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Improved Liu estimator for the beta regression model: methods, simulation and applications

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Abstract

The beta regression model (BRM) is a well-known approach to modeling a response variable that has a beta distribution. The maximum likelihood estimator (MLE) does not produce accurate results for the BRM when the data has a high degree of multicollinearity. We propose a one-parameter beta Liu estimator (OPBLE) for the BRM to tackle the weaknesses of the available Liu estimator in dealing with the issue of multicollinearity. Using the mean square error (MSE), we analytically show that the proposed estimator performs more efficiently than the MLE, beta ridge regression estimator (BRRE), and beta Liu estimator (BLE). We conduct a simulation study and use two practical examples to investigate the performance of the OPBLE. Using the findings from the simulations and empirical studies, we demonstrate the superiority of the proposed estimator over the MLE, BRRE, and BLE in the presence of multicollinearity in the regressors.

Keywords: *beta regression, mean square error, beta liu estimator, multicollinearity*

1. Introduction

The most popular model to model the ratio and rate types response variables that are well fitted to the beta distribution is the beta regression model (BRM) introduced by Ferrari and Cribari-Neto [12]. The BRM has various applications, for example, to model the proportion of income spent on food, to model the level of poverty, to model the proportion of crude oil and many others. The most frequently used method to estimate the unknown regression parameters of the BRM is the maximum likelihood estimator (MLE). Like the linear regression model (LRM), the BRM also has the assumption that explanatory variables are not perfectly linearly correlated [15, 18] but in practice, there are situations where the explanatory variables are usually correlated, especially in the field of economics, health, and social sciences. The correlation among explanatory variables is alarming for the MLE and related inferences [15, 18]. This

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situation is particularly called multicollinearity. Due to multicollinearity, the accuracy of the MLE becomes doubtful due to the large variance in the regression estimates [20]. To overcome this problem, many authors developed some biased estimation methods for the LRM, the generalized linear model (GLM), and the BRM. One of the most well-known and commonly used methods to overcome the effect of multicollinearity is ridge regression estimation (RRE). Hoerl and Kennard [13] first proposed the RRE method for the LRM, where the ridge parameter contributes the main role. Many researchers introduced different methods for finding the best value of biasing parameter k for the RRE [3, 6, 8, 9, 13, 16].

The consequence of multicollinearity on the GLM has also been discussed by numerous researchers using the ridge regression approach. Schaefer et al.[32] introduced the RRE for the logistic regression model. For the logistic regression model, several ridge parameters were also proposed by Mansson and Shukur [24]. Mansson and Shukur [25] developed a Poisson ridge regression (PRR) estimator. The RRE for the negative binomial regression model was introduced by Mansson [22]. Amin et al. [7] evaluated the effectiveness of inverse Gaussian ridge regression estimators. Abonazel and Taha[1] introduced several ridge parameter estimators for the BRM. Amin et al. [5] proposed ridge estimators for the bell regression model. Qasim et al. [31] developed some ridge parameter estimators for the BRM.

The Liu estimator (LE) [16] is another estimating technique that addresses the multicollinearity issue in comparison to the RRE. Because it is a linear function of the shrinkage parameter, the LE is chosen over the RRE. Some researchers developed different methods for the LE in the LRM [15, 18, 29]. For the GLM, several researchers worked on the LE in contrast to ridge regression [23, 29, 31, 35]. Varathan and Wijekoon [35] developed LE for the logistic regression model. Some researchers worked on the biased estimation methods in the BRM to reduce the impact of multicollinearity on the ML estimates. Initially, the LE for the BRM was considered by Karlsson et al. [15]. Then, Qasim et al. [30] considered the RRE for the BRM. Algamal and Abonazel [4] introduced the Liu-type estimator for the BRM which was the combination of ridge and Liu parameters. Abonazel et al. [2] worked on the two-parameter estimator in the BRM. Abonazel et al. [1] proposed the Dawoud–Kibria estimator for thee BRM. Akram et al. [3] considered the Kibria–Lukman estimator for the BRM. Mustafa et al.[26] suggested the best ridge parameter for the BRM with different link functions. The LE has the weakness that it often produces negative values of the Liu parameter. This limitation greatly disturbs the efficiency and effectiveness of the LE. To overwhelm this restriction, some researchers suggested adjusted LE (ALE) and Liu-type estimator (LTE). Some literature is also available on ALE and LTE for the LRM [14, 21].

Various researchers made different modifications to the biased estimators for the LRM and the GLM. Recently, Seifollahi et al. [33] proposed the jackknife Liu-type estimator for the BRM. Karlsson et al. [15] introduced the BRM's LE but it has a limitation in that mostly the shrinkage parameter of the LE produces negative values which affect the estimation procedure. Therefore, the current study introduces a new estimator that we call a modified one-parameter LE of the BRM to overcome the limitation of the available LE. We derive the MSE of the available and proposed estimators. We also give theoretical comparisons of these estimators under the MSE criterion. The rest of the article follows the methodology of the BRM with all notations, mathematical formulation of the BRM, its estimation, and proposed estimator is illustrated in section 2. In this section, we also covered the theoretical properties of the suggested estimator and how it stacks up against alternative estimators. In section 3, we give the complete computational details of biased estimation methods and evaluate the performance of the shrinkage parameter estimators using Monte Carlo simulation with different multicollinearity levels, sample sizes, regressors, and dispersion. In section 4, two real-world applications are used to assess the effectiveness of the suggested estimator. The concluding remarks of this study are given in section 5.

2. Methodology

2.1. The BRM

Suppose that $y_1, y_2, \ldots, y_n$ are independent random variables with n observations that follow a beta distribution with parameters $u,\ v\ > 0$, then the probability density function (pdf) of the beta distributions has the following form

$$f(y; u, v) = \frac{\Gamma(u, v)}{\Gamma(u)\Gamma(v)} y^{u-1}(1-y)^{v-1}, \quad y \in (0,1) \tag{1}$$

where $\Gamma(\cdot)$ represents the gamma function. The mean and variance of equation (1) are, respectively, found to be

$$E(y) = \frac{u}{u+v} \quad \text{and} \quad \mathrm{Var}(y) = \frac{uv}{(u+v)^2(u+v+1)}$$

.

Now suppose that $\mu = \dfrac{u}{u+v}$ and $\varphi = u + v$, and equation (1) is reparametrized by defining $u = \mu\varphi$ and $v = \varphi - \varphi\mu$, as

$$f(y; \mu, \varphi) = \frac{\Gamma(\varphi)}{\Gamma(\mu\varphi)\Gamma(\varphi - \varphi\mu)} y^{\mu\varphi-1}(1-y)^{\varphi-\varphi\mu-1}, \quad y \in (0,1),\ 0 < \mu < 1, \quad \varphi > 0 \tag{2}$$

where $y \sim \beta(\mu, \varphi)$, μ indicate the mean of the response variable, and the precision parameter is denoted by φ. According to new parameterization, the mean and variance of the response variable are given as $(y) = \mu$, $\mathrm{Var}(y) = \dfrac{(\mu)(1-\mu)}{(1+\varphi)}$, respectively.

The value of the dispersion parameter can be obtained by taking the reciprocal of φ which is $\delta = \varphi^{-1}$. The link function for the BRM is expressed as

$$g(\mu_i) = \eta_i = x_i^t \beta \tag{3}$$

where x_i represents the ith row of X that is a data matrix of p regressors with order $n \times (p+1)$ including intercept, $\beta = (\beta_0,\ \beta_1, \beta_2,\ \ldots,\ \beta_p)^t$ is the vector of regression coefficients with $(p+1) \times 1$ order, $g(.)$ is the BRM link function and the linear predictor is denoted by η_i. The link function is strictly monotonic and two times differentiable, i.e., $g(.) : (0,1) \rightarrow \mathbb{R}$. Several link functions are appropriate for fitting the BRM. Ferrari and Cribari-Neto [11] introduced the logit link function for the BRM and it is the most widely used link function. It can be expressed as

$$g(\mu_i) = \log\left(\frac{\mu_i}{1-\mu_i}\right)$$

Using the logit link function the conditional mean of the dependent variable can be written as

$$\mu_i = \frac{\exp(x_i^t\beta)}{1+\exp(x_i^t\beta)}$$

where μ is the function of η. To estimate the BRM, first, we consider equation (2)'s log-likelihood function, that is:

$$l(\beta) = \sum_{i=1}^{n} l_i(\mu_i, \varphi) = \sum_{i=1}^{n} \{\log \mathit{\Gamma}(\varphi) - \log \mathit{\Gamma}(\mu_i\varphi) - \log \mathit{\Gamma}(\varphi - \mu_i\varphi)$$
$$+ (\varphi\mu_i - 1)\log y_i + (\varphi - \mu_i\varphi - 1)\log(1 - y_i)\}$$

Suppose that the estimated value of β using MLE is $\widehat{\beta}$. Our main interest in the study is to find the unknown parameter of a β vector. The estimation method known as MLE is commonly employed to determine the values of regression coefficients β_j that are unknown. The $\mathcal{S}(\beta)$ indicates the score function of equation (4) and it is given as

$$\mathcal{S}(\beta) = \delta X^t T\left(y^* - \mu^*\right) \tag{4}$$

where $y^* = \log\dfrac{y}{1-y}$, $\mu^* = \psi(\mu\delta) - \psi\left((1-\mu)\delta\right)$, X is a design matrix with dimension $n \times (p+1)$, $T = \operatorname{diag}\left[\dfrac{1}{g'(\mu_1)}, \ldots, \dfrac{1}{g'(\mu_n)}\right]$, the digamma function denoted as ψ and $g(\cdot)$ is the logit link function. Let $\eta_i = g(\mu_i) = \log\left(\dfrac{\mu_i}{1-\mu_i}\right) = x_i^t\beta$, where x_i correspond to the ith row of the data matrix, and represents the $(p+1)\times 1$ vector of regression coefficients with intercept and $p+1$ represents the explanatory variables including the intercept. Since equation (4) is nonlinear in β, it requires the utilization of an iterative reweighted method. By the work of Abonazel and Taha [1], the iterative method provides a way to compute β that is

$$\beta_{r+1} = \beta_r + \left\{I_r^{\beta\beta}\right\}^{-1} \mathcal{S}(\beta_r) \tag{5}$$

The iteration denoted by $r = 0, 1, 2, \ldots$, continues until convergence is achieved. In each iteration $I_r^{\beta\beta}$ represents the information matrix of β. In the final iteration [1], equation (5) can be expressed as

$$\widehat{\beta}_{\mathrm{MLE}} = \left(X^t\widehat{V}X\right)^{-1} X^t\widehat{V}z \tag{6}$$

where $V = \operatorname{diag}(v, v_2, \ldots, v_n)$, $v_i = \dfrac{\delta\left(\psi'(\mu_i\delta) - \psi'\left((1-\mu_i)\delta\right)\right)}{g'^2(\mu_i)}$ and $z = \eta V^{-1}T(y^* - \mu^*)$, where V and z are measured on the last iteration. Fisher scoring iterative procedure used for the evaluation of V and z. The estimation of the matrix MSE (MMSE) and scalar MSE can be achieved by considering $\alpha = \zeta^t\widehat{\beta}_{\mathrm{MLE}}$ and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{p+1})$ which is equal to $\zeta(X^t\widehat{V}X)\zeta^t$, where ζ indicates the orthogonal matrix and whose columns are the eigenvectors of $X^t\widehat{V}X$; i.e., $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{p+1})$, where ζ_j represents the jth eigenvectors of $X^t\widehat{V}X$ and $\lambda_1 \geq \lambda_2 \geq, \ldots, \lambda_{p+1} \geq 0$ represents the eigenvalues of the matrix $X^t\widehat{V}X$. Furthermore α_j for all $j = 1, 2, \ldots, p+1$ corresponds to the jth element of $\zeta^t\widehat{\beta}_{\mathrm{MLE}}$.

Subsequently the covariance and MMSE of the $\widehat{\beta}_{\text{MLE}}$ are given by the following expression

$$\operatorname{Cov}\left(\widehat{\beta}_{\text{MLE}}\right) = \widehat{\delta}\left(X^t\widehat{V}X\right)^{-1} \tag{7}$$

$$\operatorname{MMSE}\left(\widehat{\beta}_{\text{MLE}}\right) = \zeta\Lambda^{-1}\zeta^t \tag{8}$$

The scalar MSE of the $\widehat{\beta}_{\text{MLE}}$ can be expressed as

$$\operatorname{MSE}\left(\widehat{\beta}_{\text{MLE}}\right) = E\left(\widehat{\beta}_{\text{MLE}} - \beta\right)^t\left(\widehat{\beta}_{\text{MLE}} - \beta\right) = \widehat{\delta}\left[\operatorname{tr}\left(\zeta\Lambda^{-1}\zeta^t\right)\right] = \widehat{\delta}\sum_{j=1}^{p+1}\frac{1}{\lambda_j} \tag{9}$$

where λ_j is the *j*th eigenvalue of the matrix $X^t\widehat{V}X$.

When the explanatory variables are correlated, then the X^tVX matrix becomes ill-conditioned. Due to this problem, the MSE of the MLE is inflated because of the small eigenvalues. The MLE of the BRM becomes inflated and convoluted in the presence of multicollinearity, making it difficult to make a valid inference. To deal with the problem of multicollinearity, some researchers introduced some biased estimators for the BRM.

2.2. Beta ridge regression estimator

For the BRM, Qasim et al. [31] introduced the mean square error (BRRE) which can be defined as

$$\widehat{\beta}_{\text{BRRE}} = G_k\widehat{\beta}_{\text{MLE}} \tag{10}$$

where $G_k = \left(X^t\widehat{V}X + kI_{p+1}\right)^{-1}\left(X^t\widehat{V}X\right)$, $k > 0$. The shrinkage parameter of the BRRE is denoted by k and I_{p+1} refers to the identity matrix with dimension $(p+1)\times(p+1)$. When k approaches to 0, then $\widehat{\beta}_{\text{BRRE}} = \widehat{\beta}_{\text{MLE}}$ We can express the bias vector and covariance matrix of equation (10) as

$$\operatorname{Bias}\left(\widehat{\beta}_{\text{BRRE}}\right) = -k\zeta\Lambda_k^{-1}\beta \tag{11}$$

$$\operatorname{Cov}\left(\widehat{\beta}_{\text{BRRE}}\right) = \widehat{\delta}\zeta\Lambda_k^{-1}\Lambda\Lambda_k^{-1}\zeta^t \tag{12}$$

Therefore, the MMSE of the BBRE is defined as

$$\begin{aligned}\operatorname{MMSE}\left(\widehat{\beta}_{\text{BRRE}}\right) &= G_k\left(X^t\widehat{V}X\right)^{-1}G_k^t + \operatorname{Bias}\left(\widehat{\beta}_{\text{BRRE}}\right)\operatorname{Bias}\left(\widehat{\beta}_{\text{BRRE}}\right)^t \\ &= \widehat{\delta}\zeta\Lambda_k^{-1}\Lambda\Lambda_k^{-1}\zeta^t + k^2\zeta\Lambda_k^{-1}\beta\beta^t\Lambda_k^{-1}\zeta^t\end{aligned} \tag{13}$$

In equation (13), Λ_k is a diagonal matrix defined as $\operatorname{diag}\left(\lambda_1 + k,\ \lambda_2 + k,\ \ldots,\ \lambda_{p+1} + k\right)$, and $\Lambda = \operatorname{diag}\left(\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_{p+1}\right)$ which is equal to $\zeta\left(X^t\widehat{V}X\right)\zeta^t$, where the orthogonal matrix with its eigenvectors columns of $X^t\widehat{V}X$ is denoted by ζ. Therefore, we apply the $\operatorname{tr}(.)$ operator on equation (13) to find the scalar MSE of the BRRE as

$$\operatorname{MSE}\left(\widehat{\beta}_{\text{BRRE}}\right) = \operatorname{tr}\left\{\operatorname{MMSE}\left(\widehat{\beta}_{\text{BRRE}}\right)\right\} = \widehat{\delta}\sum_{j=1}^{p+1}\frac{\lambda_j}{\left(\lambda_j + k\right)^2} + k^2\sum_{j=1}^{p+1}\frac{{\alpha_j}^2}{\left(\lambda_j + k\right)^2},\quad \alpha = \zeta^t\widehat{\beta}_{\text{MLE}} \tag{14}$$

2.3. Beta Liu estimator

Another estimator called the beta Liu estimator (BLE) in contrast to BRRE for the BRM was introduced by Karlsson et al. [15]. It is defined as

$$\widehat{\beta}_{\text{BLE}} = q_d \widehat{\beta}_{\text{MLE}} \tag{15}$$

where $q_d = \left(X^t \widehat{V} X + I_{p+1}\right)^{-1} \left(X^t \widehat{V} X + dI_{p+1}\right)$ and $d\ [0,1]$ is a Liu parameter. We can define the bias vector and covariance matrix of equation (15) as

$$\text{Bias}\left(\widehat{\beta}_{\text{BLE}}\right) = \zeta(d-1)\Lambda_I^{-1}\beta \tag{16}$$

$$\text{Cov}\left(\widehat{\beta}_{\text{BLE}}\right) = \widehat{\delta}\zeta\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1}\zeta^t \tag{17}$$

Therefore, the MMSE of the BLE is defined as

$$\begin{aligned}\text{MMSE}\left(\widehat{\beta}_{\text{BLE}}\right) &= q_d \left(X^t \widehat{V} X\right)^{-1} q_d^t + \text{Bias}\left(\widehat{\beta}_{\text{BLE}}\right) \text{Bias}\left(\widehat{\beta}_{\text{BLE}}\right)^t \\ &= \widehat{\delta}\zeta\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1}\zeta^t + (d-1)^2\zeta\Lambda_I^{-1}\beta\beta^t\Lambda_I^{-1}\zeta^t\end{aligned} \tag{18}$$

where Λ_I indicates $\text{diag}\left(\lambda_1 + I,\ \lambda_2 + I,\ \ldots,\ \lambda_{p+1} + I\right)$, and

$$\Lambda_d = \text{diag}\left(\lambda_1 + d,\ \lambda_2 + d,\ \ldots,\ \lambda_{p+1} + d\right)$$

The BLE's scalar MSE can be expressed as

$$\text{MSE}\left(\widehat{\beta}_{\text{BLE}}\right) = \text{tr}\left\{\text{MMSE}\left(\widehat{\beta}_{\text{BLE}}\right)\right\} = \widehat{\delta}\sum_{j=1}^{p+1} \frac{(\lambda_j + d)^2}{\lambda_j\left(\lambda_j + 1\right)^2} + (d-1)^2 \sum_{j=1}^{p+1} \frac{{\alpha_j}^2}{\left(\lambda_j + 1\right)^2} \tag{19}$$

2.4. Proposed estimator for the BRM

One of the limitations of the shrinkage parameter by Liu [16] is that it can return a negative value most of the time, affecting the LE's performance. To deal with this issue, we introduced a one-parameter BLE (OPBLE) for the BRM as

$$\widehat{\beta}_{\text{OPBLE}} = B_d \widehat{\beta}_{\text{MLE}} \tag{20}$$

where $B_d = (X^t\hat{V}X + I_{p+1})^{-1}(X^t\hat{V}X - d^* I_{p+1})$ and $0 < d^* < 1$ indicates the Liu parameter of the OPBLE. We can express the bias, covariance and MMSE of the OPBLE as

$$\text{Bias}\left(\widehat{\beta}_{\text{OPBLE}}\right) = -\left(d^* - 1\right)\left(X^t \widehat{V} X + I_{p+1}\right)^{-1} \beta \tag{21}$$

$$\begin{aligned}\text{Cov}\left(\widehat{\beta}_{\text{OPBLE}}\right) &= \widehat{\delta}(X^t \widehat{V} X + I_{p+1})^{-1}(X^t \widehat{V} X - d^* I_{p+1})\left(X^t \widehat{V} X\right)^{-1} \\ &\times (X^t \widehat{V} X - d^* I_{p+1})\left(X^t \widehat{V} X + I_{p+1}\right)^{-1}\end{aligned} \tag{22}$$

Now the MMSE of the OPBLE is defined by

$$\begin{aligned}\operatorname{MMSE}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right) &= \operatorname{Cov}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right) + \operatorname{Bias}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right)\operatorname{Bias}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right)^t \\ &= \widehat{\delta} B_d \left(X^t\widehat{V}X\right)^{-1} B_d^t + (d^*-1)^2\left(X^t\widehat{V}X + I_{p+1}\right)^{-1}\beta\beta^t\left(X^t\widehat{V}X + I_{p+1}\right)^{-1}\end{aligned} \tag{23}$$

The scalar MSE of the OPBLE can be written as

$$\operatorname{MSE}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right) = \operatorname{tr}\left\{\operatorname{MMSE}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right)\right\} = \widehat{\delta}\sum_{j=1}^{p+1}\frac{(\lambda_j - d^*)^2}{\lambda_j(\lambda_j+1)^2} + (d^*+1)^2\sum_{j=1}^{p+1}\frac{{\alpha_j}^2}{(\lambda_j+1)^2} \tag{24}$$

2.5. Theoretical comparison of the BRM estimators

Lemma 1 ([11]). Let G be a positive definite matrix, $G > 0$, and suppose that α specifies several vectors, then $G - \alpha\alpha^t \geq 0$ only if $\alpha^t G^{-1}\alpha \leq 1$.

Lemma 2 ([34]). Assume we have two estimators for the parameter θ denoted as $\widehat{\theta}_1 = B_1 y$ and $\widehat{\theta}_2 = B_2 y$. It is assumed that $D = \operatorname{Cov}\left(\widehat{\theta}_1\right) - \operatorname{Cov}\left(\widehat{\theta}_2\right) > 0$, where $\operatorname{Cov}\left(\widehat{\theta}_1\right)$ and $\operatorname{Cov}\left(\widehat{\theta}_2\right)$ are, respectively, the covariance matrices of $\widehat{\theta}_1$ and $\widehat{\theta}_2$. Then the condition, $\operatorname{MMSE}\left(\widehat{\theta}_1\right) - \operatorname{MMSE}\left(\widehat{\theta}_2\right) > 0$ holds true only when $c_2^t(D + c_2 c_2^t)^{-1}c_2 < 1$. In this case c_2 represent the bias and $\operatorname{MMSE}\left(\widehat{\theta}_j\right) = \operatorname{Cov}\left(\widehat{\theta}_j\right) + c_j c_j^t$, where the bias vector of $\widehat{\theta}_j$ is denoted by c_j.

Theorem 1. In the BRM, if $d^* > 0$, it indicates that the estimator $\widehat{\beta}_{\mathrm{OPBLE}}$ performs superior to $\widehat{\beta}_{\mathrm{MLE}}$, that is $\Delta_1 = \operatorname{MMSE}\left(\widehat{\beta}_{\mathrm{MLE}}\right) - \operatorname{MMSE}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right) > 0$ if and only if,

$$b_{\mathrm{OPBLE}}^t\left[\begin{matrix}(\widehat{\delta}(X^t\widehat{V}X)^{-1} - \widehat{\delta}\left(X^t\widehat{V}X + I_{p+1}\right)^{-1}(X^t\widehat{V}X - d^*I_{p+1})\left(X^t\widehat{V}X\right)^{-1} \\ (X^t\widehat{V}X - d^*I_{p+1})\left(X^t\widehat{V}X + I_{p+1}\right)^{-1})\end{matrix}\right]^{-1} b_{\mathrm{OPBLE}} < 1$$

where $b_{\mathrm{OPBLE}} = -(d^*-1)\left(X^t\widehat{V}X + I_{p+1}\right)^{-1}\beta$.

Proof. The discrepancy between the MMSE functions of the MLE and the OPBLE is determined by

$$\Delta_1 = \left[\begin{matrix}\widehat{\delta}\left(X^t\widehat{V}X\right)^{-1} - \widehat{\delta}\left(X^t\widehat{V}X + I_{p+1}\right)^{-1}(X^t\widehat{V}X - d^*I_{p+1})\left(X^t\widehat{V}X\right)^{-1} \\ (X^t\widehat{V}X - d^*I_{p+1})\left(X^t\widehat{V}X + I_{p+1}\right)^{-1} - b_{\mathrm{OPBLE}}b_{\mathrm{OPBLE}}^t\end{matrix}\right] \tag{25}$$

$$\begin{aligned}\Delta_1 = \widehat{\delta}\left(X^t\widehat{V}X\right)^{-1}&\left[\begin{matrix}I_{p+1} - \left(X^t\widehat{V}X + I_{p+1}\right)^{-1}\left(X^t\widehat{V}X - d_0 I_{p+1}\right)\left(X^t\widehat{V}X - d^*I_{p+1}\right) \\ \left(X^t\widehat{V}X + I_{p+1}\right)^{-1}\end{matrix}\right] \\ &- b_{\mathrm{OPBLE}}b_{\mathrm{OPBLE}}^t\end{aligned} \tag{26}$$

However, the difference of the scalar MSEs for equation (26) can be expressed as

$$\mathrm{MSE}\left(\widehat{\beta}_{\mathrm{MLE}}\right)-\mathrm{MSE}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right)=\widehat{\delta}\zeta\mathrm{diag}\left\{\frac{1}{\lambda_j}-\frac{\left(\lambda_j-d^*\right)^2}{\lambda_j\left(\lambda_j+1\right)^2}\right\}_{j=1}^{p+1}\zeta^t-b_{\mathrm{OPBLE}}^t b_{\mathrm{OPBLE}} \quad (27)$$

$$\mathrm{MSE}\left(\widehat{\beta}_{\mathrm{MLE}}\right)-\mathrm{MSE}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right)=\widehat{\delta}\zeta\mathrm{diag}\left\{\frac{\left(\lambda_j+1\right)^2-\left(\lambda_j-d^*\right)^2}{\lambda_j\left(\lambda_j+1\right)^2}\right\}_{j=1}^{p+1}\zeta^t-b_{\mathrm{OPBLE}}^t b_{\mathrm{OPBLE}}. \quad (28)$$

The expression is given in equation (26), that is, $I_{p+1}-\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}\left(X^t\widehat{V}X-d^*I_{p+1}\right)\times\left(X^t\widehat{V}X-d^*I_{p+1}\right)\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}$ is p.d. $\left(\lambda_j+1\right)^2>\left(\lambda_j-d^*\right)^2$ which is further corresponding to $2\lambda_j\left(d^*+1\right)-d^{*2}+1)>0$. Using MMSE and scalar MSE criteria, it is, therefore sufficient to prove that OPBLE showed better results than the MLE when $0<d^*<1$, $\forall j=1,2,\ldots,p+1$. □

Theorem 2. In the context of BRM, if $d^*>0$, it implies that the estimator $\widehat{\beta}_{\mathrm{OPBLE}}$ is superior to $\widehat{\beta}_{\mathrm{BRRE}}$, that is $\Delta_2=\mathrm{MMSE}\left(\widehat{\beta}_{\mathrm{BRRE}}\right)-\mathrm{MMSE}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right)>0$ if and only if,

$$b_{\mathrm{OPBLE}}^t\left[\left(\begin{matrix}\widehat{\delta}\left(X^t\widehat{V}X+kI_{p+1}\right)^{-1}\left(X^t\widehat{V}X\right)\left(X^t\widehat{V}X+kI_{p+1}\right)^{-1}-\widehat{\delta}\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}\\ \left(X^t\widehat{V}X-d^*I_{p+1}\right)\left(X^t\widehat{V}X\right)^{-1}\left(X^t\widehat{V}X-d^*I_{p+1}\right)\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}\end{matrix}\right)\right]^{-1}$$

$\times b_{\mathrm{OPBLE}}<1$, where $b_{\mathrm{OPBLE}}=-\left(d^*-1\right)\left(X^tVX+I_{p+1}\right)^{-1}\beta$.

Proof. We compare the MMSEs of the BRRE and the OPBLE and their MMSEs difference is given as

$$\Delta_2=\left[\begin{matrix}\widehat{\delta}\left(X^t\widehat{V}X+kI_{p+1}\right)^{-1}\left(X^t\widehat{V}X\right)\left(X^t\widehat{V}X+kI_{p+1}\right)^{-1}\\ -\widehat{\delta}\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}\left(X^t\widehat{V}X-d^*I_{p+1}\right)\\ \left(X^t\widehat{V}X\right)^{-1}\left(X^t\widehat{V}X-d^*I_{p+1}\right)\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}\end{matrix}\right] \quad (29)$$
$$+\ b_{\mathrm{BRRE}}\,b_{\mathrm{BRRE}}^t-b_{\mathrm{OPBLE}}\,b_{\mathrm{OPBLE}}^t$$

where $b_{\mathrm{BRRE}}=-k\left(X^t\widehat{V}X+kI_{p+1}\right)^{-1}$. Though, the scalar MSEs difference of equation (29) is

$$\mathrm{MSE}\left(\widehat{\beta}_{\mathrm{BRRE}}\right)-\mathrm{MSE}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right)=\widehat{\delta}\zeta\mathrm{diag}\left\{\frac{\lambda_j}{\left(\lambda_j+k\right)^2}-\frac{\left(\lambda_j-d^*\right)^2}{\lambda_j\left(\lambda_j+1\right)^2}\right\}_{j=1}^{p+1}\zeta^t \quad (30)$$
$$+b_{\mathrm{BRRE}}^t b_{\mathrm{BRRE}}-b_{\mathrm{OPBLE}}^t b_{\mathrm{OPBLE}}$$

$$\mathrm{MSE}\left(\widehat{\beta}_{\mathrm{BRRE}}\right)-\mathrm{MSE}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right)=\widehat{\delta}\zeta\mathrm{diag}\left\{\frac{\lambda_j{}^2\left(\lambda_j+1\right)^2-\left(\lambda_j-d^*\right)^2\left(\lambda_j+k\right)^2}{\lambda_j\left(\lambda_j+k\right)^2\left(\lambda_j+1\right)^2}\right\}_{j=1}^{p+1}\zeta^t \quad (31)$$
$$+b_{\mathrm{BRRE}}^t b_{\mathrm{BRRE}}-b_{\mathrm{OPBLE}}^t b_{\mathrm{OPBLE}}$$

Since $b_{\text{BRRE}}^t b_{\text{BRRE}}$ in equation (31) is positive definite. So, it is sufficient to demonstrate that

$$\begin{bmatrix} \widehat{\delta}\left(X^t\widehat{V}X+kI_{p+1}\right)^{-1}\left(X^t\widehat{V}X\right)\left(X^t\widehat{V}X+kI_{p+1}\right)^{-1}-\widehat{\delta}\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}(X^t\widehat{V}X-d^*I_{p+1}) \\ \left(X^t\widehat{V}X\right)^{-1}(X^t\widehat{V}X-d^*I_{p+1})\left(X^t\widehat{V}X+I_{p+1}\right)^{-1} \end{bmatrix}$$
$$+\, b_{\text{BRRE}}b_{\text{BRRE}}^t - b_{\text{OPBLE}}b_{\text{OPBLE}}^t$$

is p.d. if $\lambda_j^2\left(\lambda_j^2+1+2\lambda_j\right) > \left[\lambda_j^2+d^{*2}-2\lambda_j d^*)(\lambda_j^2+k^2+2\lambda_j k)\right] > 0$, where j belongs to $1,...p+1$). At this stage, if $k>0$ and $0<d^*<1$, the completion of the theorem is achieved with the assistance of Lemma 1 and Lemma 2. □

Theorem 3. In the BRM, if $0<d<1,$ and $0<d^*<1$, this condition implies that the estimator $\widehat{\beta}_{\text{OPBLE}}$ is superior to $\widehat{\beta}_{\text{BLE}}$, that is $\Delta_3=\text{MMSE}\left(\widehat{\beta}_{\text{BLE}}\right)-\text{MMSE}\left(\widehat{\beta}_{\text{OPBLE}}\right)>0$ if and only if,

$$b_{\text{OPBLE}}^t\left[\widehat{\vartheta}\begin{pmatrix} \widehat{\delta}\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}\left(X^t\widehat{V}X+kI_{p+1}\right)^{-1}\left(X^t\widehat{V}X+dI_{p+1}\right)\left(X^t\widehat{V}X\right)^{-1} \\ \left(X^t\widehat{V}X+dI_{p+1}\right)\left(X^t\widehat{V}X+kI_{p+1}\right)^{-1}-\widehat{\delta}\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}\left(X^t\widehat{V}X-d^*I_{p+1}\right) \\ \left(X^t\widehat{V}X\right)^{-1}\left(X^t\widehat{V}X-d^*I_{p+1}\right)\left(X^t\widehat{V}X+I_{p+1}\right)^{-1} \end{pmatrix}c_1c_1^t\right]^{-1}$$
$$\times b_{\text{OPBLE}}<1$$

where $b_{\text{OPBLE}}=-\left(d^*-1\right)\left(X^tVX+I_{p+1}\right)^{-1}\beta$.

Proof. The distinction between MMSEs of the BLE and OPBLE is defined as

$$\Delta_3=\begin{bmatrix} \widehat{\delta}\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}\left(X^t\widehat{V}X+dI_{p+1}\right)\left(X^t\widehat{V}X\right)^{-1}\left(X^t\widehat{V}X+dI_{p+1}\right) \\ \left(X^t\widehat{V}X+I_{p+1}\right)^{-1}-\widehat{\delta}\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}\left(X^t\widehat{V}X-d^*I_{p+1}\right) \\ \left(X^t\widehat{V}X\right)^{-1}\left(X^t\widehat{V}X-d^*I_{p+1}\right)\left(X^t\widehat{V}X+I_{p+1}\right)^{-1} \end{bmatrix} \tag{32}$$
$$+\ b_{\text{BLE}}b_{\text{BLE}}^t-b_{\text{OPBLE}}b_{\text{OPBLE}}^t$$

$$\Delta_3=\widehat{\delta}\left(X^t\widehat{V}X\right)^{-1}$$
$$\times\begin{bmatrix} \left(X^t\widehat{V}X+I_{p+1}\right)^{-1}\left(X^t\widehat{V}X+dI_{p+1}\right)\left(X^t\widehat{V}X+dI_{p+1}\right)\left(X^t\widehat{V}X+I_{p+1}\right)^{-1} \\ -\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}\left(X^t\widehat{V}X-d^*I_{p+1}\right)\left(X^t\widehat{V}X-d^*I_{p+1}\right)\left(X^t\widehat{V}X+I_{p+1}\right)^{-1} \end{bmatrix} \tag{33}$$
$$+\,b_{\text{BLE}}b_{\text{BLE}}^t-b_{\text{OPBLE}}b_{\text{OPBLE}}^t$$

where $b_{\text{BLE}}=(d-1)\left(X^t\widehat{V}X+I_{p+1}\right)^{-1}\beta$. While the scalar MSEs difference of equation (33) can be defined as

$$\text{MSE}\left(\widehat{\beta}_{\text{BLE}}\right)-\text{MSE}\left(\widehat{\beta}_{\text{OPBLE}}\right)=\widehat{\delta}\zeta\text{diag}\left\{\frac{(\lambda_j+d)^2}{\lambda_j(\lambda_j+1)^2}-\frac{(\lambda_j-d^*)^2}{\lambda_j\left(\lambda_j+1\right)^2}\right\}_{j=1}^{p+1}\zeta^t \tag{34}$$
$$+\,b_{\text{BLE}}^t b_{\text{BLE}}-b_{\text{OPBLE}}^t b_{\text{OPBLE}}$$

$$\begin{aligned}&\mathrm{MSE}\left(\widehat{\beta}_{\mathrm{BLE}}\right)-\mathrm{MSE}\left(\widehat{\beta}_{\mathrm{OPBLE}}\right)\\&\quad=\widehat{\delta}\zeta\mathrm{diag}\left\{\frac{\left(d^2-d^{*2}\right)+2\lambda_j\left(d+d^*\right)}{\lambda_j\left(\lambda_j+1\right)^2}\right\}_{j=1}^{p+1}\zeta^t+b_{\mathrm{BLE}}^t b_{\mathrm{BLE}}-\ b_{\mathrm{OPBLE}}^t b_{\mathrm{OPBLE}}\end{aligned}\tag{35}$$

By observing that $b_{\mathrm{BLE}}b_{\mathrm{BLE}}^t$ is a non-negative definite term in equation (35), it becomes evident that

$$\begin{bmatrix}\widehat{\delta}\left(X^tVX+I_{p+1}\right)^{-1}\left(X^tVX+dI_{p+1}\right)\left(X^tVX+dI_{p+1}\right)\left(X^tVX+I_{p+1}\right)^{-1}-\widehat{\delta}\left(X^tVX+I_{p+1}\right)^{-1}\\\left(X^tVX-d^*I_{p+1}\right)\left(X^tVX-d^*I_{p+1}\right)\left(X^tVX+I_{p+1}\right)^{-1}\end{bmatrix}$$

$$-b_{\mathrm{OPBLE}}b_{\mathrm{OPBLE}}^t$$

is non-negative. Eq (34) is p.d. if $\left(\lambda_j+d\right)^2-\left(\lambda_j-d^*\right)^2>0$ which is equivalent to

$$\left(d^2-d^{*2}\right)+2\lambda_j\left(d+d^*\right)>0$$

Thus, if $0<d<1$ and $0<d^*<1$, therefore the theorem is considered to be complete. □

2.6. Selection of the biasing parameter for the OPBLE

The selection of the biasing parameter for the OPBLE is an important issue. Hence, selecting an appropriate value for the biasing parameter is imperative to achieve the desired objective. There is no hard and fast rule for finding the ideal value for the biasing parameter. We select the biasing parameter by following the suggestions of Hoerl and Kennard [13], Kibria [17] and some other authors. For comparison, we choose the following biasing parameter for the BRRE

$$k=\frac{p+1}{\sum\limits_{j=1}^{p+1}\alpha_j^2}\tag{36}$$

We also consider another ridge parameter as given Algamal and Abonazel [4] and we call it the optimum biased parameter as given by

$$k_{\mathrm{opt}}=\frac{\widehat{\delta}}{\sum\limits_{j=1}^{p+1}\alpha_j^2}\tag{37}$$

Similarly, we consider the following liu parameter for the BLE

$$d=\max\left(0,\min d_j\right)\tag{38}$$

where $d_j=\dfrac{\alpha_j^2-\widehat{\delta}}{\left(\dfrac{\widehat{\delta}}{\lambda_j}\right)+\alpha_j^2}$. It is possible to determine the optimal value of the biasing parameter for the OPBLE by taking the partial derivative of equation (24) w.r.t. d^*, equating to zero and obtaining the jth term as

$$d_j^*=\frac{\lambda_j(\hat{\delta}+\alpha_j^2)}{1+\lambda_j\alpha_j^2}\tag{39}$$

For this study, we take a minimum value of equation (39) as

$$d^{*} = \min\left(d_j^{*}\right) \tag{40}$$

3. Monte Carlo simulation

A Monte Carlo simulation study will be executed to check the effectiveness of the proposed modified Liu estimator. The following formula is used to generate the correlated regressors

$$x_{ij} = \left(1-\rho^{2}\right)^{\left(\frac{1}{2}\right)} z_{ij} + \rho z_{i(j+1)}, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, p,$$

where ρ^2 represent the level of correlation between predictors and random numbers z_{ij} are generated from the standard normal distribution. Our study is concerned that the dependent variable y follows the beta distribution which is generated as $y_i \sim B(\mu(x_i), \delta), i = 1, 2, \ldots, n$, where $\mu(x_i)$ indicates the mean function of the response variable. The regression parameter values of β are chosen under the common restriction in the simulation study which is $\beta^t\beta = 1$ [17]. We considered various values for different factors including sample size, dispersion, level of multicollinearity, and the number of regressors to assess the performance of the BRM's estimators. The assumed values of these factors are reported in Table 1. The generated data is replicated 2000 times. The robust optimization problem is defined as follows:

Table 1. Summary of Monte Carlo simulation design

Factor	Notation	Assumed values
Number of regressors	p	$4, 8, 12$
Number of replicates	m	2000
Dispersion parameter	δ	$0.5, 2, 4, 6, 8$
Sample size	n	$25, 50, 100, 200, 300, 500$
Degree of correlation	ρ	$0.8, 0.9, 0.95, 0.99$

MSE is employed as the criterion for assessing the performance of the proposed estimator and other existing estimators of the BRM under different conditions, which is computed as

$$MSE = \frac{\sum_{i=1}^{m}\left(\widehat{\beta}_i - \beta\right)^{t}\left(\widehat{\beta}_i - \beta\right)}{m}$$

where $\widehat{\beta}$ is the estimator of β and m is the number of replications used in the Monte Carlo simulation study. All the calculations are performed using R software. We also used the R package betareg() for the support of numerical evaluation.

3.1. Results and discussion

Tables 2–16 provide the estimated MSEs for various conditions of the MLE, BRRE, BLE, and proposed OPBLE. We noticed that the multicollinearity affects the simulated MSEs of the biased estimators. An increase in correlation also increases the MSEs of the MLE, BRRE, BLE, and OPBLE. This is not true for smaller sample sizes, i.e., $n = 25$ and dispersion 0.5. When we increase the collinearity from 0.8 to 0.99, the results of the simulation show a direct relationship between MSEs and multicollinearity. We also observed that the proposed OPBLE performed more efficiently than other estimators because it has less MSE.

To check the influence of sample size on the MSEs of the MLE, BRRE, BLE, and OPBLE, we run a simulation using sample sizes $n = 25,\ 50,\ 100,\ 200,\ 300\ and\ 500$ while other factors remain fixed. The simulated results while other factors remain fixed at $\delta = 0.5,\ p = 4,\ 8,\ 12$ are presented in Tables 2, 7, and 12. The sample size has an inverse impact on the MSEs of the MLE as well as other estimators. The MSEs of all estimators decreased mostly when we increased the sample size from 25 to 200. For $\delta > 1,\ p = 4,\ 8,\ 12$, the MSEs of all estimators are given in Tables 3–6, 8–11, and 13–16. It can be noticed that as the sample size increases, the MSEs of the MLE, BRRE and BLE decrease. While the MSE of our proposed estimator for these simulation conditions a little bit increases with the sample sizes but is not larger than the MSEs of all other estimators. Moreover, the behavior of our proposed estimator is more consistent than other considered estimators. In general, for most of the cases, the proposed OPBLE shows a better performance than the MLE, BRRE, and BLE.

Table 2. Estimated MSEs for $p = 4$ and $\delta = 0.5$.

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	1.6267	1.4242	1.5424	1.538	1.2809
	0.9	2.3371	1.9417	2.167	2.1735	1.5665
	0.95	4.116	3.274	3.7377	3.8725	2.187
	0.99	16.4548	12.2894	14.5023	6.2118	3.7447
50	0.8	0.8545	0.8177	0.8414	0.8153	0.8055
	0.9	1.0963	0.9974	1.0642	1.0614	0.9632
	0.95	1.695	1.4735	1.6229	1.6414	1.331
	0.99	6.2032	4.9659	5.7697	5.4936	2.5118
100	0.8	0.5466	0.5416	0.5448	0.5421	0.54
	0.9	0.682	0.6566	0.6748	0.6768	0.6545
	0.95	0.9191	0.857	0.9022	0.9115	0.845
	0.99	2.8285	2.3508	2.6883	2.8027	1.8328
200	0.8	0.5133	0.5119	0.5128	0.5132	0.5115
	0.9	0.6116	0.6027	0.6093	0.6081	0.603
	0.95	0.7505	0.7302	0.7454	0.7449	0.7297
	0.99	1.8719	1.7071	1.8285	1.8553	1.5889
300	0.8	0.4482	0.4475	0.448	0.4481	0.4473
	0.9	0.529	0.5241	0.5278	0.5271	0.5244
	0.95	0.6491	0.6379	0.6464	0.6461	0.6383
	0.99	1.5292	1.4321	1.5046	1.5218	1.3843
500	0.8	0.4148	0.4145	0.4147	0.4147	0.4145
	0.9	0.475	0.4727	0.4744	0.4741	0.473
	0.95	0.5769	0.5716	0.5756	0.5754	0.572
	0.99	1.1595	1.1124	1.1479	1.1563	1.1002

Results also demonstrate that increasing the number of explanatory variables has a direct impact on the simulated MSEs of all estimators. Tables 2–16 show that if we increase the number of regressors while holding all the other factors fixed including dispersion parameter, multicollinearity, and sample size, then the MSEs increase for the MLE, BRRE, BLE, and OPBLE. From these tables, it also observed that the proposed estimator for the BRM outperforms other estimators for all conditions.

When the level of dispersion is increased, most often increases the MSEs of all estimators in the BRM. This suggests a direct correlation between the MSEs of estimators and the dispersion parameter. Furthermore, in most of the situations, the proposed OPBLE exhibits superior performance compared to the other estimators being compared.

Table 3. Estimated MSEs for $p = 4$ and $\delta = 2$

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	5.6727	4.3963	4.4811	3.6722	2.5208
	0.9	6.6938	5.0402	5.1576	4.5585	3.0298
	0.95	7.8123	5.4778	5.675	5.7851	3.4965
	0.99	17.4372	8.4249	9.2387	8.4085	4.1447
50	0.8	4.6691	4.1156	4.2446	3.6365	2.9453
	0.9	5.1195	4.4935	4.6294	4.0129	3.2971
	0.95	5.5564	4.7533	4.9378	4.4522	3.5604
	0.99	9.0352	5.985	6.4775	7.7683	4.0575
100	0.8	3.9845	3.7545	3.8283	3.5449	3.2927
	0.9	4.3575	4.1084	4.1865	3.8922	3.593
	0.95	4.7215	4.4132	4.5063	4.2407	3.8113
	0.99	6.2113	5.0902	5.3458	5.6983	4.0462
200	0.8	4.1118	3.9951	4.0376	3.8853	3.7353
	0.9	4.4874	4.364	4.4092	4.2493	4.0727
	0.95	4.7409	4.5991	4.6509	4.4978	4.2621
	0.99	5.5914	5.1224	5.2716	5.3315	4.3712
300	0.8	4.0358	3.9581	3.9871	3.8846	3.7807
	0.9	4.333	4.2526	4.2832	4.1782	4.0692
	0.95	4.5576	4.4678	4.5017	4.3982	4.2572
	0.99	5.2307	4.9395	5.0404	5.0629	4.4437
500	0.8	3.9603	3.9155	3.9328	3.8708	3.8151
	0.9	4.2459	4.2001	4.2181	4.1549	4.0971
	0.95	4.4295	4.3796	4.3991	4.3367	4.2698
	0.99	4.9205	4.7718	4.8267	4.8245	4.4982

4. Applications

In this section, we assess the performance of the proposed estimator and compare its performance with the available estimators using two real-life applications. The BRM's coefficients using MLE, BRRE, BLE, and OPBLE are obtained using equations (6), (10), (15), and (20), respectively. The scalar MSEs of MLE, BRRE, BLE, and OPBLE are obtained using equation (9), (14), (19), and (24). The value of the ridge parameter of the BRRE is estimated using equation (36), the value of the BLE's Liu parameter is estimated using equation (38), and the value of the modified Liu parameter for the OPBLE is computed using equation (40).

Table 4. Estimated MSEs for $p = 4$ and $\delta = 4$

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	10.9029	8.313	7.0937	5.6152	3.8426
	0.9	12.085	9.1487	7.994	6.7026	4.9833
	0.95	13.061	9.4236	8.426	8.0954	5.72
	0.99	23.5898	11.816	10.5557	9.3159	6.5346
50	0.8	9.0335	7.9328	7.6178	6.1175	4.3582
	0.9	9.7213	8.5351	8.1916	6.6575	5.3508
	0.95	10.4674	9.0672	8.7335	7.3121	6.3666
	0.99	13.3232	9.6661	9.3111	10.0046	7.4787
100	0.8	8.2511	7.7764	7.6921	6.8609	5.865
	0.9	8.6697	8.1852	8.1049	7.2658	6.3384
	0.95	9.0805	8.5416	8.4521	7.6458	6.802
	0.99	10.6487	9.1846	9.0578	9.1847	7.7419
200	0.8	8.4303	8.193	8.1683	7.7018	7.0609
	0.9	8.7619	8.5235	8.5031	8.0448	7.403
	0.95	9.2039	8.9425	8.9198	8.465	7.7617
	0.99	10.1681	9.4972	9.4632	9.3991	8.2662
300	0.8	8.2777	8.1203	8.1072	7.7892	7.3442
	0.9	8.6945	8.5357	8.5252	8.2069	7.7824
	0.95	8.9958	8.8258	8.8144	8.5023	8.0462
	0.99	9.5893	9.2073	9.1902	9.0828	8.3491
500	0.8	8.1828	8.0917	8.0856	7.8918	7.6414
	0.9	8.564	8.4731	8.4687	8.2769	8.0344
	0.95	8.8271	8.7321	8.7281	8.539	8.2919
	0.99	9.265	9.0656	9.0602	8.9732	8.4717

Table 5. Estimated MSEs for $p = 4$ and $\delta = 6$

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	14.7185	10.973	8.0793	6.665	4.3535
	0.9	15.654	11.656	8.9923	7.7714	5.7363
	0.95	16.694	12.0191	9.5434	9.4094	6.7056
	0.99	25.6592	13.6293	11.1415	11.6787	7.7544
50	0.8	12.4739	10.8608	9.7074	7.5978	5.0652
	0.9	13.2995	11.5857	10.3435	8.1811	6.6357
	0.95	13.8202	11.923	10.7149	8.6799	7.9298
	0.99	17.013	12.7442	11.4866	11.7198	9.5562
100	0.8	11.5609	10.8485	10.3895	9.0351	7.068
	0.9	12.1141	11.3889	10.9241	9.5528	7.8997
	0.95	12.4309	11.6569	11.2012	9.9079	8.7034
	0.99	14.0997	12.371	11.7944	11.5182	10.3657
200	0.8	11.7032	11.353	11.1646	10.3699	9.0352
	0.9	12.2563	11.9015	11.7138	10.9188	9.5995
	0.95	12.6721	12.2939	12.0982	11.3153	10.1259
	0.99	13.5851	12.7919	12.5021	12.1719	11.1892
300	0.8	11.6845	11.4472	11.3191	10.7512	9.7662
	0.9	12.1288	11.8934	11.7741	11.215	10.3061
	0.95	12.3722	12.1263	12.0052	11.4634	10.6105
	0.99	12.9842	12.5077	12.3339	12.0609	11.3076
500	0.8	11.4841	11.3473	11.2762	10.9308	10.3646
	0.9	12.0031	11.8668	11.7994	11.4565	10.9138
	0.95	12.1705	12.0323	11.9687	11.6381	11.1308
	0.99	12.7686	12.5086	12.4064	12.2198	11.5714

Table 6. Estimated MSEs for $p = 4$ and $\delta = 8$

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	17.7771	12.9607	8.2386	7.2881	4.5796
	0.9	19.1419	13.9393	9.1831	8.4445	6.1148
	0.95	20.4014	14.3687	9.808	10.6906	7.0699
	0.99	29.99	15.9577	11.6509	11.5559	8.1024
50	0.8	15.4373	13.2619	10.8196	8.4597	5.3986
	0.9	16.4621	14.1952	11.6896	9.146	7.4735
	0.95	16.9495	14.4739	12.0139	9.6762	9.0233
	0.99	19.5518	15.021	12.84	12.8288	10.8735
100	0.8	14.3452	13.3974	12.3693	10.5723	7.6284
	0.9	15.0445	14.0819	13.039	11.1983	8.9224
	0.95	15.3479	14.3323	13.2916	11.5509	10.1703
	0.99	16.8528	14.8748	13.744	13.063	12.2195
200	0.8	14.5481	14.0728	13.5897	12.4593	10.1642
	0.9	15.0708	14.6015	14.1522	13.0386	10.983
	0.95	15.4441	14.9512	14.4906	13.4098	11.7923
	0.99	16.503	15.5588	14.9629	14.3714	13.4981
300	0.8	14.3788	14.0642	13.7546	12.9484	11.288
	0.9	14.9167	14.6051	14.3128	13.5198	12.0323
	0.95	15.2359	14.9131	14.6183	13.8444	12.5996
	0.99	15.9811	15.379	14.9741	14.5465	13.7678
500	0.8	14.2057	14.024	13.8498	13.3503	12.3687
	0.9	14.7367	14.5569	14.3917	13.9005	12.985
	0.95	15.0466	14.8624	14.6962	14.2143	13.3771
	0.99	15.4625	15.1498	14.9228	14.6338	13.9063

Table 7. Estimated MSEs for $p = 8$ and $\delta = 0.5$

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	7.9249	6.5184	7.4923	7.0793	5.8303
	0.9	10.4156	8.2095	9.7746	9.5866	6.6309
	0.95	10.9676	7.1222	9.5734	10.3493	3.6347
	0.99	54.577	35.8714	48.1431	26.3426	9.3
50	0.8	1.6661	1.5407	1.6403	1.5377	1.5355
	0.9	2.5048	2.1784	2.4442	2.1469	2.1135
	0.95	3.9392	3.2184	3.8069	3.162	2.8597
	0.99	17.411	13.3494	16.631	12.3559	5.35
100	0.8	1.3449	1.3163	1.3403	1.3412	1.3214
	0.9	1.9781	1.8993	1.9665	1.9626	1.9059
	0.95	3.125	2.9547	3.1009	3.1046	2.9232
	0.99	11.4429	10.4586	11.3028	11.4011	8.6275
200	0.8	1.0582	1.0542	1.0576	1.053	1.055
	0.9	1.3347	1.3123	1.3318	1.324	1.3182
	0.95	1.913	1.8613	1.9064	1.9086	1.8671
	0.99	8.3474	7.9999	8.3015	8.323	7.7153
300	0.8	0.8432	0.8431	0.8432	0.8431	0.8428
	0.9	1.0791	1.07	1.078	1.079	1.0731
	0.95	1.4624	1.4381	1.4594	1.4607	1.4442
	0.99	4.2847	4.1217	4.2646	4.2791	4.0391
500	0.8	0.7035	0.7029	0.7034	0.7032	0.7031
	0.9	0.8546	0.851	0.8542	0.8536	0.8525
	0.95	1.0502	1.0422	1.0493	1.0486	1.0452
	0.99	2.241	2.1857	2.2344	2.2374	2.1872

Table 8. Estimated MSEs for p = 8 and δ = 2

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	19.5656	15.5296	16.6683	18.1347	13.6347
	0.9	21.2689	16.29	17.7462	19.9524	14.2943
	0.95	13.7061	6.4912	7.8259	10.6906	4.1427
	0.99	45.1537	13.7749	18.1899	17.1508	4.398
50	0.8	4.8108	3.7557	4.2869	3.7096	2.7147
	0.9	5.7907	4.4336	5.1053	4.6285	3.2698
	0.95	7.1938	5.1026	6.061	6.0716	3.6825
	0.99	17.194	8.0899	11.3708	16.3888	4.1737
100	0.8	4.5322	4.126	4.3922	4.0597	3.6973
	0.9	5.207	4.723	5.0426	4.7415	4.2238
	0.95	6.2664	5.5286	6.0062	5.7938	4.7636
	0.99	13.3323	9.4986	11.7306	12.7758	5.7439
200	0.8	4.2066	4.0111	4.1453	3.9768	3.8023
	0.9	4.6818	4.4672	4.6172	4.4601	4.2405
	0.95	5.2491	4.9576	5.1606	5.0374	4.6548
	0.99	18.1175	16.5224	17.5221	18.0993	14.784
300	0.8	4.0156	3.8918	3.9785	3.8628	3.761
	0.9	4.4348	4.3018	4.396	4.2853	4.155
	0.95	4.8997	4.7256	4.8484	4.7525	4.5396
	0.99	7.2296	6.3341	6.9289	7.104	5.3607
500	0.8	4.0113	3.9384	3.9899	3.9236	3.8646
	0.9	4.3887	4.3148	4.3675	4.3027	4.2356
	0.95	4.6743	4.5901	4.6504	4.5889	4.5031
	0.99	5.8613	5.5292	5.7576	5.7697	5.1964

Table 9. Estimated MSEs for p = 8 and δ = 4

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	28.5676	22.2188	22.2754	23.5617	19.0484
	0.9	30.7882	23.3811	23.452	25.8207	20.3857
	0.95	19.2761	8.9954	8.2488	13.8258	5.5846
	0.99	45.9419	13.0482	12.6691	14.1962	5.9767
50	0.8	9.2092	7.0696	7.4385	5.9706	4.1369
	0.9	9.8922	7.5621	8.0139	6.802	5.2542
	0.95	11.6515	8.4586	8.9694	8.4646	6.3376
	0.99	20.9636	10.1788	11.355	19.0143	7.0692
100	0.8	8.5928	7.7657	8.1172	7.1312	5.9442
	0.9	9.4029	8.5056	8.9057	8.0139	6.7724
	0.95	10.2674	9.0554	9.5907	9.0136	7.2547
	0.99	17.4921	11.8723	13.7623	16.5022	8.3247
200	0.8	7.9764	7.5872	7.7769	7.2812	6.6232
	0.9	8.5386	8.1358	8.3406	7.8853	7.219
	0.95	9.0188	8.5244	8.7837	8.4239	7.581
	0.99	27.3647	25.0115	25.9423	27.1948	22.7907
300	0.8	7.8557	7.6024	7.728	7.38	6.9246
	0.9	8.3013	8.0492	8.1819	7.8657	7.4283
	0.95	8.7786	8.4729	8.6345	8.3622	7.7982
	0.99	11.2586	9.8497	10.5013	10.9177	8.1452
500	0.8	7.949	7.7974	7.8744	7.6683	7.3679
	0.9	8.3143	8.1683	8.2464	8.0579	7.781
	0.95	8.6058	8.4489	8.5348	8.359	8.0663
	0.99	9.7521	9.2544	9.5107	9.5048	8.491

Table 10. Estimated MSEs for p = 8 and δ = 6

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	34.8583	26.2625	24.3811	26.1273	21.7235
	0.9	36.6105	27.3061	25.8299	28.3968	23.0281
	0.95	22.7772	10.5494	8.2257	18.3726	6.125
	0.99	49.9893	14.6085	12.2166	19.3216	6.6605
50	0.8	12.7231	9.5149	9.1497	7.4051	5.0729
	0.9	13.3887	10.0343	9.7824	8.2928	6.6152
	0.95	14.784	10.6566	10.4819	9.8417	7.7449
	0.99	24.3262	12.4254	12.3179	21.2951	8.916
100	0.8	11.752	10.5095	10.7536	9.135	6.9523
	0.9	12.5236	11.2391	11.5498	10.1073	8.2495
	0.95	13.4472	11.8632	12.2747	11.2152	9.2857
	0.99	21.0634	14.3352	15.5175	19.4434	10.7216
200	0.8	11.0058	10.4242	10.5888	9.7219	8.3485
	0.9	11.5973	11.0173	11.2084	10.4182	9.2044
	0.95	12.2633	11.5706	11.8068	11.143	9.8361
	0.99	34.188	31.3128	31.9158	33.744	28.8356
300	0.8	10.9412	10.5611	10.6738	10.0534	9.0784
	0.9	11.4692	11.0969	11.2224	10.6586	9.7653
	0.95	11.9134	11.4885	11.6401	11.1552	10.2149
	0.99	14.4042	12.7077	13.2117	13.7821	10.7516
500	0.8	10.9814	10.7565	10.8281	10.46	9.8124
	0.9	11.4094	11.1958	11.2734	10.9402	10.3665
	0.95	11.7393	11.5173	11.603	11.2918	10.749
	0.99	12.9337	12.2875	12.5174	12.5009	11.1612

Table 11. Estimated MSEs for p = 8 and δ = 8

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	39.4485	29.1499	25.838	27.754	23.3622
	0.9	42.2345	30.5004	26.8285	30.3397	24.7902
	0.95	26.8379	12.0367	7.9948	19.1334	6.2365
	0.99	53.9231	16.0798	11.7998	20.6348	6.7055
50	0.8	15.546	11.3393	9.915	8.4121	5.6398
	0.9	16.5495	12.1113	10.7511	9.3636	7.5158
	0.95	17.723	12.6035	11.2989	10.8514	8.7455
	0.99	26.842	14.2449	13.11	22.5399	9.9944
100	0.8	14.568	12.9007	12.8679	10.6527	7.6357
	0.9	15.2537	13.5696	13.6212	11.6129	9.4201
	0.95	16.2816	14.2718	14.423	12.9028	10.9117
	0.99	23.7233	16.3099	16.7184	21.2734	12.6689
200	0.8	13.6063	12.8216	12.878	11.6189	9.3709
	0.9	14.2825	13.5165	13.6226	12.4595	10.6169
	0.95	14.9084	14.0326	14.1838	13.203	11.5659
	0.99	39.4125	36.0321	36.2848	38.6312	33.2095
300	0.8	13.6338	13.1222	13.1689	12.2498	10.5996
	0.9	14.1248	13.6313	13.7061	12.8809	11.4628
	0.95	14.5505	14.0052	14.1029	13.3859	12.0691
	0.99	17.3247	15.35	15.6409	16.3236	13.1747
500	0.8	13.5198	13.2231	13.2626	12.7229	11.6176
	0.9	13.9883	13.7071	13.76	13.2702	12.3242
	0.95	14.3118	14.022	14.0855	13.6325	12.7819
	0.99	15.5245	14.7646	14.9286	14.8859	13.371

Table 12. Estimated MSEs for p = 12 and δ = 0.5

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	10.8928	6.7019	9.1062	9.7983	4.3018
	0.9	19.3459	11.7872	16.2625	17.2706	6.1183
	0.95	38.1331	23.8021	32.3379	33.798	9.5264
	0.99	183.5229	106.8655	148.4445	121.4217	19.6797
50	0.8	3.038	2.4237	2.9397	2.9971	2.3434
	0.9	5.3933	4.1261	5.1923	5.3301	3.589
	0.95	9.6336	6.9705	9.2077	9.5668	4.9616
	0.99	47.0498	32.4977	44.6805	46.1444	11.2954
100	0.8	1.2083	1.1144	1.1978	1.1996	1.1427
	0.9	2.0118	1.7925	1.9872	1.988	1.8274
	0.95	3.4958	2.9985	3.4395	3.4543	2.9532
	0.99	14.0756	11.2304	13.7241	13.9409	7.7806
200	0.8	1.2652	1.2439	1.2632	1.2598	1.2526
	0.9	1.742	1.6961	1.7379	1.7314	1.7118
	0.95	2.3535	2.2596	2.3451	2.339	2.2827
	0.99	7.1028	6.5128	7.0474	7.065	6.1951
300	0.8	5.5136	5.4646	5.5096	5.5175	5.4878
	0.9	6.4657	6.4131	6.4614	6.4696	6.4346
	0.95	7.401	7.3403	7.3961	7.4049	7.3599
	0.99	6.3588	6.088	6.3364	6.3428	5.9846
500	0.8	1.1709	1.1654	1.1704	1.1685	1.1678
	0.9	1.4301	1.4216	1.4294	1.4263	1.4253
	0.95	1.73	1.718	1.729	1.7258	1.7228
	0.99	3.5037	3.4429	3.4989	3.4983	3.4481

Table 13. Estimated MSEs for p = 12 and δ = 2

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	15.7344	5.689	6.5827	15.0675	3.848
	0.9	24.0933	8.6085	10.2454	22.6646	4.8819
	0.95	36.5893	12.3591	15.7387	35.9252	5.5171
	0.99	154.0891	39.6507	55.7465	40.5199	5.492
50	0.8	6.9195	5.1304	6.2452	5.6457	4.2239
	0.9	9.1811	6.4019	8.0279	8.1925	5.2089
	0.95	12.7737	7.8807	10.5738	12.2569	6.0566
	0.99	42.8938	16.9143	29.3522	39.1001	7.4509
100	0.8	5.0074	4.3697	4.844	4.3299	3.8921
	0.9	6.021	5.2024	5.8084	5.3594	4.613
	0.95	7.4166	6.0943	7.0348	6.726	5.2016
	0.99	16.767	9.9695	14.2217	15.9574	6.4379
200	0.8	5.0084	4.7328	4.9508	4.7153	4.5195
	0.9	5.5868	5.2869	5.5258	5.3046	5.0593
	0.95	6.2566	5.8497	6.1715	5.9792	5.5488
	0.99	10.2234	8.1806	9.6772	9.9515	6.6249
300	0.8	17.1732	16.9419	17.1284	17.168	16.7532
	0.9	18.3936	18.1471	18.3476	18.3854	17.958
	0.95	19.3102	19.0068	19.2531	19.3007	18.7779
	0.99	9.8072	8.4513	9.5053	9.6419	7.1987
500	0.8	4.9477	4.8438	4.9283	4.8344	4.7603
	0.9	5.4241	5.32	5.4054	5.3143	5.2369
	0.95	5.761	5.6459	5.7407	5.6549	5.5574
	0.99	7.4043	6.958	7.317	7.306	6.6045

Table 14. Estimated MSEs for p = 12 and δ = 4

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	21.8358	7.1535	6.4472	20.5903	4.6675
	0.9	30.7594	10.0924	8.8312	29.832	5.5149
	0.95	44.2534	13.1111	11.7057	33.1373	6.0881
	0.99	159.2169	28.5049	28.0352	44.7997	6.2795
50	0.8	12.0233	8.81	10.0192	8.6437	6.8612
	0.9	14.1017	10.0314	11.5637	11.6706	8.193
	0.95	18.3279	11.6437	13.8208	17.1138	9.2776
	0.99	44.3526	16.0826	22.1142	32.5165	10.1276
100	0.8	9.78	8.4833	9.2114	7.7882	6.4229
	0.9	10.9201	9.4477	10.2885	9.0204	7.5765
	0.95	12.3044	10.2405	11.3448	10.3849	8.4906
	0.99	21.5375	12.6608	16.1504	19.3437	9.5907
200	0.8	9.0392	8.5128	8.8644	8.2446	7.5227
	0.9	9.4471	8.9257	9.2861	8.7416	8.0514
	0.95	10.0521	9.4162	9.8537	9.3772	8.5279
	0.99	13.8808	11.2056	12.8019	13.2881	9.3392
300	0.8	27.0489	26.6041	26.908	26.9038	25.6606
	0.9	28.4281	27.9831	28.296	28.2946	27.1283
	0.95	29.3497	28.8316	29.1968	29.2236	27.9624
	0.99	13.9408	11.8298	13.2074	13.5598	9.6193
500	0.8	9.0259	8.8205	8.9643	8.7074	8.3618
	0.9	9.4699	9.2753	9.4161	9.1834	8.8744
	0.95	9.8656	9.6515	9.8077	9.5909	9.2569
	0.99	11.4922	10.7162	11.2547	11.2459	9.787

Table 15. Estimated MSEs for p = 12 and δ = 6

n	ρ	MLE	BRRE		BLE	OPBLE
			$k1$	kopt	d	$d*$
25	0.8	25.7839	7.822	6.0674	20.3246	4.8335
	0.9	36.0008	10.8222	7.8172	24.7202	5.6426
	0.95	50.2725	13.9913	10.6588	28.1797	6.211
	0.99	172.3873	26.2148	20.1025	34.7578	6.4978
50	0.8	15.6422	11.1608	11.9955	10.4718	8.3063
	0.9	18.1839	12.8003	13.8443	14.2297	10.1095
	0.95	22.2654	14.274	15.6177	20.3646	11.2281
	0.99	50.2077	18.3165	21.7789	33.8359	12.0673
100	0.8	13.5587	11.5938	12.3323	10.0384	7.8388
	0.9	14.846	12.6923	13.5464	11.4528	9.6825
	0.95	16.4255	13.588	14.6507	13.0083	11.09
	0.99	25.27	15.3518	17.6509	21.6516	12.4316
200	0.8	12.3032	11.5305	11.9494	10.892	9.4207
	0.9	12.6669	11.9225	12.3555	11.4432	10.1941
	0.95	13.1703	12.3111	12.8174	12.0353	10.7658
	0.99	16.7537	13.6885	15.1862	15.7639	11.7053
300	0.8	34.1021	33.443	33.8109	33.7003	31.3009
	0.9	35.5849	34.9336	35.3136	35.2222	33.1019
	0.95	36.535	35.8177	36.2457	36.1987	34.1664
	0.99	17.1971	14.608	16.0009	16.5558	11.9272
500	0.8	12.1939	11.8924	12.0701	11.6261	10.9065
	0.9	12.6848	12.4015	12.5784	12.1817	11.5531
	0.95	13.1241	12.8178	13.0123	12.6437	12.0021
	0.99	14.8272	13.8444	14.4222	14.3931	12.5163

Table 16. Estimated MSEs for p = 12 and δ = 8

n	ρ	MLE	BRRE		BLE	OPBLE
25	0.8	30.0596	8.1133	5.8973	25.881	4.8666
	0.9	39.0035	11.0563	7.7142	27.0869	5.7753
	0.95	54.0946	14.2224	9.6198	31.2964	6.1617
	0.99	180.5057	25.7764	17.8996	36.0216	6.6411
50	0.8	19.0266	13.2297	13.3192	12.1292	9.3609
	0.9	21.4541	14.8014	15.0926	16.3695	11.2027
	0.95	25.3127	16.1986	16.7121	23.5701	12.3763
	0.99	53.1576	20.0377	21.4693	37.6457	13.3993
100	0.8	16.6409	14.0148	14.5397	11.5569	8.837
	0.9	17.9558	15.1642	15.8171	13.0935	11.2563
	0.95	19.7392	16.1544	16.9434	14.8264	12.9657
	0.99	29.0784	17.8916	19.364	24.3167	14.6174
200	0.8	14.9089	13.8846	14.3055	12.818	10.5135
	0.9	15.5227	14.535	14.99	13.664	11.8306
	0.95	15.8652	14.77	15.3031	14.2022	12.5489
	0.99	19.8695	16.16	17.5945	18.4119	13.7773
300	0.8	39.4939	38.611	38.9852	38.7583	34.9027
	0.9	41.1584	40.3056	40.7043	40.4996	37.2998
	0.95	42.2964	41.3686	41.8112	41.6737	38.7916
	0.99	20.292	17.2047	18.5279	19.3429	14.1742
500	0.8	14.8812	14.4796	14.6695	14.0183	12.7622
	0.9	15.4514	15.0765	15.2716	14.6874	13.649
	0.95	15.9304	15.5288	15.7445	15.2097	14.2226
	0.99	17.624	16.4481	17.0252	16.9747	14.8086

4.1. Gasoline yield data

The performance of the proposed OPBLE over the MLE, BRRE, and BLE is evaluated by considering the gasoline yield dataset which is taken from Prater [28]. This application has four explanatory variables that may affect crude oil (gasoline yield). These explanatory variables include crude oil gravity (x_1), vapor pressure on crude oil (x_2), the temperature at which 10 percent of crude oil has vaporized (x_3) , the temperature at which all gasoline is vaporized (x_4) . Lemonte et al. [19] found that the gasoline yield is well-fitted to the beta distribution. So, the BRM is a more suitable model for this data. Qasim et al. [31] and Pirmohammadi and Bidram [27] showed that the explanatory variables are multicollinear. Therefore, we are considering this data to see the performance of our proposed estimator. Table 17 presents the estimated coefficients and MSEs of the MLE, BRRE, BLE, and OPBLE. Table 17 indicates that the proposed OPBLE has a minimum MSE than other estimators. Therefore, we can say that the proposed OPBLE shows better performance than the MLE, BRRE, and BLE.

Table 17. Estimated coefficients and MSEs for the gasoline yield data

Terms	MLE	BRRE		BLE	OPBLE
		$k1$	kopt	d	$d*$
Intercept	–2.6949	0.0007	0.0616	0.0282	–0.0006
x1	0.0045	–0.0096	–0.0099	–0.0098	–0.0098
x2	0.0304	–0.0427	–0.0444	–0.0435	–0.0419
x3	–0.011	–0.0187	–0.0189	–0.0188	–0.0187
x4	0.0106	0.0106	0.0106	0.0106	0.0106
MSE	21110.6	28	7.96	21.06	18.07

4.2. Body fat data

We consider another application to illustrate the superiority of the proposed OPBLE which is the body fat dataset. In this application, there are 252 observations with one response and 14 explanatory variables. The response variable denoted as y, represents the percentage of body fat. The explanatory variables include density determined from underwater weighing (x_1), age (x_2), weight (x_3) , height (x_4), neck circumference (x_5), chest circumference (x_6), abdomen 2 circumference (x_7), hip circumference (x_8), thigh circumference (x_9), knee circumference (x_{10}), ankle circumference (x_{11}), biceps extended circumference (x_{12}), forearm circumference (x_{13}), and wrist circumference (x_{14}). These explanatory variables are used to evaluate the impact of these factors on the percentage of body fat. This data is already used by Dunder and Gengiz [10] and Amin et al. [6] in the BRM. To test the multicollinearity among explanatory variables, we use the condition index (CI) which is mathematically defined as CI = (min/max)$^{(1/2)}$ where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of X^tWX excluding intercept. We found that the CI of this dataset is 13,762.92. This indicates the existence of severe multicollinearity among the explanatory variables. So, we use this dataset to evaluate the performance of the proposed estimator. Table 18 presents the estimated coefficients and MSEs of the MLE, BRRE with two biasing parameters, BLE, and OPBLE. The proposed BRRE with optimum biasing parameter outperforms the other compared estimators in the sense of minimal MSE. The second best estimator for this is the OPBLE as compared to the BRRE with the first biasing parameter, MLE, and BLE. These results are also compatible with the simulation results because, for $n > 200$, there is a minor difference among MSEs of the considered estimators and the application has a sample of size 252.

Table 18. Estimated coefficients and MSEs for the body fat data

Terms	MLE	BRRE		BLE	OPBLE
		$k1$	kopt	d	$d*$
Intercept	30.6659	–0.0013	–0.4207	–0.0356	–0.1401
x1	–30.6599	–0.0022	–0.4332	–0.0365	–0.1835
x2	0.0018	0.0043	0.0041	0.0042	0.0041
x3	0.0005	0.0029	0.0006	0.0027	0.0021
x4	0.0048	–0.0123	–0.0103	–0.0125	–0.0117
x5	–0.004	–0.0387	–0.0343	–0.0385	–0.0374
x6	–0.001	–0.0171	–0.0152	–0.0172	–0.0163
x7	–0.0036	0.0682	0.0674	0.0684	0.0677
x8	–0.0056	–0.04	–0.0361	–0.0401	–0.0384
x9	0.0087	0.0103	0.0112	0.0102	0.0106
x10	0.0058	–0.0055	–0.002	–0.0051	–0.0045
x11	–0.0074	0.0029	0.0051	0.0032	0.0027
x12	0.0047	0.0202	0.0205	0.0204	0.02
x13	–0.0025	0.0277	0.0282	0.0277	0.0272
x14	0.0085	–0.1272	–0.1155	–0.1221	–0.1184
MSE	153898.3	12482.37	367.55	44.66	42.8

5. Conclusion

Different researchers developed some biased estimators for the BRM to deal with the issue of multicollinearity. So, we intended to propose an estimator for the BRM to deal with the multicollinearity issue in a more consistent way than the other available biased estimators. This estimator is called OPBLE. By theoretical comparison, we show that the proposed OPBLE outperforms the MLE and other well-known estimators such as BRRE and BLE. The performance of the proposed OPBLE is also evaluated with the help of a simulation study using MSE as a performance evaluation criterion. In the simulation experiment, multiple factors are considered to assess the performance of the proposed estimator. Through simulation study results, it is determined that the proposed estimator exhibited higher efficiency compared to the MLE and other existing biased estimators. Finally, we used two real-life applications to demonstrate the efficiency of the proposed OPBLE where the proposed estimator mostly dominates the MLE, BRRE and BLE in a sense of smaller MSE. We suggest the researchers use this estimator for the BRM in the presence of multicollinearity. Furthermore, one can also work on robust biased estimators to deal with the simultaneous issues of outliers and multicollinearity.

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