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The Laplacian energy of an intuitionistic fuzzy rough graph and its utilisation in decision-making

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Abstract

An intuitionistic fuzzy rough model is a hybrid model that combines intuitionistic fuzzy sets and rough sets, addressing soft computing and ambiguity. It uses lower and upper approximation spaces in various fields, including science, technology, database systems, computer networks, and expert system architecture. The matrix of adjacency of an intuitionistic fuzzy rough graph is described in the article. The matrix of adjacency also provides the upper and lower bounds for Laplacian energy. These are used to define the Laplacian energies of intuitionistic fuzzy rough graphs and the weight function of Laplacian energy within these graphs. The intuitionistic fuzzy rough preference relation method is used for processing the intuitionistic fuzzy rough weighted average. This technique is applied in data visualizations, which make data understandable by displaying it in a graphical or pictorial style. It supports decision-making and provides factual justification. This approach benefits any field that needs creative methods for presenting large volumes of complex data. Modern computer graphics have significantly influenced visualization.

Keywords: *Laplacian energy (LE), intuitionistic fuzzy rough graph (IFRG), data visualization, and decision-making*

1. Introduction

Graph theory, developed by Euler in 1736, is a versatile tool used in various disciplines, including geometry, transportation, and engineering. Fuzzy sets and fuzzy logic were introduced by Zadeh in 1965 [32], convey ambiguity and uncertainty, and are studied in various disciplines like medicine, management, and artificial intelligence. A new degree, known as the degree of non-membership, was introduced to the fuzzy sets idea by Atanassov [4] in 1986 to resolve the ambiguity and uncertainty around the membership degree. Rosenfeld [25] pioneered fuzzy graphs, a concept first introduced by Kaufmann [17] in 1973. He proposed fuzzy relations and developed the basic framework of fuzzy graphs. Atanassov [3] is credited with developing intuitionistic fuzzy graphs and fuzzy relations.

Jamil et al. [16] use Sombor indices to study intuitionistic fuzzy graphs, which are generalizations of fuzzy graphs, to assess vaccination centers during the pandemic, highlighting their practical implications

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and efficiency. Pawlak 's [21] rough set theory is considered innovative in soft processing tools as it uniquely addresses ambiguity. Rough set theory uses lower and upper approximation ideas to extract data from systems of data and represent it as decision rules, addressing ambiguity in the universe. In Chakrabarty et al. [7], the fuzziness of imperfect sets was explored. The concept of a fuzzy rough set [11] has been developed to enhance decision-making efficiency. He et al. [14] explore rough graphs for uncertainty problems, class connections, subgraphs, weighted rough graphs, and an algorithm for exploring class optimal trees in weighted rough graphs. Akram and Arshad [1] introduced fuzzy rough graphs with application.

Cornelis et al. [9] described intuitionistic fuzzy rough sets. Xu et al. [31] investigated the intuitionistic fuzzy rough sets model based on operators. Types of intuitionistic fuzzy rough approximation operations were studied by Zhou and Wu [35], Haq et al. [13] tackle semantic issues with incomplete information by classifying types, proposing a fuzzy decision table, and developing rule extraction methods. Zhang and Zhu [34] developed three asymmetric models for resolving uncertain problems in an intuitionistic fuzzy environment, combining traditional PROMETHEE and TOPSIS procedures, demonstrating their effectiveness. Tiwari and Lohani [28] introduce a novel interval-valued intuitionistic fuzzy rough set (IVIFRS) system for conflict analysis in decision-making, utilizing rough set and interval-valued intuitionistic fuzzy set theories to measure disputes and identify conflicting attributes. Hussain et al. [15] introduce a TOPSIS method utilizing Dombi operations for aggregating averaging and geometric operators, specifically for MCGDM, designed for diagnosing severe COVID-19 patients. With an emphasis on fuzzy preference relations for effective alternative assessment, comparison, selection, and ordering, Borzecka [6] investigates fuzzy multi-criteria decision-making utilizing Zadeh's linguistic method. To compare criterion significance, prove compositional qualities and novel relational features, provide consistent choices, and rank options, Peneva and Popchev [22] investigate the weighted aggregation of fuzzy preference relations for decision-making issues.

Akram et al. [19, 33] introduced several innovative ideas in intuitionistic fuzzy rough graphs for decision-making. The Akram and Zafar monograph [2] explores hybrid models, specifically fuzzy digraphs, for complex problem-solving, highlighting their applications in decision-making and overcoming the uncertainty issues of traditional methods. Mahmood et al. [18] and Mazarbhuiya and Shenify [20] present innovative techniques for precise disease diagnosis and anomaly detection, combining rough set theory and intuitionistic fuzzy set theory for high accuracy rates. Tiwari [29] introduces a novel intuitionistic fuzzy (IF)-assisted mutual information concept that effectively handles noise, uncertainty, and vagueness in real-valued datasets, improving phospholipids positive molecule prediction. Salamat et al. [26] describe the utilization of intuitionistic fuzzy sets and rough sets in informal computing for sparse data, aiming for accuracy, accessibility, and cost-effectiveness.

Gutman introduced the concept of graph energy in chemistry, discovering lower and upper bounds on the total electron energy of certain molecules. Chemistry's relationship between molecular graph energy and overall electron energy is crucial, as a graph with n nodes has energy $2(n-1)$, while a graph with separated vertices has energy zero. Gutman and Zhou [12] defined the Laplacian energy (LE) of a graph as the sum of the absolute values of discrepancies between the average vertex degree and its Laplacian eigenvalues. Rahimi Sharbaf and Fayazi [24] defined a fuzzy graph's Laplacian energy. Basha and Kartheek [5] generalised the Laplacian energy of an intuitionistic fuzzy graph from the Laplacian energy of a fuzzy graph. Poonia and Bajaj [23] introduce a decision-making methodology for ranking alternatives in fuzzy graphs or directed graphs for hydropower plant site selection, analyzing its originality, comparative remarks, advantageous features, and limitations. Sarkar introduced advanced software to the pharmaceutical industry for tasks such as record-keeping, quality control, and clinical trial data management [27]. Dave et al. provide a concise discussion on data visualization, highlighting various tools and techniques for effectively communicating insights [10].

1.1. Challenges and motivation

Based on the overhead review, it can be concluded that an intuitionistic fuzzy rough graph (IFRG) aims to provide a reliable and adaptable mathematical framework for analyzing systems under imprecision and uncertainty in practical applications. IFRGs are also often used to reflect uncertainty in decision problems. Therefore, combinations of this tool with multi-criteria decision-making (MCDM) methods are continuously being developed. Due to the difficulty of choosing the proper MCDM techniques, further work on their development should be carried out. Moreover, a frequently addressed issue in IFRGs is the aggregation of expert knowledge. Numerous aggregation operators need to be continuously investigated. Given the above constraints associated with IFRGs, the motivation of this study is as follows:

- There is a lack of comparative analysis between the results obtained from IFRGs in some works.
- Specific terms concentrate on a specific purpose that combines domain expertise.
- Works considering IFRGs in real-life problems are missing.
- Few studies consider using MCDM techniques in conjunction with IFRGs.

1.2. Contributions and novelties

This paper focuses on using the Laplacian energy of an intuitionistic fuzzy rough graph approach in a data visualization problem. Laplacian energy is used to determine decision weights based on the decision-maker's preferences. Three decision-makers were involved in the process and presented their strategies using an intuitionistic fuzzy rough graph. Then, evaluate the aggregated preferences of the decision-makers. These preferences were also aggregated into four functions.

The paper is organized as follows: Section 2 contains fundamental concepts related to intuitionistic fuzzy rough graphs. In Section 3, we describe an intuitionistic fuzzy rough graph's Laplacian energy with an example. In Section 4, the lower and upper bounds for an intuitionistic fuzzy rough graph's Laplacian energy are also determined by its properties. The Laplacian energy of an intuitionistic fuzzy rough digraph is described in Section 5. Data visualization serves as a real-world example, and we use a comparison study to demonstrate these concepts in Section 6. Finally, Section 7 presents conclusions and future research directions.

2. Preliminaries

Definition 1. Considering that F is an ordered tuple (P_i, P_iR, Q_i, Q_iS), a non-empty set with a fuzzy rough graph of G [1].

1. P_i denotes a fuzzy relation to F.
2. Q_i denotes a fuzzy relation to $D \subseteq F \times F$.

3. $P_iR = \left(\underline{P_i}R, \overline{P_i}R\right)$ denotes a fuzzy rough set to F.
4. $Q_iS = \left(\underline{Q_i}S, \overline{Q_i}S\right)$ denotes a fuzzy rough relation to F.
5. Therefore, $G = \left(\left(\underline{P_i}R, \overline{P_i}R\right), \left(\underline{Q_i}S, \overline{Q_i}S\right)\right)$ are lower and upper approximations on fuzzy rough graphs of $G, \forall\, c, d \in F$.

$$\left(\underline{Q_i}S\right)(cd) \;\leq\; \min\left\{\left(\underline{P_i}R\right)(c),\; \left(\underline{P_i}R\right)(d)\right\}, \left(\overline{Q_i}S\right)\;(cd)\;\leq \min\left\{\left(\overline{P_i}R\right)(c),\; \left(\overline{P_i}R\right)(d)\right\}$$

Definition 2. A four-ordered tuple (X_i, X_iU, Y_i, Y_iV) is an intuitionistic fuzzy rough graph G on a nonempty set F [2].
1. X_i represents an intuitionistic fuzzy relation to F,
2. Y_i represents an intuitionistic fuzzy relation to $D \subseteq F \times F$,
3. $X_iU = \left(\underline{X_i}U, \overline{X_i}U\right)$ represents an intuitionistic fuzzy rough set to F,
4. $Y_iV = \left(\underline{Y_i}V, \overline{Y_i}V\right)$ represents an intuitionistic fuzzy rough set to F.
5. Thus $G = \left(\underline{G}, \overline{G}\right) = (X_iU, Y_iV)$ is an intuitionistic fuzzy rough graph. Here $\underline{G} = \left(\underline{X_i}U, \overline{X_i}U\right)$ and $\overline{G} = \left(\underline{Y_i}V, \overline{Y_i}V\right)$ are lower and upper approximations of the intuitionistic fuzzy rough graph of $G, \forall\, l, m \in F$

$$\left(\underline{Y_i}V\right)^+(lm) \leq \min\left\{\left(\underline{X_i}U\right)^+(l),\; \left(\underline{X_i}U\right)^+(m)\right\}$$

$$\left(\underline{Y_i}V\right)^-(lm) \leq \max\left\{\left(\underline{X_i}U\right)^-(l),\; \left(\underline{X_i}U\right)^-(m)\right\}$$

$$\left(\overline{Y_i}V\right)^+(lm) \leq \min\left\{\left(\overline{X_i}U\right)^+(l),\; \left(\overline{X_i}U\right)^+(m)\right\}$$

$$\left(\overline{Y_i}V\right)^-(lm) \leq \max\left\{\left(\overline{X_i}U\right)^-(l),\; \left(\overline{X_i}U\right)^-(m)\right\}$$

3. Laplacian energy of an intuitionistic fuzzy rough graph

Definition 3. The matrix of adjacency $A(\xi) = (A(\mu_N(p_ip_j)),\; A(\nu_N(p_ip_j)))$ of an IFRG $\xi = \left(\underline{\xi}, \overline{\xi}\right)$ $= (M, N)$ is defined as a square matrix $A(\xi) = [a_{ij}]$, where $a_{ij} = (\mu_N(p_ip_j), \nu_N(p_ip_j))$.

Here, $\mu_N(p_ip_j), \nu_N(p_ip_j)$ represent the degree of the relationship and degree of non-relationship between p_i and p_j, respectively.

Definition 4. Consider p_i as a vertex of the intuitionistic fuzzy rough graph $\xi = (\underline{\xi}, \overline{\xi}) = (M, N)$. The degree of u is denoted as

$$d_\xi(p_i) = \sum_{p_ip_j \in E(\xi)} \mu(p_ip_j)$$

Definition 5. Let $\xi = (\underline{\xi}, \overline{\xi}) = (M, N)$ be an IFRG on n vertices. The degree matrix, $D(\xi) = (D(\mu_N(p_ip_j)), D(\nu_N(p_ip_j))) = [d_{ij}]$, of ξ is a $n \times n$ diagonal matrix defined as:

$$d_{ij} = \begin{cases} d_\xi(p_i), & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Definition 6. The Laplacian matrix of an IFRG $\xi = (\underline{\xi}, \overline{\xi}) = (M, N)$ is defined as $L(\xi) = (L(\mu_N(p_ip_j)), L(\nu_N(p_ip_j))) = D(\xi) - A(\xi)$, $A(\xi)$ is an adjacency matrix and $D(\xi)$ is a degree matrix of an IFRG ξ.

Definition 7. The Laplacian matrix of spectrum of an IFRG $L(\xi)$ is denoted by (R_L, S_L), here $R_L = L(\mu_N(p_ip_j))$ and $S_L = L(\nu_N(p_ip_j))$ are the sets of Laplacian eigenvalues.

Definition 8. The Laplacian energy of an IFRG $\xi = (\underline{\xi}, \overline{\xi}) = (M, N)$ is defined as:

$$L\underline{E}(\underline{\xi}) = (L\underline{E}(\underline{\mu_N}(p_ip_j)), L\underline{E}(\underline{\nu_N}(p_ip_j))) = \left(\sum_{i=1}^{n} |\underline{\rho_i}|, \sum_{i=1}^{n} |\underline{\sigma_i}|\right)$$

$$L\overline{E}(\overline{\xi}) = (L\overline{E}(\overline{\mu_N}(p_ip_j)), L\overline{E}(\overline{\mu_N}(p_ip_j))) = \left(\sum_{i=1}^{n} |\overline{\rho_i}|, \sum_{i=1}^{n} |\overline{\sigma_i}|\right)$$

Here

$$\underline{\rho_i} = \underline{\phi_i} - \frac{2\sum_{1\le i<j\le n} \underline{\mu_N}(p_ip_j)}{n}, \ \underline{\sigma_i} = \underline{\psi_i} - \frac{2\sum_{1\le i<j\le n} \underline{\nu_N}(p_ip_j)}{n}$$

$$\overline{\rho_i} = \overline{\phi_i} - \frac{2\sum_{1\le i<j\le n} \overline{\mu_N}\,(p_ip_j)}{n}, \overline{\sigma_i} = \overline{\psi_i} - \frac{2\sum_{1\le i<j\le n} \overline{\nu_N}\,(p_ip_j)}{n}$$

Definition 9. If the weight $w_r = \left(\underline{w_r}, \overline{w_r}\right)$ of a wide range of Laplacian energy of an intuitionistic fuzzy rough graph is calculated using the formula below.

For lower,

$$\underline{w_r} = \left(\left(\underline{w_\mu}\right)_r, \left(\underline{w_\nu}\right)_r\right) = \left(\frac{L\underline{E}\left(\left(\underline{D_\mu}\right)_r\right)}{\sum_{i=1}^{S} L\underline{E}\left(\underline{D_\mu}\right)_i}, \frac{L\underline{E}\left(\left(\underline{D_\nu}\right)_r\right)}{\sum_{i=1}^{S} L\underline{E}\left(\underline{D_\nu}\right)_i}\right)$$

For upper,

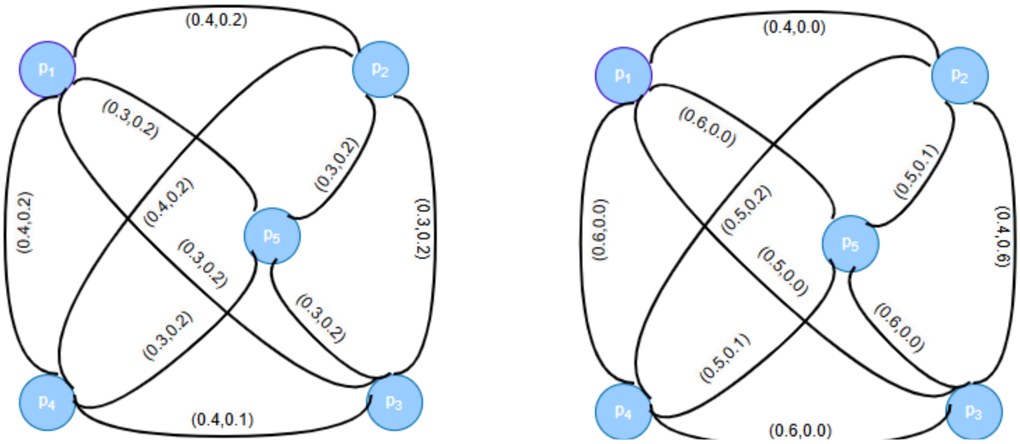
$$\overline{w_r} = \left(\left(\overline{w_\mu}\right)_r, \left(\overline{w_\nu}\right)_r\right) = \left(\frac{L\overline{E}\left(\left(\overline{D_\mu}\right)_r\right)}{\sum_{i=1}^{S} L\overline{E}\left(\overline{D_\mu}\right)_i}, \frac{L\overline{E}\left(\left(\overline{D_\nu}\right)_r\right)}{\sum_{i=1}^{S} L\overline{E}\left(\overline{D_\nu}\right)_i}\right)$$

$r = 1, 2, \ \ldots, \ s.$

Example 1. Let a graph

$G = (V, E)$, where $V = \{p_1, p_2, p_3, p_4, p_5\}$, $\quad E = \{p_1p_2, p_2p_3, p_3p_4, p_4p_5, p_5p_1, p_1p_3, p_1p_4, p_2p_4, p_2p_5\}$

Let $\xi = (\underline{\xi}, \overline{\xi}) = (M, N)$ be an IFRG on V as show in Figure 1.



a) **b)**

Figure 1. Intuitionistic fuzzy rough graphs: a) lower, b) upper

The adjacency matrix, degree matrix and Laplacian matrix of a lower IFRG shown in Figure 1a) are as follows:

$$A(\underline{\xi}) = \begin{bmatrix} (0.0,0.0) & (0.4,0.2) & (0.3,0.2) & (0.4,0.2) & (0.3,0.2) \\ (0.4,0.2) & (0.0,0.0) & (0.3,0.2) & (0.4,0.2) & (0.3,0.2) \\ (0.3,0.2) & (0.3,0.2) & (0.0,0.0) & (0.4,0.1) & (0.3,0.2) \\ (0.4,0.2) & (0.4,0.2) & (0.4,0.1) & (0.0,0.0) & (0.3,0.2) \\ (0.3,0.2) & (0.3,0.2) & (0.3,0.2) & (0.3,0.2) & (0.0,0.0) \end{bmatrix}$$

$$D(\underline{\xi}) = \begin{bmatrix} (1.4,0.8) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (1.4,0.8) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (1.3,0.7) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (1.5,0.7) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (1.2,0.8) \end{bmatrix}$$

$$L(\underline{\xi}) = \begin{bmatrix} (1.4,0.8) & (-0.4,-0.2) & (-0.3,-0.2) & (-0.4,-0.2) & (-0.3,-0.2) \\ (-0.4,-0.2) & (1.4,0.8) & (-0.3,-0.2) & (-0.4,-0.2) & (-0.3,-0.2) \\ (-0.3,-0.2) & (-0.3,-0.2) & (1.3,0.7) & (-0.4,-0.1) & (-0.3,-0.2) \\ (-0.4,-0.2) & (-0.4,-0.2) & (-0.4,-0.1) & (1.5,0.7) & (-0.3,-0.2) \\ (-0.3,-0.2) & (-0.3,-0.2) & (-0.3,-0.2) & (-0.3,-0.2) & (1.2,0.8) \end{bmatrix}$$

The Laplacian spectrum and the Laplacian energy of a lower IFRG $\underline{\xi}$

$$\text{Laplacian spec}\,(\underline{\mu_N}(p_ip_j)) = (0.0000, 1.5000, 1.6000, 1.8000, 1.9000)$$

$$\text{Laplacian spec}\,(\underline{\nu_N}(p_ip_j)) = (0.0000, 0.8000, 1.0000, 1.0000, 1.0000)$$

Therefore,

$$\text{Laplacian spec}(\underline{\xi}) = ((0,0),(1.5000,0.8000),(1.6000,1.0000),(1.8000,1.0000),(1.9000,1.0000))$$

Now,

$$L\underline{E}(\underline{\xi}) = (L\underline{E}(\underline{\mu_N}(p_ip_j)),\ L\underline{E}(\underline{\nu_N}(p_ip_j))) = (2.7200, 1.5200)$$

The adjacency matrix, degree matrix and Laplacian matrix of an upper IFRG shown in Figure 1b) are:

$$A(\overline{\xi}) = \begin{bmatrix} (0.0,0.0) & (0.4,0.0) & (0.5,0.0) & (0.6,0.0) & (0.6,0.0) \\ (0.4,0.0) & (0.0,0.0) & (0.4,0.6) & (0.5,0.2) & (0.5,0.1) \\ (0.5,0.0) & (0.4,0.6) & (0.0,0.0) & (0.6,0.0) & (0.6,0.0) \\ (0.6,0.0) & (0.5,0.2) & (0.6,0.0) & (0.0,0.0) & (0.5,0.1) \\ (0.6,0.0) & (0.5,0.1) & (0.6,0.0) & (0.5,0.1) & (0.0,0.0) \end{bmatrix}$$

$$D(\overline{\xi}) = \begin{bmatrix} (2.1,\ 0.0) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (1.8,0.9) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (2.1,\ 0.6) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (2.2,0.3) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (2.2,0.2) \end{bmatrix}$$

$$L(\overline{\xi}) = \begin{bmatrix} (2.1,\ 0.0) & (-0.4,-0.0) & (-0.5,-0.0) & (-0.6,-0.0) & (-0.6,-0.0) \\ (-0.4,-0.0) & (1.8,0.9) & (-0.4,-0.6) & (-0.5,-0.2) & (-0.5,-0.1) \\ (-0.5,-0.0) & (-0.4,-0.6) & (2.1,\ 0.6) & (-0.6,-0.0) & (-0.6,-0.0) \\ (-0.6,-0.0) & (-0.5,-0.2) & (-0.6,-0.0) & (2.2,0.3) & (-0.5,-0.1) \\ (-0.6,-0.0) & (-0.5,-0.1) & (-0.6,-0.0) & (-0.5,-0.1) & (2.2,0.2) \end{bmatrix}$$

The Laplacian spectrum and the Laplacian energy of an upper IFRG $\overline{\xi}$

$$\text{Laplacian spec}\,(\overline{\mu_N}(p_ip_j)) = (0.0000, 2.2298, 2.6000, 2.7000, 2.8702)$$

$$\text{Laplacian spec}\,(\overline{\nu_N}(p_ip_j)) = (0.0000, 0.0000, 0.2284, 0.3766, 1.3950)$$

Therefore,

$$\text{Laplacian spec}\,(\overline{\xi}) = ((0,0),(2.2298,0.0000),(2.6000,0.2284),(2.7000,0.3766),(2.8702,1.3950))$$

Now,

$$L\overline{E}(\overline{\xi}) = (L\overline{E}(\overline{\mu_N}(p_ip_j)),\ L\overline{E}(\overline{\nu_N}(p_ip_j))) = (4.1600, 1.9900)$$

The Laplacian energy of an intuitionistic fuzzy rough graph is

$$LE(\xi) = (L\underline{E}(\underline{\xi}),\ \ L\overline{E}(\overline{\xi})) = ((2.7200, 1.5200), (4.1600, 1.9900))$$

4. Results

Theorem 1. Let $\xi = (\underline{\xi}, \overline{\xi}) = (M, N)$ be an IFRG, and let $L(\xi)$ be the Laplacian matrix ξ. If $\eta_1 \geq \eta_2 \geq \eta_3 \geq \cdots \geq \eta_n$ and $\theta_1 \geq \theta_2 \geq \theta_3 \geq \cdots \geq \theta_n$ are the eigenvalues of $L(\mu_N(p_i p_j))$ and $L(\nu_N(p_i p_j))$, respectively, Then

$$(1) \qquad \sum_{i=1,\eta_i \in R_L}^{n} \eta_i = 2 \sum_{1 \leq i < j \leq n} \mu_N(p_i p_j)$$

$$\sum_{i=1,\theta_i \in S_L}^{n} \theta_i = 2 \sum_{1 \leq i < j \leq n} \nu_N(p_i p_j)$$

$$(2) \qquad \sum_{i=1,\eta_i \in R_L}^{n} {\eta_i}^2 = 2 \sum_{1 \leq i < j \leq n} (\mu_N(p_i p_j))^2 + \sum_{i=1}^{n} d^2_{\mu_N(p_i p_j)}(p_i)$$

$$\sum_{i=1,\theta_i \in S_L}^{n} {\theta_i}^2 = 2 \sum_{1 \leq i < j \leq n} (\nu_N(p_i p_j))^2 + \sum_{i=1}^{n} d^2_{\nu_N(p_i p_j)}(p_i)$$

Proof. 1. Since $L(\xi)$ contains non-negative Laplacian eigenvalues and is a symmetric matrix, such that:

$$\sum_{i=1,\eta_i \in R_L}^{n} \eta_i = \operatorname{tr}(L(\xi)) = \sum_{i=1}^{n} d_{\mu_{N(p_i p_j)}}(p_i) = 2 \sum_{1 \leq i < j \leq n} \mu_N(p_i p_j)$$

Thus,

$$\sum_{i=1,\eta_i \in R_L}^{n} \eta_i = 2 \sum_{1 \leq i < j \leq n} \mu_N(p_i p_j)$$

Similarly, we have to prove

$$\sum_{i=1,\theta_i \in S_L}^{n} \theta_i = 2 \sum_{1 \leq i < j \leq n} \nu_N(p_i p_j)$$

2. According to the definition of a Laplacian matrix, we have:

$$L(\mu_N(p_i p_j)) = \begin{bmatrix} d_{\mu_N(p_i p_j)}(p_1) & -\mu_N(p_1 p_2) & \cdots & -\mu_N(p_1 p_n) \\ -\mu_N(p_2 p_1) & d_{\mu_N(p_i p_j)}(p_2) & \cdots & -\mu_N(p_2 p_n) \\ \vdots & \vdots & \ddots & \vdots \\ -\mu_N(p_n p_1) & -\mu_N(p_n p_2) & \ldots & d_{\mu_N(p_i p_j)}(p_n) \end{bmatrix}$$

According to the trace properties of a matrix, we have:

$$\operatorname{tr}((L(\mu_N(p_i p_j)))^2) = \sum_{i=1,\eta_i \in R_L}^{n} \eta_i^2$$

where

$$\operatorname{tr}((L(\mu_N(p_ip_j)))^2) = (d^2_{\mu_{N(p_ip_j)}}(p_1) + \mu_N^2(p_1p_2) + \cdots + \mu_N^2(p_1p_n))$$
$$+ (\mu_N^2(p_2p_1) + \ d^2_{\mu_{N(p_ip_j)}}(p_2) + \cdots + \mu_N^2(p_2p_n)) + \cdots + (\ (\mu_N^2(p_np_1)$$
$$+ \mu_N^2(p_np_2) + \cdots + d^2_{\mu_{N(p_ip_j)}}(p_n)) = 2 \sum_{1\le i<j\le n} (\mu_N(p_ip_j))^2 + \sum_{i=1}^{n} d^2_{\mu_N(p_ip_j)}(p_i)$$

Therefore,

$$\sum_{i=1,\eta_i\in R_L}^{n} {\eta_i}^2 = 2 \sum_{1\le i<j\le n} (\mu_N(p_ip_j))^2 + \sum_{i=1}^{n} d^2_{\mu_N(p_ip_j)}(p_i)$$

Likewise, we have to prove that

$$\sum_{i=1,\theta_i\in S_L}^{n} {\theta_i}^2 = 2 \sum_{1\le i<j\le n} (\nu_N(p_ip_j))^2 + \sum_{i=1}^{n} d^2_{\nu_N(p_ip_j)}(p_i)$$

□

Theorem 2. Let ξ =($\underline{\xi}, \overline{\xi}$) = (M, N) be an IFRG, and let $L(\xi)$ be the Laplacian matrix of ξ. If $\eta_1 \ge \eta_2 \ge \eta_3 \ge \cdots \ge \eta_n$ and $\theta_1 \ge \theta_2 \ge \theta_3 \ge \cdots \ge \theta_n$ are the eigenvalues of $L(\mu_N(p_ip_j))$ and $L(\nu_N(p_ip_j))$ respectively, and

$$\rho_i = \eta_i - \frac{2 \sum\limits_{1\le i<j\le n} \mu_N(p_ip_j)}{n}, \quad \sigma_i = \theta_i - \frac{2 \sum\limits_{1\le i<j\le n} \nu_N(p_ip_j)}{n}$$

then

$$(1) \sum_{i=1}^{n} \rho_i = 0, \quad \sum_{i=1}^{n} \sigma_i = 0, \quad (2) \quad \sum_{i=1}^{n} {\rho_i}^2 = 2P_\mu, \quad \sum_{i=1}^{n} {\sigma_i}^2 = 2P_\nu$$

where

$$P_\mu = \sum_{1\le i<j\le n} (\mu_N(p_ip_j))^2 + \frac{1}{2} \sum_{i=1}^{n} \left(d_{\mu_N(p_ip_j)}(p_i) - \frac{2 \sum\limits_{1\le i<j\le n} \mu_N(p_ip_j)}{n} \right)^2$$

$$P_\nu = \sum_{1\le i<j\le n} (\nu_N(p_ip_j))^2 + \frac{1}{2} \sum_{i=1}^{n} \left(d_{\nu_N(p_ip_j)}(p_i) - \frac{2 \sum\limits_{1\le i<j\le n} \nu_N(p_ip_j)}{n} \right)^2$$

Example 1. Consider IFRG, ξ =($\underline{\xi}, \overline{\xi}$) = (M, N) and V = $\{p_1, p_2, p_3, p_4, p_5\}$ as shown in Figure 1. Then From Theorem 2, we have lower approximations as follows:

$$(1) \quad \sum_{i=1}^{5} \underline{\rho_i} = 0, \sum_{i=1}^{5} \underline{\sigma_i} = 0$$

$$(2) \quad \sum_{i=1}^{5} \underline{\rho_i}^2 = 2.4120 = 2(1.2060) = 2\underline{P_\mu}, \sum_{i=1}^{5} \underline{\sigma_i}^2 = 0.7520 = 2(0.3760) = 2\underline{P_\nu}$$

From Theorem 2, we have upper approximations as follows:

$$(1) \quad \sum_{i=1}^{5} \overline{\rho_i} = 0, \sum_{i=1}^{5} \overline{\sigma_i} = 0$$

$$(2) \quad \sum_{i=1}^{5} \overline{\rho_i^2} = 5.6280 = 2(2.8140) = 2\overline{P_\mu}, \sum_{i=1}^{5} \overline{\sigma_i^2} = 1.3400 = 2(0.6700) = 2\overline{P_\nu}$$

Theorem 3. Let ξ =($\underline{\xi}, \overline{\xi}$) = (M, N) be an IFRG on n vertices, let $L(\xi)$ = $(L(\mu_N(p_ip_j)), L(\nu_N(p_ip_j)))$ be the Laplacian matrix of ξ. Then

$$(1) \quad L\underline{E}(\mu_N(p_ip_j)), L\overline{E}(\mu_N(p_ip_j))$$

$$\leq \left(2n \sum_{1\leq i<j\leq n} (\mu_N(p_ip_j))^2 + n \sum_{i=1}^{n} \left(d_{\mu_N(p_ip_j)}(p_i) - \frac{2 \sum_{1\leq i<j\leq n} \mu_N(p_ip_j)}{n}\right)^2\right)^{1/2}$$

$$(2) \quad L\underline{E}(\nu_N(p_ip_j)), L\overline{E}(\nu_N(p_ip_j))$$

$$\leq \left(2n \sum_{1\leq i<j\leq n} (\nu_N(p_ip_j))^2 + n \sum_{i=1}^{n} \left(d_{\nu_N(p_ip_j)}(p_i) - \frac{2 \sum_{1\leq i<j\leq n} \nu_N(p_ip_j)}{n}\right)^2\right)^{1/2}$$

Proof. Apply Cauchy–Schwarz inequality to n numbers $(1, 1, \ldots, 1)$ and $(|\rho_1|, |\rho_2|, \ldots, |\rho_n|)$,

$$\sum_{i=1}^{n} |\rho_i| \leq \sqrt{n} \left(\sum_{i=1}^{n} |\rho_i|^2\right)^{1/2}$$

$$L\underline{E}(\mu_N(p_ip_j)) \leq n^{1/2} (2P_\mu)^{1/2} = (2nP_\mu)^{1/2}$$

Since

$$P_\mu = \sum_{1\leq i<j\leq n} (\mu_N(p_ip_j))^2 + \frac{1}{2} \sum_{i=1}^{n} \left(d_{\mu_N(p_ip_j)}(p_i) - \frac{2 \sum_{1\leq i<j\leq n} \mu_N(p_ip_j)}{n}\right)^2$$

Therefore

$$L\underline{E}(\mu_N(p_ip_j)) \leq \left(2n \sum_{1\leq i<j\leq n} (\mu_N(p_ip_j))^2 + n\sum_{i=1}^{n}\left(d_{\mu_N(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i<j\leq n}\mu_N(p_ip_j)}{n}\right)^2\right)^{1/2}$$

Similarly, we have to prove $L\overline{E}(\mu_N(p_ip_j)), L\underline{E}(\nu_N(p_ip_j)), L\overline{E}(\nu_N(p_ip_j))$ □

Example 2. Consider IFRG, ξ =($\underline{\xi}, \overline{\xi}$) = (M, N) and V = $\{p_1, p_2, p_3, p_4, p_5\}$ as shown in Figure 1. Then:

From Theorem 3, we have a lower approximation as follows

$$L\underline{E}(\underline{\mu_N}(p_ip_j)) \leq (2(5)(1.1800) + 5(0.0520))^{1/2} \text{ implies } 2.7200 \leq 3.4728$$

$$L\underline{E}(\underline{\nu_N}(p_ip_j)) \leq (2(5)(0.3700) + 5(0.0120))^{1/2} \text{ implies } 1.5200 \leq 1.9391$$

From Theorem 3, we have an upper approximation as follows

$$L\overline{E}(\overline{\mu_N}(p_ip_j)) \leq (2(5)(2.7600) + 5(0.1080))^{1/2} \text{ implies } 4.1600 \leq 5.3047$$

$$L\overline{E}(\overline{\nu_N}(p_ip_j)) \leq (2(5)(0.4200) + 5(0.5000))^{1/2} \text{ implies } 1.9900 \leq 2.5884$$

Theorem 4. Let ξ =($\underline{\xi}, \overline{\xi}$) = (M, N) be an IFRG on n vertices, let $L(\xi) = (L(\mu_N(p_ip_j)), L(\nu_N(p_ip_j)))$ be the Laplacian matrix of ξ. Then

(1) $L\underline{E}(\mu_N(p_ip_j))$,$L\overline{E}(\mu_N(p_ip_j))$

$$\geq 2\left(\sum_{1\leq i<j\leq n} (\mu_N(p_ip_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_N(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i<j\leq n}\mu_N(p_ip_j)}{n}\right)^2\right)^{1/2}$$

(2) $L\underline{E}(\nu_N(p_ip_j))$,$L\overline{E}(\nu_N(p_ip_j))$

$$\geq 2\left(\sum_{1\leq i<j\leq n} (\nu_N(p_ip_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\nu_N(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i<j\leq n}\nu_N(p_ip_j)}{n}\right)^2\right)^{1/2}$$

Proof.

$$\left(\sum_{i=1}^{n}|\rho_i|\right)^{\mathbf{2}} = \sum_{i=1}^{n}|\rho_i|^2 + 2\sum_{1\leq i<j\leq n}|\rho_i|\,|\rho_i| \geq 4P_\mu,\ \ L\underline{E}(\mu_N(p_ip_j)) \geq 2\,(P_\mu)^{1/2}$$

Since

$$P_\mu = \sum_{1\le i<j\le n} (\mu_N(p_ip_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_N(p_ip_j)}(p_i) - \frac{2\sum_{1\le i<j\le n}\mu_N(p_ip_j)}{n}\right)^2$$

Therefore,

$$L\underline{E}(\mu_N(p_ip_j)) \;\ge\; 2\left(\sum_{1\le i<j\le n} (\mu_N(p_ip_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_N(p_ip_j)}(p_i) - \frac{2\sum_{1\le i<j\le n}\mu_N(p_ip_j)}{n}\right)^2\right)^{1/2}$$

Similarly, we have to show that $L\overline{E}(\mu_N(p_ip_j)), L\underline{E}(\nu_N(p_ip_j)), L\overline{E}(\nu_N(p_ip_j))$. □

Example 3. Consider IFRG, $\xi = (\underline{\xi}, \overline{\xi}) = (M, N)$ and $V = \{p_1, p_2, p_3, p_4, p_5\}$ as shown in Figure 1. Then From Theorem 4, we have lower approximations as follows:

$$L\underline{E}(\underline{\mu_N}(p_ip_j)) \ge 2\left(1.1800 + \frac{1}{2}(0.0520)\right)^{1/2} \text{ implies } 2.7200 \ge 2.1964.$$

$$L\underline{E}(\underline{\nu_N}(p_ip_j)) \ge 2\left(0.3700 + \frac{1}{2}(0.0120)\right)^{1/2} \text{ implies } 1.5200 \ge 1.2264.$$

From Theorem 4, we have upper approximations as follows:

$$L\overline{E}(\overline{\mu_N}(p_ip_j)) \ge 2\left(2.7600 + \frac{1}{2}(0.1080)\right)^{1/2} \text{ implies } 4.1600 \ge 3.3550.$$

$$L\overline{E}(\overline{\nu_N}(p_ip_j)) \ge 2\left(0.4200 + \frac{1}{2}(0.5000)\right)^{1/2} \text{ implies } 1.9900 \ge 1.6371.$$

Theorem 5. Let $\xi = (\underline{\xi}, \overline{\xi}) = (M, N)$ be an IFRG on n vertices, and let $L(\xi) = (L(\mu_N(p_ip_j)), L(\nu_N(p_ip_j)))$ be the Laplacian matrix of ξ. Then

(1) $L\underline{E}(\mu_N(p_ip_j)),\ L\overline{E}(\mu_N(p_ip_j)) \le$

$$|\rho_1| + \left((n-1)\left(2\sum_{1\le i<j\le n}(\mu_N(p_ip_j))^2 + \sum_{i=1}^{n}\left(d_{\mu_N(p_ip_j)}(p_i) - \frac{2\sum_{1\le i<j\le n}\mu_N(p_ip_j)}{n}\right)^2 - \rho_1^2\right)\right)^{1/2}$$

(2) $L\underline{E}(\nu_N(p_ip_j)),\ L\overline{E}(\nu_N(p_ip_j))$

$$\le |\sigma_1| + \left((n-1)\left(2\sum_{1\le i<j\le n}(\nu_N(p_ip_j))^2 + \sum_{i=1}^{n}\left(d_{\nu_N(p_ip_j)}(p_i) - \frac{2\sum_{1\le i<j\le n}\nu_N(p_ip_j)}{n}\right)^2 - \sigma_1^2\right)\right)^{1/2}$$

Proof. Applying Cauchy–Schwarz inequality to $(1, 1, \ldots, 1)$ and $(|\rho_1|, |\rho_2|, \ldots, |\rho_n|)$, we get:

$$\sum_{i=1}^{n} |\rho_i| \leq \left(n \sum_{i=1}^{n} |\rho_i|^2 \right)^{1/2}, \sum_{i=2}^{n} |\rho_i| \leq \left((n-1) \sum_{i=2}^{n} |\rho_i|^2 \right)^{1/2}$$

$$L\underline{E}(\mu_N(p_ip_j)) \leq |\rho_1| + \left((n-1)(2P_\mu - \rho_1^2)\right)^{1/2}$$

Since,

$$P_\mu = \sum_{1\leq i<j\leq n} (\mu_N(p_ip_j))^2 + \frac{1}{2} \sum_{i=1}^{n} \left(d_{\mu_N(p_ip_j)}(p_i) - \frac{2 \sum_{1\leq i<j\leq n} \mu_N(p_ip_j)}{n} \right)^2$$

Therefore,

$$L\underline{E}(\mu_N(p_ip_j)) \leq |\rho_1|$$
$$+ \left((n-1) \left(2 \sum_{1\leq i<j\leq n} (\mu_N(p_ip_j))^2 + \sum_{i=1}^{n} \left(d_{\mu_N(p_ip_j)}(p_i) - \frac{2 \sum_{1\leq i<j\leq n} \mu_N(p_ip_j)}{n} \right)^2 - \rho_1^2 \right) \right)^{1/2}$$

Similarly, we have to prove $L\overline{E}(\mu_N(p_ip_j)), L\underline{E}(\nu_N(p_ip_j)), L\overline{E}(\nu_N(p_ip_j))$. □

Example 4. Consider IFRG, $\xi = (\underline{\xi}, \overline{\xi}) = (M, N)$ and $V = \{p_1, p_2, p_3, p_4, p_5\}$ as shown in Figure 1. Then

From Theorem 5, we have a lower approximation bound as follows:

$$L\underline{E}(\underline{\mu_N}(p_ip_j)) \leq 1.36 + ((5-1)(2.3600 + 0.0520 - 1.8496))^{1/2} \text{ implies } 2.7200 \leq 2.8599.$$

$$L\underline{E}(\underline{\nu_N}(p_ip_j)) \leq 0.76 + ((5-1)(0.7400 + 0.0120 - 0.5776))^{1/2} \text{ implies } 1.5200 \leq 1.5952.$$

From Theorem 5, we have an upper approximation bound as follows:

$$L\overline{E}(\overline{\mu_N}(p_ip_j)) \leq 2.08 + ((5-1)(5.5200 + 0.1080 - 4.3264))^{1/2} \text{ implies } 4.1600 \leq 4.3618.$$

$$L\overline{E}(\overline{\nu_N}(p_ip_j)) \leq 0.40 + ((5-1)(0.8400 + 0.5000 - 0.1600))^{1/2} \text{ implies } 1.9900 \leq 2.5726.$$

Theorem 6. If the IFRG $\xi = (\underline{\xi}, \overline{\xi}) = (M, N)$ is regular, Then

$$(1) \quad L\underline{E}(\mu_N(p_ip_j)), L\overline{E}(\mu_N(p_ip_j)) \leq |\rho_1| + \left((n-1) \left(2 \sum_{1\leq i<j\leq n} (\mu_N(p_ip_j))^2 - \rho_i^2 \right) \right)^{1/2}$$

$$(2) \quad L\underline{E}(\nu_N(p_ip_j)), L\overline{E}(\nu_N(p_ip_j)) \leq |\sigma_1| + \left((n-1) \left(2 \sum_{1\leq i<j\leq n} (\nu_N(p_ip_j))^2 - \sigma_i^2 \right) \right)^{1/2}$$

Proof. Let ξ be a regular intuitionistic fuzzy rough graph and

$$d_{\mu_N(p_ip_j)}(p_i) = \frac{2\sum_{1\le i<j\le n}\mu_N(p_ip_j)}{n}$$

Substituting this value in the Theorem 5, we get

$$L\underline{E}(\mu_N(p_ip_j)) \le |\rho_1| + \left((n-1)\left(2\sum_{1\le i<j\le n}(\mu_N(p_ip_j))^2 - \rho_i^2\right)\right)^{1/2}$$

Similarly, we have to show that $L\overline{E}(\mu_N(p_ip_j)), L\underline{E}(\nu_N(p_ip_j)), L\overline{E}(\nu_N(p_ip_j))$. □

Example 5. Consider IFRG, ξ =($\underline{\xi},\overline{\xi}$) = (M, N) and $V = \{p_1, p_2, p_3, p_4, p_5\}$ as shown in Figure 1. Then From Theorem 6, we have lower approximations as follows:

$$L\underline{E}(\underline{\mu_N}(p_ip_j)) \le 1.36 + ((5-1)(2.3600 - 1.8496))^{1/2} \text{ implies } 2.7200 \le 2.7888$$

$$L\underline{E}(\underline{\nu_N}(p_ip_j)) \le 0.76 + ((5-1)(0.7400 - 0.5776))^{1/2} \text{ implies } 1.5200 \le 1.5660$$

From Theorem 6, we have upper approximations as follows:

$$L\overline{E}(\overline{\mu_N}(p_ip_j)) \le 2.08 + ((5-1)(5.5200 - 4.3264))^{1/2} \text{ implies } 4.1600 \le 4.2650$$

$$L\overline{E}(\overline{\nu_N}(p_ip_j)) \le 0.40 + ((5-1)(0.8400 - 0.1600))^{1/2} \text{ implies } 1.9900 \le 2.0492$$

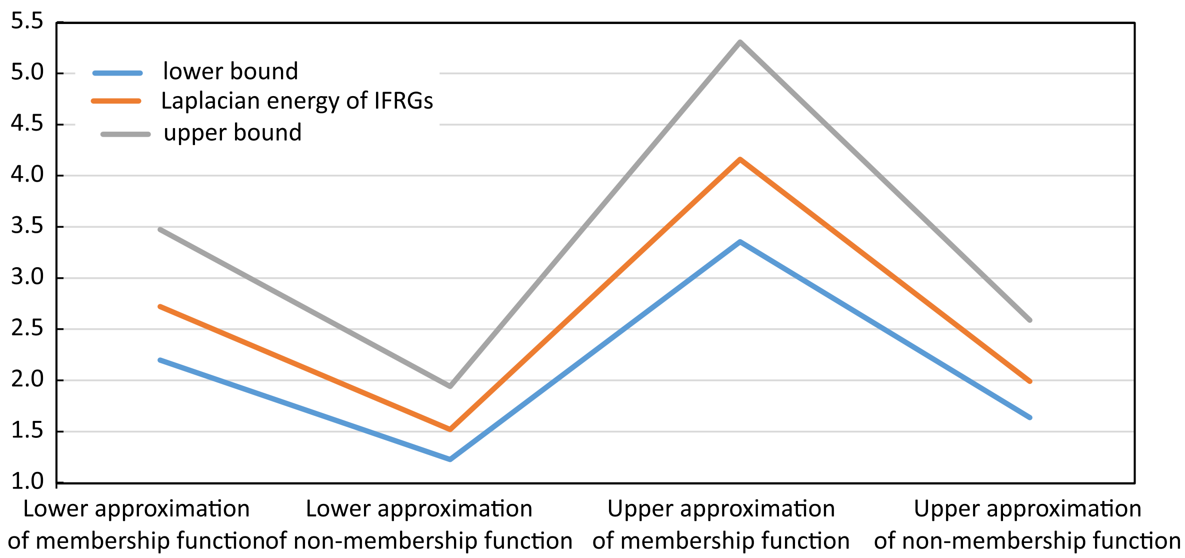


Figure 2. Laplacian energy of intuitionistic fuzzy rough graph

Figure 2 represent the lower and upper approximation of Laplacian energy of membership and non--membership of lower bound and upper bound of an intuitionistic fuzzy rough graph.

- For lower approximation membership of Laplacian energy = 2.7200. its lower bound = 2.1964 and upper bound = 3.4728.
- For lower approximation non-membership of Laplacian energy = 1.5200. its lower bound = 1.2264 and upper bound = 1.9391.

- For upper approximation membership of Laplacian energy = 4.1600. its lower bound = 3.3550 and upper bound = 5.3047.
- For upper approximation non-membership of Laplacian energy = 1.9900. its lower bound = 1.6371 and upper bound = 2.5884.

5. Laplacian energy of intuitionistic fuzzy rough digraph

Definition 10. Let $\xi = (\underline{\xi}, \overline{\xi}) = (M, \vec{N})$ be an intuitionistic fuzzy rough digraph (IFRDG) on n vertices. The out-degree matrix, $D^{\text{out}}(\xi) = (D(\mu_{\vec{N}}(p_ip_j)), D(\nu_{\vec{N}}(p_ip_j)) = [d_{ij}]$ of ξ is a $n \times n$ diagonal matrix:

$$d_{ij} = \begin{cases} d_{\xi}^{\text{out}}(p_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Definition 11. The Laplacian matrix of an IFRDG $\xi = (\underline{\xi}, \overline{\xi}) = (M, \vec{N})$ is defined as

$$L(\xi) = (L(\mu_{\vec{N}}(p_ip_j)), L(\nu_{\vec{N}}(p_ip_j)) = D^{\text{out}}(\xi) - A(\xi)$$

where $A(\xi)$ is an adjacency matrix and $D^{\text{out}}(\xi)$ is an out degree matrix of an IFRDG of D.

Definition 12. The Laplacian matrix of spectrum of an IFRG $L(\xi)$ is denoted by (R_L, S_L), here $R_L = L(\mu_{\vec{N}}(p_ip_j))$ and $S_L = L(\nu_{\vec{N}}(p_ip_j))$ are the sets of Laplacian eigenvalues.

Definition 13. The Laplacian energy of an IFRG $\xi = (\underline{\xi}, \overline{\xi}) = (M, \vec{N})$ is defined as:

$$L\underline{E}(\underline{\xi}) = (L\underline{E}(\underline{\mu_{\vec{N}}}(p_ip_j)), L\underline{E}(\underline{\mu_{\vec{N}}}(p_ip_j))) = \left(\sum_{i=1}^{n} |\underline{\rho_i}|, \sum_{i=1}^{n} |\underline{\sigma_i}|\right)$$

$$L\overline{E}(\overline{\xi}) = (L\overline{E}(\overline{\mu_{\vec{N}}}(p_ip_j)), L\overline{E}(\overline{\mu_{\vec{N}}}(p_ip_j))) = \left(\sum_{i=1}^{n} |\overline{\rho_i}|, \sum_{i=1}^{n} |\overline{\sigma_i}|\right)$$

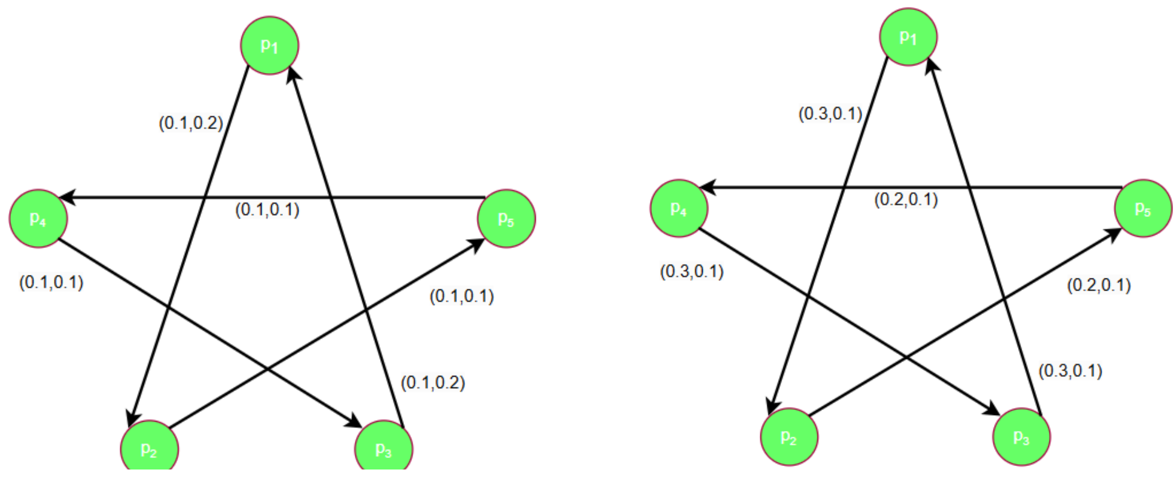
Here

$$\underline{\rho_i} = \underline{\phi_i} - \frac{2 \sum\limits_{1 \le i < j \le n} \underline{\mu_{\vec{N}}}(p_ip_j)}{n}, \underline{\sigma_i} = \underline{\psi_i} - \frac{2 \sum\limits_{1 \le i < j \le n} \underline{\nu_{\vec{N}}}(p_ip_j)}{n}$$

$$\overline{\rho_i} = \overline{\phi_i} - \frac{2 \sum\limits_{1 \le i < j \le n} \overline{\mu_{\vec{N}}}(p_ip_j)}{n}, \overline{\sigma_i} = \overline{\psi_i} - \frac{2 \sum\limits_{1 \le i < j \le n} \overline{\nu_{\vec{N}}}(p_ip_j)}{n}$$

Example 1. Consider an IFRDG $\xi = (\underline{\xi}, \overline{\xi}) = (M, \vec{N})$ on $V = \{p_1, p_2, p_3, p_4, p_5\}$ and $\vec{E} = \{p_1p_2, p_2p_5, p_5p_4, p_4p_3, p_3p_1\}$, as show in Figure (3).

From Figure 3a), the adjacency matrix, out degree matrix and Laplacian matrix of the lower IFRDG are as follows:



a) **b)**

Figure 3. Intuitionistic fuzzy rough digraphs: a) lower, b) upper

$$A(\underline{\xi}) = \begin{bmatrix} (0.0, 0.0) & (0.1,\ 0.2) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) \\ (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (0.1,\ 0.1) \\ (0.1,\ 0.2) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) \\ (0.0, 0.0) & (0.0, 0.0) & (0.1,\ 0.1) & (0.0, 0.0) & (0.0, 0.0) \\ (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (0.1,\ 0.1) & (0.0, 0.0) \end{bmatrix}$$

$$D^{\text{out}}(\underline{\xi}) = \begin{bmatrix} (0.1,\ 0.2) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) \\ (0.0, 0.0) & (0.1,\ 0.1) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) \\ (0.0, 0.0) & (0.0, 0.0) & (0.1,\ 0.2) & (0.0, 0.0) & (0.0, 0.0) \\ (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (0.1,\ 0.1) & (0.0, 0.0) \\ (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (0.1,\ 0.1) \end{bmatrix}$$

$$L(\underline{\xi}) = \begin{bmatrix} (0.1,\ 0.2) & (-0.1, -0.2) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) \\ (0.0, 0.0) & (0.1,\ 0.1) & (0.0, 0.0) & (0.0, 0.0) & (-0.1, -0.1) \\ (-0.1, -0.2) & (0.0, 0.0) & (0.1,\ 0.2) & (0.0, 0.0) & (0.0, 0.0) \\ (0.0, 0.0) & (0.0, 0.0) & (-0.1, -0.1) & (0.1,\ 0.1) & (0.0, 0.0) \\ (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (-0.1, -0.1) & (0.1,\ 0.1) \end{bmatrix}$$

The Laplacian spectrum and the Laplacian energy of an IFRDG for lower are shown in Figure 3a), as follows

$$\begin{aligned} &\text{Laplacian spec}\left(\underline{\mu_{\vec{N}}}(p_i p_j)\right) \\ &\qquad = (0.0000, 0.0691 + 0.0951i, 0.0691 - 0.0951i, 0.1809 + 0.0588i, 0.1809 - 0.0588i) \end{aligned}$$

Laplacian spec $(\underline{\nu_{\vec{N}}}(p_ip_j))$

$$= (0.2545 + 0.0716i, 0.2545 - 0.0716i, 0.0955 + 0.1174i, 0.0955 - 0.1174i, 0.0000)$$

The calculation of the components for Laplacian energy of an intuitionistic fuzzy rough digraph for lower $\underline{\xi}$ is $L\underline{E}(\underline{\xi}) = (0.5000, 0.6612)$.

From Figure 3b), the adjacency matrix, out degree and Laplacian matrix of the upper IFRDG are as follows:

$$A(\overline{\xi}) = \begin{bmatrix} (0.0,0.0) & (0.3,0.1) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (0.2,0.1) \\ (0.3,0.1) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (0.3,0.1) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (0.2,0.1) & (0.0,0.0) \end{bmatrix}$$

$$D^{\text{out}}(\overline{\xi}) = \begin{bmatrix} (0.3,0.1) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (0.2,0.1) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (0.3,0.1) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (0.3,0.1) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (0.2,0.1) \end{bmatrix}$$

$$L(\overline{\xi}) = \begin{bmatrix} (0.3,0.1) & (-0.3,-0.1) & (0.0,0.0) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (0.2,0.1) & (0.0,0.0) & (0.0,0.0) & (-0.2,-0.1) \\ (-0.3,-0.1) & (0.0,0.0) & (0.3,0.1) & (0.0,0.0) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (-0.3,-0.1) & (0.3,0.1) & (0.0,0.0) \\ (0.0,0.0) & (0.0,0.0) & (0.0,0.0) & (-0.2,-0.1) & (0.2,0.1) \end{bmatrix}$$

The Laplacian spectrum and the Laplacian energy of an IFRDG for upper are shown in Figure (3b)), as follows:

Laplacian spec $(\overline{\mu_{\vec{N}}}(p_ip_j))$

$$=(0.4701 + 0.1474i, 0.4701 - 0.1474i, 0.1799 + 0.2380i, 0.1799 - 0.2380i, 0.0000)$$

Laplacian spec$(\overline{\nu_{\vec{N}}}(p_ip_j))$

$$=(0.0000, 0.0691 + 0.0951i, 0.0691 - 0.0951i, 0.1809 + 0.0588i, 0.1809 - 0.0588i)$$

The calculation of the components for Laplacian energy of an intuitionistic fuzzy rough digraph for upper $\overline{\xi}$ is $L\overline{E}(\overline{\xi}) = (1.2754, 0.5000)$.

Hence, the Laplacian energy of an intuitionistic fuzzy rough digraph ξ is

$$LE(\xi) = (L\underline{E}(\underline{\xi}), L\overline{E}(\overline{\xi})) = ((0.5000, 0.6612), (1.2754, 0.5000))$$

5.1. Intuitionistic fuzzy rough preference relation (IFRPR)

This part presents a group decision-making method with an intuitionistic fuzzy rough preference relation. Let $C = \{c_1, c_2, \ldots, c_n\}$ be a set of n alternatives, which is evaluated by a set $e = \{e_1, e_2, \ldots, e_m\}$ of decision-makers. Without loss of the generality, assume that the measures $e_1, e_2, \ldots, e_k$ are an advantage, while rules $e_{k+1}, e_{k+2}, \ldots, e_m$ are a disadvantage. Furthermore, let $R_k = \left(r_{ij}^{(k)}\right)_{n\times n}$ be an intuitionistic fuzzy rough preference relation, where $k = 1, 2, \ldots, p$. Weight w_r are assigned to the measurements e_r so that

$$\sum_{r=1}^{n} w_r = 1$$

It is necessary to select the most optimal alternative.

For IFRGs, the cumulative grid may be obtained by intuitionistic fuzzy rough weighted averaging (IFRWA) [8], as shown below

$$\text{IFRWA}\left(R_{ij}^{(1)}, R_{ij}^{(2)}, \ldots, R_{ij}^{(n)}\right) = \left(1 - \prod_{r=1}^{s}\left(1 - \mu_{jk}^{(r)}\right)^{w_r}, \prod_{r=1}^{s}\left(\nu_{jk}^{(r)}\right)^{w_r}\right)$$

where w_r is the weight function, $R_{ij}^{(1)}, R_{ij}^{(2)}, \ldots, R_{ij}^{(n)}$ an individual intuitionistic fuzzy rough preference relation (IFRPR), μ_{jk} the membership value, and ν_{jk} the non-membership value.

Here w_r are weight of n number of Laplacian energies calculating by using following formula:

For lower

$$\underline{w_r} = \left(\left(\underline{w_\mu}\right)_r, \left(\underline{w_\nu}\right)_r\right) = \left(\frac{L\underline{E}\left(\left(\underline{D_\mu}\right)_r\right)}{\sum_{i=1}^{S} L\underline{E}\left(\underline{D_\mu}\right)_i}, \frac{L\underline{E}\left(\left(\underline{D_\nu}\right)_r\right)}{\sum_{i=1}^{S} L\underline{E}\left(\underline{D_\nu}\right)_i}\right)$$

For upper

$$\overline{w_r} = \left(\left(\overline{w_\mu}\right)_r, \left(\overline{w_\nu}\right)_r\right) = \left(\frac{L\overline{E}\left(\left(\overline{D_\mu}\right)_r\right)}{\sum_{i=1}^{S} L\overline{E}\left(\overline{D_\mu}\right)_i}, \frac{L\overline{E}\left(\left(\overline{D_\mu}\right)_r\right)}{\sum_{i=1}^{S} L\overline{E}\left(\overline{D_\nu}\right)_i}\right)$$

$r = 1, 2, \ldots, s.$

5.2. Algorithm

The algorithm for selecting the most significant data visualization.

INPUT. A set of alternatives $C = \{c_1, c_2, \ldots, c_n\}$, a set of experts $e = \{e_1, e_2, \ldots, e_m\}$ and built of intuitionistic fuzzy rough preference relation for lower and upper of $R_k = \left(r_{ij}^{(k)}\right)_{n\times n}$ for each expert k.

OUTPUT. The process of selecting the best visualization.

1. Start

2. Determine the Laplacian energy of every IFRDG C_k, $k = 1, 2, \ldots, n.$

$$\text{for lower}\quad L\underline{E}\left(\underline{\xi}\right)=\left|\underline{\phi_i}-\frac{2\sum\limits_{1\le i<j\le n}\underline{\mu_{\vec{N}}}\left(p_ip_j\right)}{n}\right|,L\underline{E}\left(\underline{\xi}\right)=\left|\underline{\psi_i}-\frac{2\sum\limits_{1\le i<j\le n}\underline{\nu_{\vec{N}}}\left(p_ip_j\right)}{n}\right|$$
$$\text{for upper}\quad L\overline{E}(\overline{\xi})=\left|\overline{\phi_i}-\frac{2\sum\limits_{1\le i<j\le n}\overline{\mu_{\vec{N}}}(p_ip_j)}{n}\right|,L\overline{E}(\overline{\xi})=\left|\overline{\psi_i}-\frac{2\sum\limits_{1\le i<j\le n}\overline{\nu_{\vec{N}}}(p_ip_j)}{n}\right| \tag{1}$$

3. Compute the weight vector for specialists based on the Laplacian energy of IFRDG, by utilizing

$$\text{for lower}\quad \underline{w_r}=\left(\left(\underline{w_\mu}\right)_r,\ \left(\underline{w_\nu}\right)_r\right)=\left(\frac{L\underline{E}\left(\left(\underline{D_\mu}\right)_r\right)}{\sum\limits_{i=1}^{S}L\underline{E}\left(\underline{D_\mu}\right)_i},\frac{L\underline{E}\left(\left(\underline{D_\nu}\right)_r\right)}{\sum\limits_{i=1}^{S}L\underline{E}\left(\underline{D_\nu}\right)_i}\right),\quad r=1,2,\,\dots,\,s$$
$$\text{for upper}\quad \overline{w_r}=\left(\left(\overline{w_\mu}\right)_r,\ \left(\overline{w_\nu}\right)_r\right)=\left(\frac{L\overline{E}\left(\left(\overline{D_\mu}\right)_r\right)}{\sum\limits_{i=1}^{S}L\overline{E}\left(\overline{D_\mu}\right)_i},\frac{L\overline{E}\left(\left(\overline{D_\nu}\right)_r\right)}{\sum\limits_{i=1}^{S}L\overline{E}\left(\overline{D_\nu}\right)_i}\right),\quad r=1,2,\,\dots,\,s \tag{2}$$

4. Compute the intuitionistic fuzzy rough weighted averaging by utilizing

$$\text{IFRWA}\left(R_{ij}^{(1)},R_{ij}^{(2)},\,\dots,\,R_{ij}^{(n)}\right)=\left(1-\prod_{r=1}^{s}\left(1-\mu_{jk}^{(r)}\right)^{w_r},\prod_{r=1}^{s}\left(\nu_{jk}^{(r)}\right)^{w_r}\right) \tag{3}$$

5. Utilize the IFRWA operator to fuse all the individuals IFRPRs $R_k = \left(r_{ij}^{(k)}\right)_{n\times n}$ $(k=1,2,\,\dots,\,p)$ into the collective IFRPR $R = (r_{ij})_{n\times n}$.

6. Calculate their scores using the score function

$$s_{ij}=\mu_{ij}-\nu_{ij} \tag{4}$$

7. Determine the net degree of preference of visualization c_i over the other schemes by utilizing:

$$\sigma\left(c_i\right)=\sum_{r=1}^{n}w_r\left(\sum_{\substack{j=1\\ j\ne i}}^{n}\left(r_{ij}^{(r)}-r_{ji}^{(r)}\right)\right),\quad i=1,2,\dots,n \tag{5}$$

8. Rank all the visualization c_i $(i=1,2,\,\dots,\,n)$ according to $\sigma\left(c_i\right)$ $(i=1,2,\,\dots,\,n)$.

9. Output the best data visualization.

10. Stop.

6. Application of Laplacian energy from an IFRDG in decision-making

Group decision-making is a crucial approach to identifying the best option from finite alternatives, playing a significant role in societal development and problem-solving.

To demonstrate our proposed ideas of the intuitionistic fuzzy rough graph in decision-making, we utilize a practical example: the design of a data visualization technique.

6.1. The selection of data visualization for sales performance analysis

Study case. This section examines a case study involving the proposed Laplacian Energy of the IFRDG, applied to a real-world problem of selecting data visualization techniques for sales performance analysis. As mentioned (cf. Section 1), data visualization for sales performance often presents a complex challenge, necessitating selecting the most reliable strategy from various options.

The preference relation technique is extensively used to rank alternatives, with decision-makers providing their preferences regarding the available alternatives or criteria. When the information in the preference relation is expressed as an intuitionistic fuzzy rough number (IFRNs), the concept of the intuitionistic fuzzy rough preference relation (IFRPR) can be redefined analogously.

Data visualization. Data visualization [10] may significantly improve communication and comprehension of data. When data is presented in numerical or textual representations, people may not instantly see patterns, trends, or outliers. However, effective visualization approaches may help people do just that. Visualizing data helps you comprehend and analyze it. By visualizing simplifies raw data in charts, graphs, and maps, trends, patterns, and insights may be shown. Data visualization simplifies complicated information, and clarifies results, making it crucial in corporate analysis and scientific investigation.

A company wants to analyze the sales performance of its products over the past year to make informed decisions on marketing strategies, inventory management, and product development. By employing these data visualization techniques, decision-makers can quickly and intuitively grasp the key insights from the sales data, leading to more informed decisions regarding marketing strategies, inventory management, and product development. There are four different data visualizations c_i $(i = 1, 2, 3, 4)$ to select from c_1, c_2, c_3, c_4. Therefore, monthly sales data for each product category, regional sales breakdown, seasonal trends, and promotions data, only the best of these would be chosen. To conduct the evaluation process based on the opinions of three independently appointed experts $(e)_k$, $(k = 1, 2, 3)$. Their comparative opinions, derived from their experience, were represented using intuitionistic fuzzy rough numbers. Additionally, intuitionistic fuzzy rough preference relations were organized into matrices as the initial step for selecting data visualization. With the proposed Laplacian energy of IFRDG utilizing these preference relations, we offer an algorithm to solve the stated visualization problem, as illustrated in Figure 4.

For lower approximation. The experts compare the involved factors and provide the initial information for computing in the form of lower intuitionistic fuzzy rough reference relations. These are

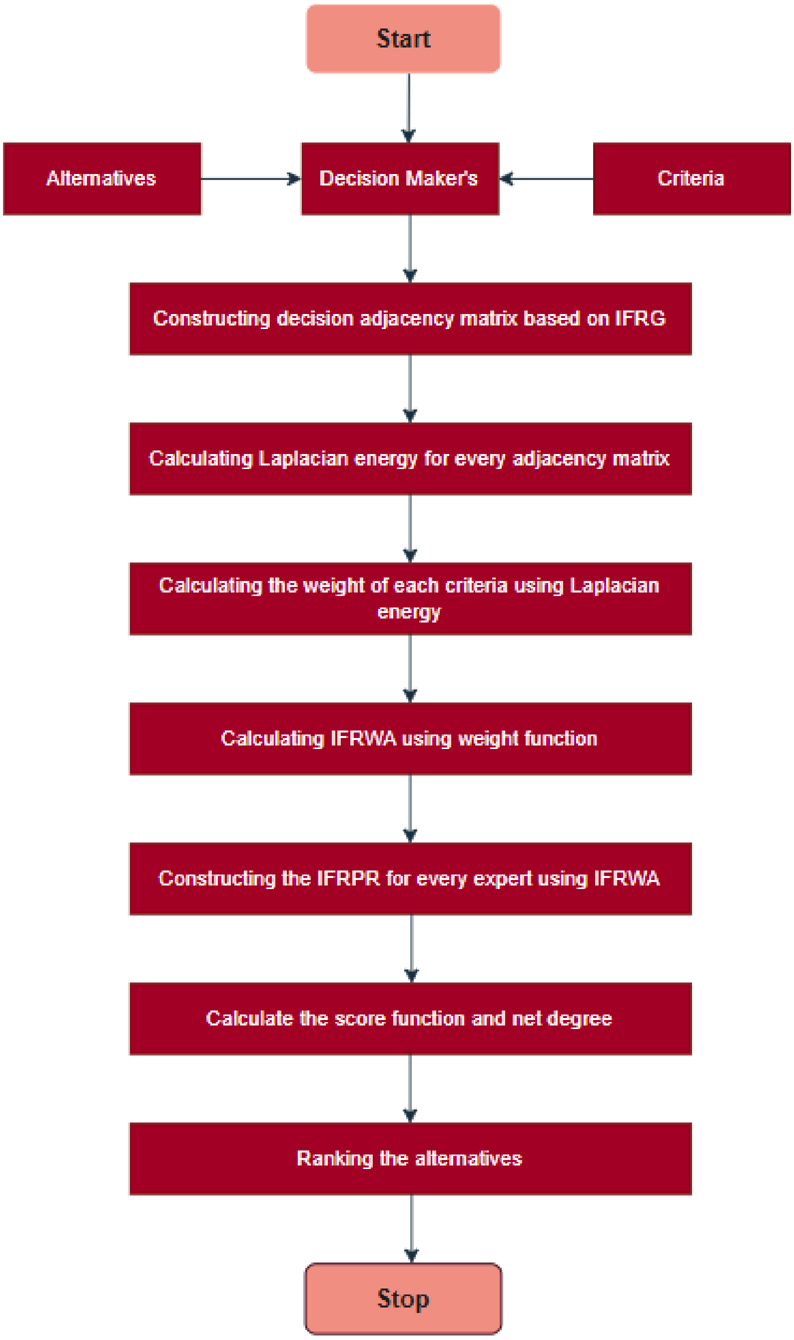


Figure 4. The procedure of ranking the alternatives for decision-making assessment

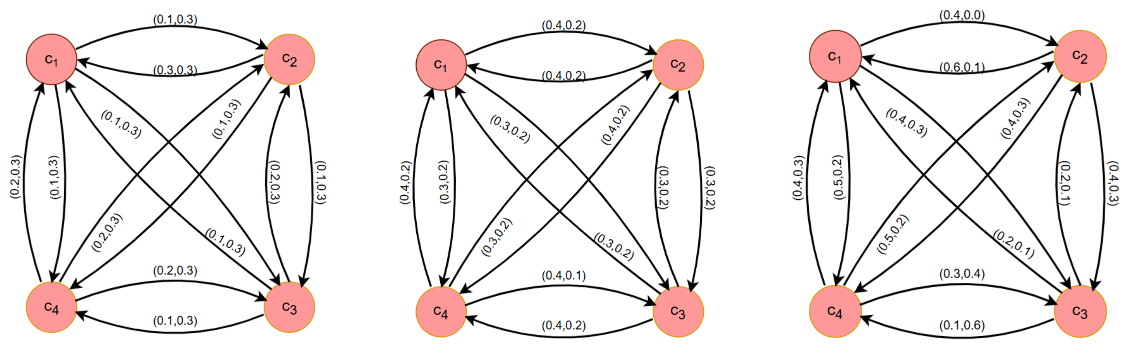
represented as matrices $R_k = \left(r_{ij}^{(k)}\right)_{4\times 4}$ $(k = 1, 2, 3, \ldots, p)$ as shown in Figure (5) and described below:

$$\underline{R_1} = \begin{bmatrix} (0.0, 0.0) & (0.1,\ 0.3) & (0.1,\ 0.3) & (0.1,\ 0.3) \\ (0.3, 0.3) & (0.0, 0.0) & (0.1,\ 0.3) & (0.1,\ 0.3) \\ (0.1,\ 0.3) & (0.2, 0.3) & (0.0, 0.0) & (0.1,\ 0.3) \\ (0.2, 0.3) & (0.2, 0.3) & (0.2, 0.3) & (0.0, 0.0) \end{bmatrix}$$

$$\underline{R}_2 = \begin{bmatrix} (0.0,0.0) & (0.4,0.2) & (0.3,0.2) & (0.3,0.2) \\ (0.4,0.2) & (0.0,0.0) & (0.3,0.2) & (0.4,0.2) \\ (0.3,0.2) & (0.3,0.2) & (0.0,0.0) & (0.4,0.2) \\ (0.4,0.2) & (0.3,0.2) & (0.4,0.1) & (0.0,0.0) \end{bmatrix}$$

$$\underline{R}_3 = \begin{bmatrix} (0.0,0.0) & (0.4,0.0) & (0.4,0.3) & (0.5,0.2) \\ (0.6,0.1) & (0.0,0.0) & (0.4,0.3) & (0.4,0.3) \\ (0.2,0.1) & (0.2,0.1) & (0.0,0.0) & (0.1,\ 0.6) \\ (0.4,0.3) & (0.5,0.2) & (0.3,0.4) & (0.0,0.0) \end{bmatrix}$$

The IFRDG for lower C_k corresponding to IFRPRs in matrices R_k, $k = 1, 2, 3$.



a) **b)** **c)**

Figure 5. Intuitionistic fuzzy rough graph for lower: a) $\underline{G_1}$, b) $\underline{G_2}$, c) $\underline{G_3}$

The Laplacian energy of each IFRDG for lower is calculated by using equation (1) as

$$\underline{LE}(\underline{R_1}) = (0.9000, 1.8000), \underline{LE}(\underline{R_2}) = (2.6725, 1.1500), \underline{LE}(\underline{R_3}) = (2.6985, 1.6191)$$

Then, the weight of each expert can be calculated by using equation (2) as:

$$\underline{w_1} = (0.1435, 0.3940), \underline{w_2} = (0.4262, 0.2517), \underline{w_3} = (0.4303, 0.3544)$$

Hence, the experts weight vector e_k $k = 1, 2, 3$ is calculated as:

$$\underline{w} = ((0.1435, 0.3940), (0.4262, 0.2517), (0.4303, 0.3544))$$

Utilize the aggregation operator to fuse all the individual IFRPRs $R_k = \left(r_{ij}^{(k)}\right)_{4\times4}$ $(k = 1, 2, 3, 4)$ into the collective IFRPR $R = (r_{ij})_{4\times4}$. Here, we apply the lower approximation of the intuitionistic fuzzy rough weighted averaging (IFRWA) operator [8] to fuse the individual lower IFRPR. Thus, we have by using equation (3) as

$$\underline{R} = \begin{bmatrix} (0.0000, 0.0000) & (0.3641,\ 0.0000) & (0.3209, 0.2709) & (0.3721,\ 0.2346) \\ (0.4848, 0.1835) & (0.0000, 0.0000) & (0.3209, 0.2709) & (0.3641,\ 0.2709) \\ (0.2314, 0.1835) & (0.2443, 0.1835) & (0.0000, 0.0000) & (0.2428, 0.3463) \\ (0.3747, 0.2709) & (0.3826, 0.2346) & (0.3318, 0.2519) & (0.0000, 0.0000) \end{bmatrix}$$

Calculate their scores function by using equation (4) as

$$s_{ij} = \begin{bmatrix} 0.0000 & 0.3641 & 0.0500 & 0.1375 \\ 0.3013 & 0.0000 & 0.0500 & 0.0932 \\ 0.0479 & 0.0608 & 0.0000 & -0.1035 \\ 0.1038 & 0.1480 & 0.0799 & 0.0000 \end{bmatrix}$$

The net flow of c_i [30], i.e., the net degree of preference of c_i over the other schemes by using equation (5). Therefore, the net flows of the four schemes are

$$\sigma(c_1) = 0.0986, \sigma(c_2) = -0.1284, \sigma(c_3) = -0.1747, \sigma(c_4) = 0.2045$$

which gives the ranking of $c_4 \succ c_1 \succ c_2 \succ c_3$. Thus, the best visualization is c_4.

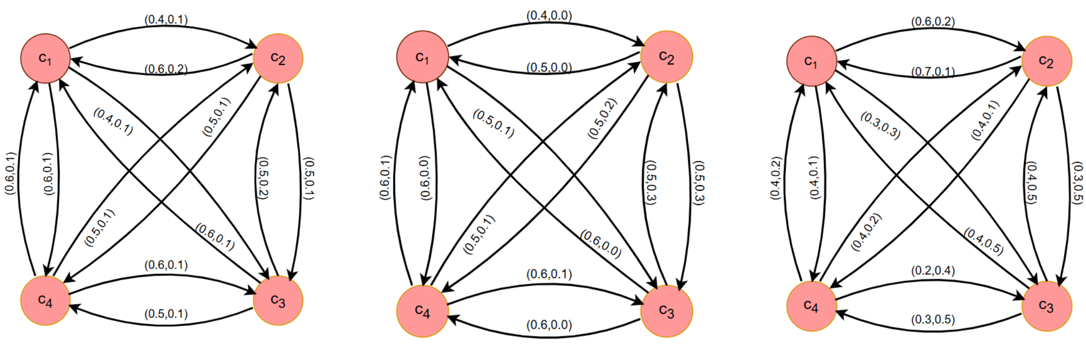
For upper approximation. The experts compare the involved factors and provide the initial information for computing in the form of upper intuitionistic fuzzy rough reference relations. These are represented as matrices $R_k = \left(r_{ij}^{(k)}\right)_{4\times 4}$ $(k = 1, 2, 3, \ldots, p)$ as shown in Figure (6) and described below

$$\overline{R_1} = \begin{bmatrix} (0.0, 0.0) & (0.4, 0.1) & (0.4, 0.1) & (0.6, 0.1) \\ (0.6, 0.2) & (0.0, 0.0) & (0.5, 0.1) & (0.5, 0.1) \\ (0.6, 0.1) & (0.5, 0.2) & (0.0, 0.0) & (0.5, 0.1) \\ (0.6, 0.1) & (0.5, 0.1) & (0.6, 0.1) & (0.0, 0.0) \end{bmatrix}$$

$$\overline{R_2} = \begin{bmatrix} (0.0, 0.0) & (0.4, 0.0) & (0.5, 0.1) & (0.6, 0.0) \\ (0.5, 0.0) & (0.0, 0.0) & (0.5, 0.3) & (0.5, 0.2) \\ (0.6, 0.0) & (0.5, 0.3) & (0.0, 0.0) & (0.6, 0.0) \\ (0.6, 0.1) & (0.5, 0.1) & (0.6, 0.1) & (0.0, 0.0) \end{bmatrix}$$

$$\overline{R_3} = \begin{bmatrix} (0.0, 0.0) & (0.6, 0.2) & (0.3, 0.3) & (0.4, 0.1) \\ (0.7, 0.1) & (0.0, 0.0) & (0.3, 0.5) & (0.4, 0.1) \\ (0.4, 0.5) & (0.4, 0.5) & (0.0, 0.0) & (0.3, 0.5) \\ (0.4, 0.2) & (0.4, 0.2) & (0.2, 0.4) & (0.0, 0.0) \end{bmatrix}$$

The IFRDG for upper C_k corresponding to IFRPRs in matrices R_k, $k = 1, 2, 3$.



a) **b)** **c)**

Figure 6. Intuitionistic fuzzy rough graph for upper: a) $\overline{G_1}$, b) $\overline{G_2}$, c) $\overline{G_3}$

The Laplacian energy of each IFRDG for upper is calculated by using equation (1) as

$$L\overline{E}(\overline{R_1}) = (3.1500, 0.7000), L\overline{E}(\overline{R_2}) = (3.2000, 0.9168), L\overline{E}(\overline{R_3}) = (2.4000, 2.0000)$$

Then, the weight of each expert can be calculated by using equation (2) as

$$\overline{w_1} = (0.3600, 0.1935), \overline{w_2} = (0.3657, 0.2535), \overline{w_3} = (0.2743, 0.5530)$$

Hence, the experts weight vector e_k $k = 1, 2, 3$ is calculated as

$$\overline{w} = ((0.3600, 0.1935), (0.3657, 0.2535), (0.2743, 0.5530))$$

Utilize the aggregation operator to fuse all the individual IFRPRs $R_k = \left(r_{ij}^{(k)}\right)_{4\times 4}$ $(k = 1, 2, 3, 4)$ into the collective IFRPR $R = (r_{ij})_{4\times 4}$. Here, we apply the lower approximation of the intuitionistic fuzzy rough weighted averaging (IFRWA) operator [8] to fuse the individual upper IFRPR. Thus, we have by using equation (3) as

$$\overline{R} = \begin{bmatrix} (0.0000, 0.0000) & (0.4632, 0.0000) & (0.4145, 0.1836) & (0.5529, 0.0000) \\ (0.5989, 0.0000) & (0.0000, 0.0000) & (0.4517, 0.3217) & (0.4744, 0.1192) \\ (0.5529, 0.0000) & (0.4744, 0.3679) & (0.0000, 0.0000) & (0.4946, 0.0000) \\ (0.5529, 0.1467) & (0.4744, 0.1467) & (0.5162, 0.2152) & (0.0000, 0.0000) \end{bmatrix}$$

Calculate their scores function by using equation (4) as:

$$s_{ij} = \begin{bmatrix} 0.0000 & 0.4632 & 0.2309 & 0.5529 \\ 0.5989 & 0.0000 & 0.1300 & 0.3552 \\ 0.5529 & 0.1065 & 0.0000 & 0.4946 \\ 0.4062 & 0.3277 & 0.3010 & 0.0000 \end{bmatrix}$$

The net flow of c_i [30], i.e., the net degree of preference of c_i over the other schemes by using equation (5). Therefore, the net flows of the four schemes are

$$\sigma(c_1) = -0.3110, \sigma(c_2) = 0.1867, \sigma(c_3) = 0.4921, \sigma(c_4) = -0.3678$$

which gives the ranking of $c_3 \succ c_2 \succ c_1 \succ c_4$. Thus, the best visualization is c_3.

6.2. Comparative analysis

This section provides a comparative analysis of selected methods for evaluating data visualization, including an intuitionistic fuzzy rough geometric aggregation function chosen for comparison. The aggregation matrices used were R_1, R_2, R_3 and the weights w_1, w_2, w_3 as determined in equation (3). The aggregation was performed using the intuitionistic fuzzy rough geometric aggregation operator [8]:

$$\text{IFRGA}\left(R_{ij}^{(1)}, R_{ij}^{(2)}, \ldots, R_{ij}^{(n)}\right) = \left(\prod_{r=1}^{s}\left(\mu_{jk}^{(r)}\right)^{w_r}, 1 - \prod_{r=1}^{s}\left(1 - \nu_{jk}^{(r)}\right)^{w_r}\right) \tag{6}$$

After aggregating the matrices, the resulting matrices were evaluated using the net degree approach. The evaluations for lower IFRG are $\sigma(c_1) = -0.1175, \sigma(c_2) = -0.1497, \sigma(c_3) = -0.2440, \sigma(c_4) = 0.3194$, ranking the alternatives as $c_4 \succ c_1 \succ c_2 \succ c_3$. For the upper IFRG, the evaluations were $\sigma(c_1) = -0.1649, \sigma(c_2) = -0.1641, \sigma(c_3) = -0.0094, \sigma(c_4) = -0.1798$, ranking order $c_3 \succ c_2 \succ c_1 \succ c_4$. Both aggregating orders are the same order in lower and upper approximation. When the lower and upper approximation order differ, the lower approximation order is considered the best because it provides a precise approximation based on the decision criteria. As a result, alternative c_4 was identified as the optimal choice, while alternative c_3 received the lowest rating and was ranked last. Thus, alternatives c_1 and c_2 have distinct rankings.

The study results indicate that the developed technique, based on Laplacian energy, is more flexible and reasonable for decision-making analysis problems. In conclusion, our research introduces the concept of the Laplacian energy of an intuitionistic fuzzy rough graph and demonstrates its application in decision-making. The findings contribute to the existing knowledge in uncertainty modeling and offer a valuable framework for addressing complex problems across various domains.

7. Conclusion

Rough set theory is a mathematical method for handling ambiguous and incomplete data. Intuitionistic fuzzy set theory addresses understanding and utilizing incomplete knowledge. Recent research suggests that integrating these two concepts can create a more expressive and adaptable framework for representing and processing incomplete information in information systems.

This paper presents a study on evaluating strategies for data visualizations. Due to the current uncertainties related to decision-makers' preferences, the intuitionistic fuzzy rough graph was chosen. The study employs the Laplacian energy of the intuitionistic fuzzy rough digraph and performs a comparative analysis using two methods for evaluating alternative strategies and four functions for aggregating intuitionistic fuzzy rough decision matrices. The results demonstrate the approach's applicability to real-world data visualization problems. The intuitionistic fuzzy rough digraph effectively captures the decision-makers' preferences, yielding credible and reliable results, unlike other directed graphs that do not account for the degree of refusal.

From the study, it is evident that the intuitionistic fuzzy rough decision matrix aggregation functions are effective. This research characterizes the Laplacian energy of an intuitionistic fuzzy rough graph using the adjacency matrix and calculates the lower and upper bounds on this energy. The Laplacian energy of the intuitionistic fuzzy rough digraph is also discussed. Future work could extend these deliberations to obtain the Laplacian energy of intuitionistic fuzzy rough hypergraphs, with appropriate comparative analyses.

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