish public HEIs do not have tuition fees, but they are financed by the government. The two main sources of income for HEIs in Poland, accounting for more than 60%, are teaching grants allocated by the Ministry of Education and Science and funds for research [36]. Ministerial subsidies are the most stable source of revenue and are allocated based on the results of these evaluations. Therefore, it is essential to measure the performance of HEIs using an accurate and objective procedure.

It is worth noting that the evaluation of scientific activity does not consider only the quantity of publications but also the quality of the journals/conferences (or publishing houses in the case of monographs) where they are published. The quality of the journals/conferences is measured periodically by the Commission for the Evaluation of Science which assigns points (20, 40, 70, 100, 140, 200) to them. Nevertheless, the ratings of the journals/conferences may change frequently, and significant variations can be observed, i.e., a journal's points may change from 70 to 200 points and vice versa. This issue not only disorients academics but also significantly impacts the evaluations of the national HEIs, their final classification, and the determination of governmental funding. Notably, REF in the UK does not rely on journal rankings but the quality of every publicly available research output is evaluated by a panel of experts, even though such a procedure is time-consuming and labor-demanding. The implementation of such a policy implies that the journal rankings do not always reflect the quality of the published manuscripts and new approaches should be employed to assess it.

DEA has been broadly employed to measure the efficiency of HEIs. Their non-profitable character, the absence of market prices that could drive the optimization process, the unknown relative importance of the factors (inputs/outputs) that describe the activities of HEIs as well as the flexibility of DEA to deal with multiple outputs set DEA as a valuable tool in this area. Relevant studies include, among others, those by Johnes and Johnes [19], Beasley [4], Athanassopoulos et Shale [2], Johnes [20], Thanassoulis et al. [42], Despotis et al. [11], Lee and Worthington [34], Koronakos et al. [29] and Lee and Johnes [33].

5.1. Assessment framework and data selection

In this section, we propose a DEA framework based on data publicly available, as an objective procedure to evaluate the teaching and research activity of Polish public HEIs. The dataset is drawn from the academic year 2020–2021. The main source of data is the governmental website RAD-on[6], which collects data on the public system in Poland from many trusted sources. The chosen factors from this source are related to standard metrics regarding the number of students, PhD students, graduates and academic staff. The second source of data is the SciVal[7] website from Elsevier, from which the Field-Weighted Citation Impact (FWCI) is selected as a proxy for the quality and the impact of the publications of each institution. Finally, data related to finances, i.e., subsidies from the Ministry of Education and Science and grants awarded by the National Science Centre (NCN) or the National Centre for Research and Development (NCBiR), are collected from the universities' annual reports or by request via email in cases where such reports were not published.

We view the teaching and research activities of the institutions as a two-stage series process as depicted in Figure 6. The first stage represents the teaching activity, and we consider as inputs the number

[6]https://radon.nauka.gov.pl/

[7]https://www.scival.com/

of students (X_1), hours devoted to teaching (X_2), and funds for the teaching for 2020 (X_3). The outputs from the first stage are the number of graduates (K), the number of Polish PhD students (Z_1), and the number of foreign PhD students (Z_2). The number of graduates is a part of the final outputs of the DMU while the numbers of PhD students (Z_1, Z_2) serve as intermediate measures. These intermediate measures, along with hours devoted to research (L_1) and funds for the research for 2020 (L_2) are the inputs of the second stage that represent the research activity. The outputs of the second stage are the Field-Weighted Citation Impact (Y_1) and attracted funds for research for 2021 (Y_2). A detailed description of these factors is provided below.

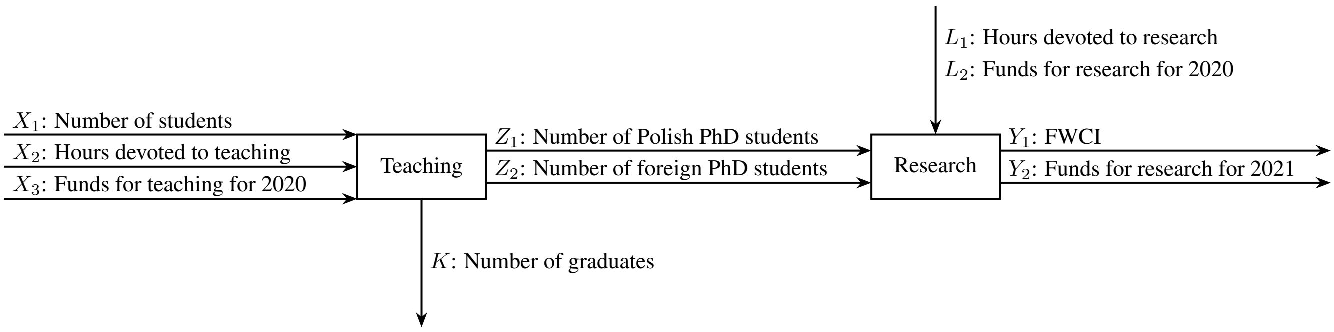

Figure 6. Teaching and research activity as a two-stage series process

X_1 – number of students. The total number of students including Polish and foreign students excluding ERASMUS students.

X_2 – hours devoted to teaching. The estimated number of hours spent by academic staff on teaching. It is calculated based on the number of academic staff in the following academic degrees or titles: professor (Pol., *profesor*), professor habilitatus (DSc) (Pol., *doktor habilitowany*), doctor (PhD) (Pol., *doktor*) and those without academic degree or title. The number of hours spent on teaching by each faculty member from each group is assessed based on the legal act [47], where 10% of the specified number of hours is assumed to be spent on administrative affairs and 90% on teaching-related activities, as shown in Table 11.

Table 11. Assumed annual number of hours spent on research, teaching, and administration affairs for each academic staff group

Activity	Professor	Professor habilitatus (DSc)	Doctor (PhD)	Without academic degree
Research	180	180	120	0
Teaching and administration	180	180	240	360
Administration	18	18	24	36
Teaching	162	162	216	324

X_3 – funds for teaching for 2020. It is assumed to be 50% of the subsidy provided by the Ministry of Education and Science for the maintenance of teaching and research for the year 2020 (the remaining 50% is assumed allocated for research). It is measured in thousands of PLN.

K – number of graduates. The total number of students graduated, both Polish and foreign.

Z_1 – number of Polish PhD students. The total number of Polish PhD students in doctoral schools (Pol. *szkoła doktorska*) and PhD programs (Pol. *studia doktoranckie*).

Z_2 – number of foreign PhD students. The total number of foreign (non-Polish) PhD students in doctoral schools (Pol. *szkoła doktorska*) and PhD programs (Pol. *studia doktoranckie*) excluding ERASMUS students.

L_1 – hours devoted to research. It is the estimated number of hours spent by academic staff on research. Again, it is based on the division of academic staff by academic degree or title as for the input X_2. Generally, it is assumed that the total number of teaching & administration hours together with hours spent on research should equal 360 for each group. The assumed annual number of hours spent on research by each faculty member from each group is provided in Table 11. Note that academic staff without an academic degree or title are assumed to be dedicated exclusively to teaching and not to conduct research.

L_2 – funds for research for 2020. It is 50% of the subsidy provided by the Ministry of Education and Science for the maintenance of teaching and research potential for year 2020. It is measured in thousands of PLN.

Y_1 – Field-Weighted Citation Impact. It is a metric provided by SciVal. On Elsevier's website, it is defined as *"the ratio of citations received relative to the expected world average for the subject field, publication type, and publication year"*. Specifically, if the index equals 1, it indicates that the entity's publications have received exactly the number of citations predicted by the global average for similar publications. However, if it is more than 1 or less than 1, it indicates above the global average or below it for similar publications, respectively.

Y_2 – funds for research for 2021. Equals to 50% of the subsidy provided by the Ministry of Education and Science for maintenance of teaching and research potential for the year 2021, plus funds granted by NCN and NCBiR in 2021 for the implementation of research projects. It is measured in thousands of PLN.

Notably, regarding research efficiency, we did not take into consideration the journal rankings of the published outcomes as their validity may be questionable. In addition, we did not include as outputs from the second stage the number of publications achieved by each institution, which is commonly used in the literature. The reason is the heterogeneity of the institutions, in terms of their disciplines, which makes the number of publication outcomes incomparable. On the contrary, as proxies for the quality and quantity of research outputs, we used the field-weighted citation impact factor and the research funds attracted during the evaluation period. FWCI is standardized by each discipline and reflects the quality and impact of the research outputs. On the other hand, the volume of research funds can serve as a proxy for the number of publications achieved, i.e., the more publications of high quality are achieved the highest the volume of research grants is expected to be.

Nevertheless, due to data unavailability, our assessment relies on some assumptions. Funds from research projects (NCN, NCBiR) in 2020 (and previous years) are excluded from the calculation of the total funds for research used in 2020. Such funds are generally used in 3–4 years and there was no information on how much of these funds were used in 2020. On the contrary, funds attracted for research in 2021 from research projects (NCN, NCBiR) are viewed as research outputs since they reflect the ability of the institutions to attract them, and their volume plays a crucial role. Furthermore, there was no information on how the subsidy provided by the Ministry of Education and Science for the maintenance of teaching and research was allocated by each institution to teaching and research. Therefore, we assumed these funds were allocated equally to research and teaching. In addition, we used the hours devoted to teaching and research as proxies of the institutions' workforce for these two activities. However, these hours are approximations based on the underlying assumptions provided in Table 11. These assumptions, to the best of our knowledge, are realistic. However, variations across different institutions may exist. Moreover, it is assumed that the foreign (non-Polish) PhD students had previous studies in the same institution. Even though this may be the general rule in Polish academia, it may not always hold. Notably, PhD students coming from other universities should be considered as an external input to the second stage and not as an intermediate measure (i.e., output from the first stage and input to the second one). Finally, due to data unavailability, factors related to the societal impact of the research outputs are not taken into consideration. Nevertheless, our aim in this application is not to assess the efficiency of the institutions but rather to propose an evaluation framework that under the presence of more rich and accurate data can potentially serve as an objective assessment framework.

Table 12. Polish institutions included in the application

DMU	English name	Polish name	Type
1	University of Warsaw	Uniwersytet Warszawski	university
2	Warsaw University of Technology	Politechnika Warszawska	technical university
3	Adam Mickiewicz University	Uniwersytet im. Adama Mickiewicza w Poznaniu	university
4	AGH University of Science and Technology	Akademia Górniczo-Hutnicza im. Stanisława Staszica w Krakowie	technical university
5	Gdansk University of Technology	Politechnika Gdańska	technical university
6	Medical University of Gdansk	Gdański Uniwersytet Medyczny	medical university
7	Wroclaw University of Science and Technology	Politechnika Wrocławska	technical university
8	Medical University of Lodz	Uniwersytet Medyczny w Łodzi	medical university
9	University of Wrocław	Uniwersytet Wrocławski	university
10	Lodz University of Technology	Politechnika Łódzka	technical university
11	Medical University of Warsaw	Warszawski Uniwersytet Medyczny	medical university
12	Nicolaus Copernicus University	Uniwersytet Mikołaja Kopernika w Toruniu	university
13	Silesian University of Technology	Politechnika Śląska	technical university
14	Warsaw School of Economics	Szkoła Główna Handlowa w Warszawie	university
15	University of Gdansk	Uniwersytet Gdański	university
16	Pomeranian Medical University	Pomorski Uniwersytet Medyczny w Szczecinie	medical university
17	Poznan University of Medical Sciences	Uniwersytet Medyczny im. Karola Marcinkowskiego w Poznaniu	medical university
18	Wroclaw Medical University	Uniwersytet Medyczny im. Piastów Śląskich we Wrocławiu	medical university
19	Medical University of Bialystok	Uniwersytet Medyczny w Białymstoku	medical university

In the academic year 2020–2021 (assessment period), 349 public HEIs were operating in Poland. However, we selected only the top 20 universities according to the Academic Ranking of Universities[8] issued by the magazine Perspektywy[9]. From this list, The Jagiellonian University was excluded due to missing financial data. The final set, composed of 19 HEIs, is provided in Table 12. Notably, there are 6 universities, 6 universities of technology, and 7 medical universities. The data for these HEIs, are provided in Table 13.

[8]https://ranking.perspektywy.pl/2022/ranking/ranking-uczelni-akademickich

[9]Perspektywy is a monthly educational magazine published since 1998 in Warsaw

Table 13. Data of the 19 HEIs according to Figure 6

DMU	X_1	X_2	X_3	Z_1	Z_2	K	L_1	L_2	Y_1	Y_2
1	40247	812484	396726.60	2159	276	8925	522480	524600.30	1.36	565928.15
2	24756	561924	286680.60	1049	52	5432	276360	351218.10	0.93	372510.15
3	35032	600156	272541.90	1057	99	7270	391200	343948.60	1.04	367313.40
4	20601	467856	296055.45	989	67	6571	268920	377820.78	1.02	386284.11
5	14130	319842	155823.00	522	48	3882	159420	175948.00	1.04	186754.70
6	6186	261738	86958.05	311	4	1083	122460	102258.85	1.33	116175.21
7	23535	473256	252155.75	807	36	6696	244920	295155.75	0.85	414620.05
8	9785	370656	133755.45	492	1	1846	163440	152203.12	1.69	168466.00
9	23033	403434	184396.55	962	59	5657	255540	245696.55	0.99	235079.70
10	12377	256068	163719.50	538	40	2962	151800	201675.90	0.90	222218.47
11	10002	432324	133881.13	411	9	2095	183840	160527.17	1.17	182845.89
12	19995	512730	210420.25	723	45	4527	280620	234609.65	1.03	238364.80
13	17530	336798	197548.40	690	48	4735	215460	214486.61	0.94	224454.53
14	10181	177066	69523.00	361	23	4349	96660	76028.91	0.88	78654.63
15	21600	391500	158467.00	838	42	6039	216600	201426.11	0.92	218635.28
16	4526	149688	54884.25	256	0	830	84240	61410.05	2.05	68784.70
17	7312	364608	105241.53	180	0	1677	161520	117908.02	1.02	139475.12
18	6372	316008	92577.52	338	1	1157	133080	102525.84	1.69	114903.15
19	5494	199692	72150.00	257	13	1097	97440	79733.00	1.12	90006.20

5.2. Results

We apply models (16)–(17) of the proposed composition approach in the data of Table 13. The overall efficiency is calculated from the multiplicative aggregation of the stage efficiency scores, i.e., $E^0 = E^1 \times E^2$. The results are provided in Table 14 and summarized in Figure 7.

Table 14. Efficiency scores of the HEIs based on models (16)–(17)

DMU	Name of HEI	E^1	E^2	E^0
1	University of Warsaw	1	0.7704	0.7704
2	Warsaw University of Technology	0.8301	0.8100	0.6723
3	Adam Mickiewicz University	0.7325	0.7657	0.5609
4	AGH University of Science and Technology	1	0.8679	0.8679
5	Gdansk University of Technology	0.8289	0.7862	0.6517
6	Medical University of Gdansk	0.5846	0.9041	0.5286
7	Wroclaw University of Science and Technology	0.7969	1	0.7969
8	Medical University of Lodz	0.5933	0.8943	0.5305
9	University of Wrocław	0.9688	0.6946	0.6729
10	Lodz University of Technology	0.8711	0.9316	0.8115
11	Medical University of Warsaw	0.6050	0.8795	0.5321
12	Nicolaus Copernicus University	0.7577	0.7383	0.5594
13	Silesian University of Technology	0.8610	0.7595	0.6540
14	Warsaw School of Economics	1	0.7981	0.7981
15	University of Gdansk	0.9899	0.7880	0.7800
16	Pomeranian Medical University	1	1	1
17	Poznan University of Medical Sciences	0.6025	1	0.6025
18	Wroclaw Medical University	0.5838	0.9192	0.5366
19	Medical University of Bialystok	0.8732	0.8885	0.7758

From the 19 evaluated units, only one (DMU 16) is overall efficient, i.e., efficient both in teaching and research activities. Regarding teaching, 4 units are efficient (DMUs 1, 4, 14, 16) and in terms of research, only 3 DMUs are on the frontier (DMUs 7, 16, 17). The obtained efficiency scores allow us not only to point out the universities that are the most efficient in each activity or overall, but also to rank the institutions in terms of teaching, research, and overall efficiency. In addition, these efficiency scores provide information regarding the strategy that each institution follows. For instance, HEIs that have higher efficiency in research than teaching imply that they are more focused on research activity.

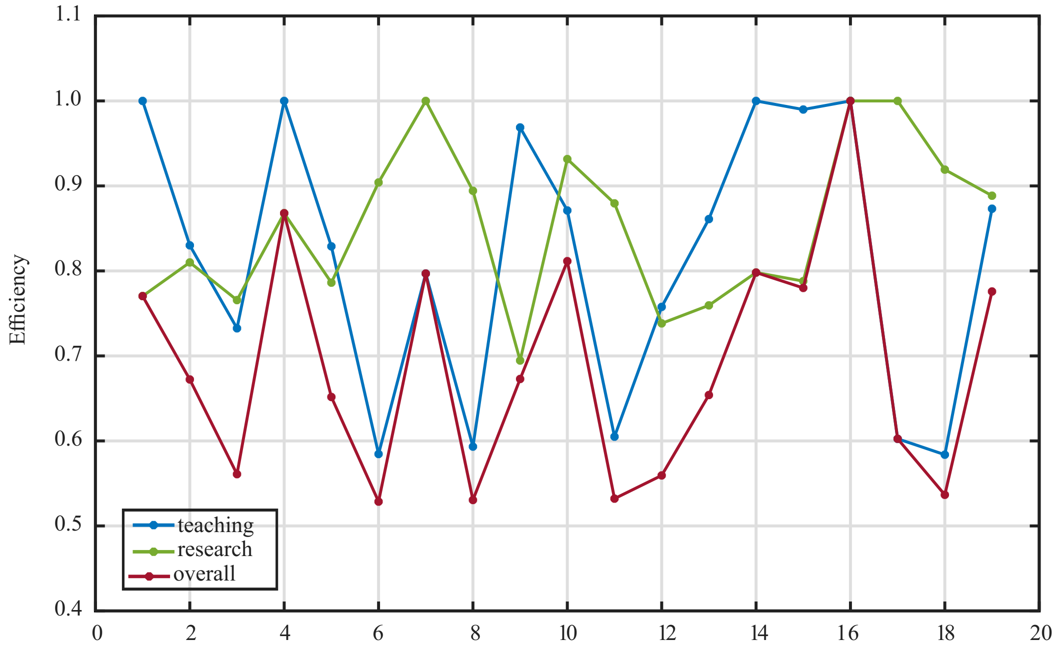

Figure 7. Efficiency scores of the HEIs based on models (16)–(17)

Notably, the teaching efficiency scores derived from the proposed framework cannot substitute the qualitative assessment of the educational programs by PKA. Nevertheless, the proposed quantitative analysis can be used as a tool to facilitate the determination of funding levels related to the maintenance of teaching. In terms of research, these results cannot be directly compared to the evaluation results of the Ministry of Science and Education for several reasons. First, in our application the assessment period was one year whereas the last assessment from the Ministry covered 5 years (2017–2021). In addition, due to data unavailability, our data are permeated with some assumptions and the criterion of societal impact was not incorporated into our assessment model. Therefore, we do not consider these results as conclusive. Nevertheless, under the presence of more accurate and rich data, such an assessment framework can provide an objective tool for the evaluation of HEIs.

6. Conclusions

In this study, we presented the pitfalls of the most established network data envelopment analysis method for units with a parallel internal structure, which to the best of our knowledge has not been explored in the literature. We further show that these pitfalls are associated with peculiarities of the decomposition models for general series structures identified in the literature. To overcome these pitfalls, we introduced an approach built upon the composition paradigm that relies on multi-objective programming. The assessment is carried out by simultaneously employing a min-max and max-min technique to estimate the divisional efficiency scores. Our approach ensures that the divisional efficiency scores are both unique and unbiased, overcoming the limitations reported for the prevalent models on the field. We further extended this approach to series and complex structures. Comparisons with other approaches under various structures and assumptions highlight the differences and advantages of the proposed approach.

We further proposed an evaluation framework for the efficiency assessment of teaching and research efficiency of HEIs in Poland. Our assessment framework departs from traditional metrics used to evaluate research activity such as number of publications and journal rankings and facilitates the field-weighted citation impact factor and research grants as proxies for the quality and quantity of publications. The results demonstrate that our method not only identifies the efficient institutions in terms of teaching and research but also provides valuable insights into the strategic focus of each institution. However, due to data unavailability and assumptions, the results should not be considered as conclusive. Nevertheless, such a framework with more rich and accurate data could serve as an objective tool for the evaluation of HEIs.

To conclude, the NDEA literature should move away from methods in which, for the sake of linearization, the weights of the sub-processes are defined as functions of the decision variables. Future research should focus on more efficient methods for identifying the nadir point in MOP and solving the proposed non-linear models as well as on methods to identify unique projections on the efficient frontier. Re-evaluating the teaching and research efficiency of Polish HEIs with more accurate data covering a longer period, investigating input congestion, and developing a data-oriented approach to allocate public funding on HEIs, based on their efficiency scores, are subjects for future research. In any case, moving away from journal rankings and facilitating more robust metrics to capture the quality of research may provide more incentives to academics to increase the quality of the published outcomes and may change the landscape regarding the predatory journals that have a considerable presence in academia.

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