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A study on the performance of a queuing system with heterogeneous arrivals and various types of breakdowns under multiple working vacations

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Abstract

The paper analyses the performance of a single server queue with heterogeneous arrivals and various types of breakdowns under multiple working vacations. Customers enter the queue according to a Poisson process with a rate that varies according to the types of customers. In both the regular busy and working during the vacation states, the server offers services with an exponential distribution. During peak times, the system may breakdown due to server unavailability, the system may breakdown at any time. The model considers systems with two types of breakdowns. In this model, batches of customers are served under the General Bulk Service Rule. The steady-state equations, the performance of measures for the systems, and particular cases of the described model are derived. Finally, in the form of tables and graphs, numerical results have been obtained.

Keywords: *breakdowns, heterogeneous arrival, multiple working vacations (MWV), queue length, bulk service*

1. Introduction

A Danish engineer, Agner Krarup Erlang, first introduced mathematical models to predict the behavior of telephone networks [9]. Erlang's work laid the foundation for what would later become known as queuing theory. Today, queuing theory remains a fundamental field of study in operations research, applied mathematics, and industrial engineering. It continues to be a valuable tool for optimizing and analyzing the performance of systems like queues and waiting lines. Queuing theory also provides techniques for optimization, such as finding the optimal number of servers or the optimal scheduling policy to minimize waiting times or maximize system efficiency.

Mathematical models are used in queuing theory to represent different types of queuing systems. These models provide a way to analyze and predict the behavior of queues and evaluate performance

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measures. Based on different assumptions and characteristics of queuing systems, each model has its own set of formulas and mathematical equations to calculate performance measures such as average waiting time, average queue length, and system utilization.

In queuing theory, the batch arrival process refers to a type of arrival pattern where multiple entities arrive simultaneously or in groups rather than individually. Instead of arrivals occurring one at a time, they occur in batches. In a batch arrival process, the inter-arrival times between batches are random variables, while the number of entities in each batch may also vary. The arrival process can be characterized by parameters such as the mean inter-arrival time between batches and the distribution of batch sizes. Analyzing and optimizing systems with batch arrival processes often requires more advanced mathematical techniques and computational methods.

The heterogeneous arrival process refers to a type of arrival pattern where the entities arriving into a queue or system are not identical or have different characteristics. Instead of all arrivals being the same, they can have different arrival rates, service requirements, and priorities. by analyzing and optimizing systems with heterogeneous arrival processes, we can easily understand the behavior of the customers.

A breakdown refers to a situation where a service facility or a server becomes unavailable or inoperable, leading to a disruption in the service process. Breakdowns can occur due to various reasons, such as equipment failure, system malfunction, or schedule maintenance. When a breakdown occurs, it can lead to increased waiting times for entities in the queue and can impact the overall performance of the system. Understanding the behavior of a queuing system under breakdown conditions helps to improve the system performance.

In this paper, a single-server queue with heterogeneous arrivals and various types of breakdowns under multiple working vacations are discussed. For this model, steady state equations, measures of performance, and analyzed the particular cases.

2. Review of literature

Many researchers developed different types of queuing and vacation models, Dutta and Choudhury [7] derived the M/M/1 traffic intensity queuing model, they different feature of the system for performance measures, introduce the simple estimators and also discussed sample size determination. Greicius and Minkevicius [10] studied the queue length with a multi-server queue, derived the limit of the queue length theorem and propose the applications under heavy traffic conditions. Bouchentouf et al. [3, 4] analyzed the single-server working vacation queuing model, impatient customers, vacation policy, decomposition properties and probability generating-functions are used for steady-state solutions and cost analysis also discussed. Batch arrival queue with breakdown discussed by Rajan et al. [23] in that model, described the bulk arrival process in two different stages. The system leads to two types of repairs for a random breakdown. Yang et al. [32] analyzed the multiple vacation with breakdowns for M/M/1 queue with two different types of servers are reliable and unreliable. Using the geometric method to formulate quasi birth-and-death (QBD) process. Cost analyses are developed and for numerical example used the canonical particle swarm algorithm. Parimala [21] analyzed the vacation model with heterogeneous services. Here, two different servers with different service rates and bulk service processes followed. Fast server and slow server were also discussed. Seenivasan et al. [26] analyzed the working vacation model with heterogeneous with unreliable servers, always available and intermittently available are considered. The matrix-geometric technique was used to construct the model. Vadivukarasi and Kalidass [28, 29] discussed the behavior of the single server vacation queue and the bulk arrival queue. M/M/1 queue with closed form transient solution and time dependent performance measures are developed. Using geometric method and stochastic decomposition derived steady-state solution. Working vacation of a single server queuing model with server breakdown presented by Agrawal et al. [1]. In that paper, two different types of server with alternate service and a steady-state solution using a geometric approach. Kalita and Choudhury [12] analyzed heterogeneous service and batch arrivals with a single server model under the vacation policy. They discussed explicit expressions for steady-state solution, also cost analysis and various performances are derived. Medhi [18] analyzed the customer's impatience and reneging model. Markov processes are used, a stochastic model and probability-generating function are derived. Baulking, discourage arrival, impatience and reneging are developed. Chandiraleka and Seenivasan [25] presented the Multiple Working Vacation single server queue with breakdown. Matrix geometric method to solve the working vacation breakdown and repair the queuing model. Polin et al. [22] discussed the heterogeneous queuing system with Markov arrivals. Renewal Markov process, infinite number of servers model and asymptotic methods was analyzed. Kumar et al. [14] analyzed the M/M/1/N queue feedback with customers reneging. Customers impatience, strategies, retention mechanisms and feedback are discussed. Yohapriyadharsini and Suvitha [33] analyzed the two kinds of working vacations with heterogeneous arrivals. Probability generating function is used, different working vacations and impatient customers are discussed. Kumar and Gupta [15] analyzed the N-policy vacation based FTC model with switching failures. Reliability metrics of a multi-unit fault tolerance (FTC) is investigated. Two different server, the immediate repair unit and standby unit are analyzed. Yiannis [5] studied the M/M/1 queue with strategic customers, delayed sensitivity and compared the strategic behaviours. Uniformly or gamma-distributed delay of customers, impact of the customers and strategic behaviour are analyzed. Heterogeneous single-server optimal control queuing models are derived by Long et al. [16] and introduced the target allocation policy. Function is convex, Gc/μ rule and function is concave, fixed priority rule are followed. Also developed hybrid routing policies. Sing et al. [27] analyzed the change point problem for heterogeneous servers in the M/M/2 queue. Also presented the results in the Monte Carlo simulation method. heterogeneous servers and impatience in with the infinite capacity queue analyzed by Satin et al. [24] considered that customers can switch the service process from slow to faster. Obtained the birth-and-death process for the upper bound of the distance between two probability distribution. Kothandaraman and Kandaiyan [13] derived the various dynamic queues and improved their effectiveness. Heterogeneous service, intermittently obtainable server using geometric method under hybrid vacation. Efrosinin et al. [8] studied the single-server heterogeneous queue with scheduling optimization. Ayyappan and Nithya [2] establish the single server, retrial, breakdown, reneging, repair and vacation queuing model. Two categories of customers priority and ordinary, are considered. Hard and soft kinds of breakdowns are analyzed, ordinary customers may renege when the server id unavailable. Probability generating and Laplace transformation are used for solving the system states. Dudin et al. [6] finite capacity single server buffer queuing system with breakdown and repair are analyzed with steady-state optimization and numerical example. A single server queue with soft failure from hardware catastrophic failures are characterized by Janani [11]. Retrial queue, additional sever, heterogeneous server, impatience and vacation are proposed by Vinitha et al. [30] two different servers with different service rate are considered. The birth-and-death process is used to solve equation using a recursive approach method.

3. Methodology

In this paper, the single-server queue with heterogeneous arrivals and various types of breakdowns under multiple working vacations are analyzed. Instead of the server being fully idle during the vacation period, the server offers service at a different rate during multiple working vacations. The model considers three different states: idle, working during the vacation, and busy. Customers enter the system in a heterogeneous process, each entry has different arrival rates with parameters λ_{iv} in the idle state, λ_{wv} in working state during the vacation state and λ_{bv} in busy state. The server provides service during the regular busy period with parameter μ_{rb} and under multiple working vacations, the server provides service with parameter μ_{wv}. The system may breakdown at any time. The system may breakdown in two different ways: working during the vacation stages is denoted as β_{v1} and during the busy period is denoted as β_{v2}. In this model, batches of customers are served under the general bulk service rule. In bulk service process, customers are served together as a batch. The number of customers in each batch can also vary. Thus, each batch of service contains a minimum a units and maximum b units of customers that are denote as (a, b). Suppose the number of customers waiting in the queue is less than a server begins a vacation at random variable V with parameter ξ. This model is denoted as heterogeneous arrival of M/M(a, b)/1/MWV queuing model with types of breakdowns.

4. Steady state equations

Consider the customers in the queue at time t, denoted as $N_c(t)$, and $L(t) = 0$, 1 or 2 according to whether the server is idle, regular busy, or working during the vacation state, respectively. Let

$$R_n^I(t) = Pr\{N_c(t) = n, L(t) = 0\}, \quad 0 \leq n \leq a-1$$

$$Q_n^V(t) = Pr\{N_c(t) = n, L(t) = 1\}, \quad n \geq 0$$

$$P_n^B(t) = Pr\{N_c(t) = n, L(t) = 2\}, \quad n \geq 0$$

When the server is idle: $L(t) = 0$, the size of the queue and system are the same.
When the server is busy or working during the vacation: $L(t) = 1$ or 2, the total number of customers in the system is the sum of the number of customers in a queue and the size of the service batches that contains particular $a \leq x \leq b$ customers.

The probabilities of the steady state are:

$$Q_n^V = \lim_{t\to\infty} Q_n^V(t), \quad R_n^I = \lim_{t\to\infty} R_n^I(t), \quad P_n^B = \lim_{t\to\infty} P_n^B(t),$$

exist and the Chapman–Kolmogrove equations satisfied by them is given by

$$\lambda_{iv} R_0^I = \mu_{rb} P_0^B + \mu_{wv} Q_0^V \tag{1}$$

$$\lambda_{iv} R_n^I = \lambda_{iv} R_{n-1}^I + \mu_{rb} P_n^B + \mu_{wv} Q_n^V, \quad 1 \leq n \leq a-1 \tag{2}$$

$$(\lambda_{wv} + \xi + \mu_{wv} + \beta_{v1}) Q_0^V = \lambda_{iv} R_{a-1}^I + \mu_{wv} \sum_{n=a}^{b} Q_n^V \tag{3}$$

$$(\lambda_{wv} + \xi + \mu_{wv} + \beta_{v1}) Q_n^V = (\lambda_{iv} + \beta_{v1}) Q_{n-1}^V + \mu_{wv} Q_{n+b}^V, \quad n \geq 1 \tag{4}$$

$$(\lambda_{bv} + \mu_{rb} + \beta_{v2}) P_0^B = \mu_{rb} \sum_{n=a}^{b} P_n^B + \xi Q_0^V \tag{5}$$

$$(\lambda_{bv} + \mu_{rb} + \beta_{v2}) P_n^B = (\lambda_{bv} + \beta_{v2}) P_{n-1}^B + \mu_{rb} P_{n+b}^B + \xi Q_n^V, \quad n \geq 1 \tag{6}$$

5. Steady state solution

To solve the steady state equation, the forward shifting operator E on P_n^B and Q_n^V are introduced then,

$$E(P_n^B) = P_{n+1}^B, \quad E(Q_n^V) = Q_{n+1}^V \quad for\ n \geq 0$$

Thus, equation (4) gives the homogeneous difference equation as follows:

$$\left(\lambda_{wv} + \beta_{v1} + \mu_{wv} E^{b+1} - (\lambda_{wv} + \beta_{v1} + \xi + \mu_{wv}) E\right) Q_n^V = 0 \tag{7}$$

The characteristic equation (7) is obtained as follows:

$$z(u) = \lambda_{wv} + \beta_{v1} + \mu_{wv} u^{b+1} - (\lambda_{wv} + \beta_{v1} + \xi + \mu_{wv}) u = 0$$

by taking $x(u) = (\lambda_{wv} + \beta_{v1} + \xi + \mu_{wv}) u$ and $y(u) = \lambda_{wv} + \beta_{v1} + \mu_{wv} u^{b+1}$, it is found that $|y(u)| < |x(u)|$ on $|u| = 1$. By Rouche's theorem, $z(u)$ has a unique root r_v inside the contour $|u| = 1$. Equation (7) has a homogeneous solution as,

$$Q_n^V = r_v^n Q_0^V \tag{8}$$

From equation (6) we get,

$$\left(\lambda_{bv} + \beta_{v2} + \mu_{rb} E^{b+1} - (\lambda_{bv} + \beta_{v2} + \mu_{rb}) E\right) P_n^B = -\xi r_v^{n+1} Q_0^V \tag{9}$$

By applying Rouche's theorem to equation (9) as,

$$\left(\lambda_{bv} + \beta_{v2} + \mu_{rb} E^{b+1} - (\lambda_{bv} + \beta_{v2} + \mu_{rb}) E\right) P_n^B = 0$$

The above equation has a unique root r with $|r| < 1$. Also, equation (9) gives a non-homogeneous solution as

$$P_n^B = \left(Z r^n - \frac{\xi r_v^{n+1}}{\lambda_{bv} + \beta_{v2} + \mu_{rb} r_v^{b+1} - (\lambda_{bv} + \beta_{v2} + \mu_{rb}) r_v} \right) Q_0^V \tag{10}$$

$$P_n^B = (Z r^n + Z^* r_v^n) Q_0^V \tag{11}$$

where

$$Z^* = \frac{\xi r_v}{\lambda_{bv}(r_v - 1) + \beta_{v2}(r_v - 1) + \mu_{rb} r_v (1 - r_v^b)} \tag{12}$$

The expression for R_n^I is obtained by adding equations (1) and (2) and substituting P_n^B and Q_n^V values,

$$R_n^I = \left(\frac{\mu_{rb}}{\lambda_{iv}}\left(\frac{Z(1 - r^{n+1})}{1 - r} + \frac{Z^*(1 - r_v^{n+1})}{1 - r_v}\right) + \frac{\mu_{wv}}{\lambda_{iv}}\frac{(1 - r_v^{n+1})}{1 - r_v}\right)Q_0^V$$

To calculate Z, consider equation (5) and substitute P_n^B and Q_n^V value,

$$Z\left((\lambda_{bv} + \mu_{rb} + \beta_{v2}) - \frac{\mu_{rb}(r^a - r^{b+1})}{1 - r}\right) = \xi - Z^*\left((\lambda_{bv} + \mu_{rb} + \beta_{v2}) - \frac{\mu_{rb}(r_v^a - r_v^{b+1})}{1 - r_v}\right) \tag{13}$$

the above expression can be simplified as,

$$\frac{Z\mu_{rb}(1 - r^a)}{1 - r} = \frac{\xi}{1 - r_v} - \frac{Z^*\mu_{rb}(1 - r_v^a)}{1 - r_v} \tag{14}$$

Hence the probability queue size of the steady-state equation in terms of Q_0^V are obtained,

$$Q_n^V = (r_v^n)Q_0^V \qquad n \geq 0 \tag{15}$$

$$P_n^B = (Zr^n + Z^* r_v^n)Q_0^V \quad n \geq 0 \tag{16}$$

where

$$Z = \frac{(1 - r)}{\mu_{rb}(1 - r^a)}\left(\frac{\xi}{1 - r_v} - \frac{Z^*\mu_{rb}(1 - r_v^a)}{1 - r_v}\right) \tag{17}$$

$$Z^* = \frac{\xi r_v}{\lambda_{bv}(r_v - 1) + \beta_{v2}(r_v - 1) + \mu_{rb} r_v (1 - r_v^b)} \tag{18}$$

and

$$R_n^I = \left(\frac{\mu_{rb}}{\lambda_{iv}}\left(\frac{Z(1 - r^{n+1})}{(1 - r)} + \frac{Z^*(1 - r_v^{n+1})}{(1 - r_v)}\right) + \frac{\mu_{wv}}{\lambda_{iv}}\frac{(1 - r_v^{n+1})}{(1 - r_v)}\right)Q_0^V \tag{19}$$

by using the normalizing condition and calculating the value of Q_0^V by

$$\sum_{n=0}^{\infty} Q_n^V + \sum_{n=0}^{\infty} P_n^B + \sum_{n=0}^{a-1} R_n^I = 1$$

By substituting P_n^B, Q_n^V and R_n^I is observed that

$$\sum_{n=0}^{\infty} r_v^n Q_0^V + \sum_{n=0}^{\infty}(Zr^n + Z^* r_v^n)Q_0^V + \sum_{n=0}^{a-1}\left(\frac{\mu_{rb}}{\lambda_{iv}}\left(\frac{Z(1 - r^{n+1})}{(1 - r)} + \frac{Z^*(1 - r_v^{n+1})}{(1 - r_v)}\right)\right.$$
$$\left. + \frac{\mu_{wv}}{\lambda_{iv}}\frac{(1 - r_v^{n+1})}{(1 - r_v)}\right)Q_0^V = 1$$

Then,

$$(Q_0^V)^{-1} = \omega(r_v, \mu_{wv}) + Z\omega(r, \mu_{rb}) + Z^*\omega(r_v, \mu_{rb}) \tag{20}$$

where

$$\omega(x,y) = \frac{1}{(1-x)}\left(1 + \frac{y}{\lambda_{iv}}(c - \frac{x(1-x^a)}{(1-x)})\right)$$

6. Performance measures

Performance measures can be evaluated by the performance of the queuing system. In this paper, we analyzed the expected queue length and characteristics of the queue.

6.1. Mean queue length

The expected queue length is given by

$$L_q = \sum_{n=1}^{\infty} n(Q_n^V + P_n^B) + \sum_{n=1}^{a-1} nR_n^I$$

By substituting P_n^B, Q_n^V and R_n^I we observe that

$$L_q = \sum_{n=1}^{\infty} n(r_v^n Q_0^V) + \sum_{n=1}^{\infty} n(Zr^n + Z^* r_v^n)Q_0^V + \sum_{n=1}^{a-1} n\left(\frac{\mu_{rb}}{\lambda_{iv}}\left(\frac{Z(1-r^{n+1})}{(1-r)}\right.\right.$$

$$\left.\left. + \frac{Z^*(1-r_v^{n+1})}{(1-r_v)}\right) + \frac{\mu_{wv}}{\lambda_{iv}}\frac{(1-r_v^{n+1})}{(1-r_v)}\right)Q_0^V$$

$$L_q = Z\omega^*(r,\mu_{rb}) + Z^*\omega^*(r_v,\mu_{rb}) + \omega^*(r_v,\mu_{wv}) \tag{21}$$

where

$$\omega^*(x,y) = \frac{x}{(1-x)^2} + \frac{y}{\lambda_{iv}(1-x)}\left(\frac{a(a-1)}{2} + \frac{ax^{a+1}(1-x) - x^2(1-x^a)}{(1-x)^2}\right)$$

and Z and Z^* are given by equations (17) and (18).

If $Pr_{(wv)}$, $Pr_{(\text{busy})}$ and $Pr_{(\text{idle})}$ denote the probability that the server in idle, regular busy and working during vacation period, then

$$Pr_{(\text{idle})} = \sum_{n=0}^{a-1} R_n^I \tag{22}$$

where the R_n^I is given by equation (19).

$$Pr_{(\text{busy})} = \sum_{n=0}^{\infty} P_n^B = \left(\frac{Z}{(1-r)} + \frac{Z^*}{(1-r_v)}\right)Q_0^V \tag{23}$$

$$Pr_{(wv)} = \sum_{n=0}^{\infty} Q_n^V = \frac{Q_0^V}{(1-r_v)} \tag{24}$$

7. Particular cases

7.1. Case 1. $M/M(a,b)/1/MWV$ heterogeneous arrival with a breakdown model

By letting $\beta_{v1} = \beta_{v2} = \beta_v$ in equations (15) to (21) become

$$Q_n^V = (r_v^n)Q_0^V \qquad n \geq 0$$

$$P_n^B = (Zr^n + Z^* r_v^n)Q_0^V \quad n \geq 0$$

where

$$Z = \frac{(1-r)}{\mu_{rb}(1-r^a)}\left(\frac{\xi}{(1-r_v)} - \frac{Z^*\mu_{rb}(1-r_v^a)}{(1-r_v)}\right)$$

$$Z^* = \frac{\xi r_v}{\lambda_{bv}(r_v-1) + \beta_v(r_v-1) + \mu_{rb}r_v(1-r_v^b)} \quad if r_v \neq r$$

and

$$R_n^I = \left(\frac{\mu_{rb}}{\lambda_{iv}}\left(\frac{Z(1-r^{n+1})}{(1-r)} + \frac{Z^*(1-r_v^{n+1})}{(1-r_v)}\right) + \frac{\mu_{wv}}{\lambda_{iv}}\frac{(1-r_v^{n+1})}{(1-r_v)}\right)Q_0^V = 0 \;\; 0 \leq n \leq a-1 \tag{25}$$

$$L_q = Z\omega^*(r,\mu_{rb}) + Z^*\omega^*(r_v,\mu_{wv}) + \omega^*(r_v,\mu_{wv})$$

where

$$\omega^*(x,y) = \frac{x}{(1-x)^2} + \frac{y}{\lambda_{iv}(1-x)}\left\{\frac{a(a-1)}{2} + \frac{ax^{a+1}(1-x) - x^2(1-x^a)}{(1-x)^2}\right\}$$

The expected queue length of analyzed model coincides with the $M/M(a,b)/1/MWV$ queuing model for heterogeneous arrival with breakdowns analyzed by P and Mary [20].

7.2. Case 2. $M/M(a,b)/1/MWV$ heterogeneous arrival model

Letting $\beta_{v1} = \beta_{v2} = 0$ in equations (15) to (21), we get

$$Q_n^V = (r_v^n)Q_0^V \qquad n \geq 0$$

$$P_n^B = (Zr^n + Z^* r_v^n)Q_0^V \quad n \geq 0$$

$$R_n^I = \left(\frac{\mu_{rb}}{\lambda_{iv}}(Zg_n(r) + Z^* g_n(r_v) + \frac{\mu_{wv}}{\lambda_{iv}}g_n(r_v)\right)Q_0^V \quad 0 \leq n \leq a-1$$

where

$$Z = \frac{(1-r)}{\mu_{rb}(1-r^a)}\left(\frac{\xi}{(1-r_v)} - \frac{Z^*\mu_{rb}(1-r_v^a)}{(1-r_v)}\right)$$

$$Z^* = \frac{\xi r_v}{\lambda_{bv}(r_v-1) + \mu_v r_v(1-r_v^b)}$$

Further

$$(Q_0^V)^{-1} = \omega(r_v,\mu_{wv}) + Z\omega(r,\mu_{rb}) + Z^*\omega(r_v,\mu_{rb})$$

where

$$\omega(x,y) = \frac{1}{(1-x)}\left(1 + \frac{y}{\lambda_{iv}}(c - \frac{x(1-x^a)}{(1-x)})\right)$$

$$L_q = Z\omega^*(r,\mu_{rb}) + Z^*\omega^*(r_v,\mu_{rb}) + \omega^*(r_v,\mu_{wv})$$

where

$$\omega^*(x,y) = \frac{x}{(1-x)^2} + \frac{y}{\lambda_{iv}(1-x)}\left(\frac{a(a-1)}{2} + \frac{ax^{a+1}(1-x) - x^2(1-x^a)}{(1-x)^2}\right)$$

The expected queue length of the analyzed model coincide with the $M/M(a,b)/1/MWV$ queuing model with heterogeneous arrival analyzed by P and Mary [19].

7.3. Case 3. M/M/1 model

By letting a=b=1, $\beta_{v1} = \beta_{v2} = 0$ and $\lambda_{iv} = \lambda_{wv} = \lambda_{bv} = \lambda_v$ in equations (15) to (19) becomes,

$$\begin{aligned} Q_n^V &= (r_v^n)Q_0^V \qquad n \geq 0 \\ P_n^B &= \frac{Z^*}{r_v}(r_v^{n+1} - r^{n+1})Q_0^V \quad n \geq 0 \\ and\ \ R_0^I &= \frac{Q_0^V}{r_v} \end{aligned}$$

where

$$r = \frac{\lambda_v}{\mu_{rb}} = \rho_v, \quad Z = -\frac{Z^*\rho_v}{r_v} \ \ and \ \ Z^* = \frac{\xi r_v}{\mu_{rb}(1-r_v)(r_v-\rho_v)}$$

The above equations coincides with the $M/M/1/MWV$ queuing model analyzed by Liu et al. [31].

7.4. Case 4. $M/M(a,b)/1/MWV$ model

By letting $\beta_{v1} = \beta_{v2} = 0$ and $\lambda_{iv} = \lambda_{wv} = \lambda_{bv} = \lambda_v$ in equations (15) to (21) and we obtain,

$$Q_n^V = (r_v^n)Q_0^V \quad n \geq 0$$

$$P_n^B = (Zr^n + Z^*r_v^n)Q_0^V \quad n \geq 0$$

$$R_n^I = \left(\frac{\mu_{rb}}{\lambda_v}(Zg_n(r) + Z^*g_n(r_v) + g_n(r_v)\right)Q_0^V \quad 0 \leq n \leq a-1$$

where

$$Z = \frac{(1-r)}{\mu_{rb}(1-r^a)}\left(\frac{\xi}{(1-r_v)} - \frac{Z^*\mu_{rb}(1-r_v^a)}{(1-r_v)}\right)$$

$$Z^* = \frac{\xi r_v}{\lambda_v(r_v-1) + \mu_{rb}r_v(1-r_v^b)}$$

Further

$$(Q_0^V)^{-1} = \omega(r_v,\mu_{wv}) + Z\omega(r,\mu_{rb}) + Z^*\omega(r_v,\mu_{rb})$$

where

$$\omega(x,y) = \frac{1}{(1-x)}\left(1 + \frac{y}{\lambda_v}(c - \frac{x(1-x^a)}{1-x})\right)$$

$$L_q = Z\omega^*(r, \mu_{rb}) + Z^*\omega^*(r_v, \mu_{rb}) + \omega^*(r_v, \mu_{wv})$$

where

$$\omega^*(x,y) = \frac{x}{(1-x)^2} + \frac{y}{\lambda_v(1-x)}\left(\frac{a(a-1)}{2} + \frac{ax^{a+1}(1-x) - x^2(1-x^a)}{(1-x)^2}\right)$$

Thus, observed that our specified model coincides with the classical $M/M(a,b)/1/MWV$ queuing model analyzed by Mary and Begum [17].

8. Numerical analysis

In the numerical section, consider the various parameters like the regular service rate (μ_{rb}), the service rate during multiple working vacations (μ_{wv}), the heterogeneous arrival rate ($\lambda_{iv}, \lambda_{wv}$ and λ_{bv}) and the breakdowns (β_{v1}) and (β_{v2}). With the aid of the above parameters, numerical analysis is carried out to evaluate the performance measures of the specified model. For that purpose, consider equations (7) and (9) and that lie in the interval [0 1].

Table 1. Expected parameters of the system of heterogeneous multiple working vacations with types of breakdown

μ_{wv}	ξ	r_v	L_q	L_s	W_q	W_s
	0.02	0.9939	158.94	159.24	39.2446	39.319
	0.04	0.9877	76.747	77.047	18.95	19.024
0.05	0.06	0.9822	51.892	52.192	12.8129	12.887
	0.08	0.9763	38.254	38.554	9.44546	9.5195
	0.10	0.9714	31.237	31.537	7.71281	7.7869
	0.02	0.9921	122.22	122.52	30.178	30.252
	0.04	0.982	51.972	52.272	12.8327	12.907
0.10	0.06	0.9784	42.49	42.79	10.4912	10.565
	0.08	0.9721	32.300	32.600	7.97533	8.0494
	0.10	0.9659	26.046	26.346	6.43122	6.5053
	0.02	0.9896	92.35	92.65	22.8024	22.877
	0.04	0.9809	48.794	49.094	12.0478	12.122
0.15	0.06	0.973	33.727	34.027	8.32753	8.4016
	0.08	0.9659	26.238	26.538	6.47857	6.5526
	0.10	0.9593	21.693	21.993	5.35628	5.4304
	0.02	0.9851	63.912	64.212	15.7808	15.855
	0.04	0.9746	36.330	36.630	8.97025	9.0443
0.20	0.06	0.9659	26.461	26.761	6.53367	6.6077
	0.08	0.9581	21.187	21.487	5.23148	5.3056
	0.10	0.951	17.908	18.208	4.42178	4.4958

By assuming $\lambda_{iv} = 3.9$, $\lambda_{bv} = 4.05$, $\mu_{rb} = 0.9$, β_{v1}=0.001 and $\beta_{v2} = 0.002$ and by varying μ_{wv} and ξ, the characteristic values are calculated and tabulated. The effect of ξ and μ_{wv} on various performance measures of the heterogeneous multiple working vacations with types of breakdowns model are shown in Table 1. As ξ and μ_{wv} values increase, the length of the queue, length of the system, the waiting time of the queue and waiting time of the system decrease gradually.

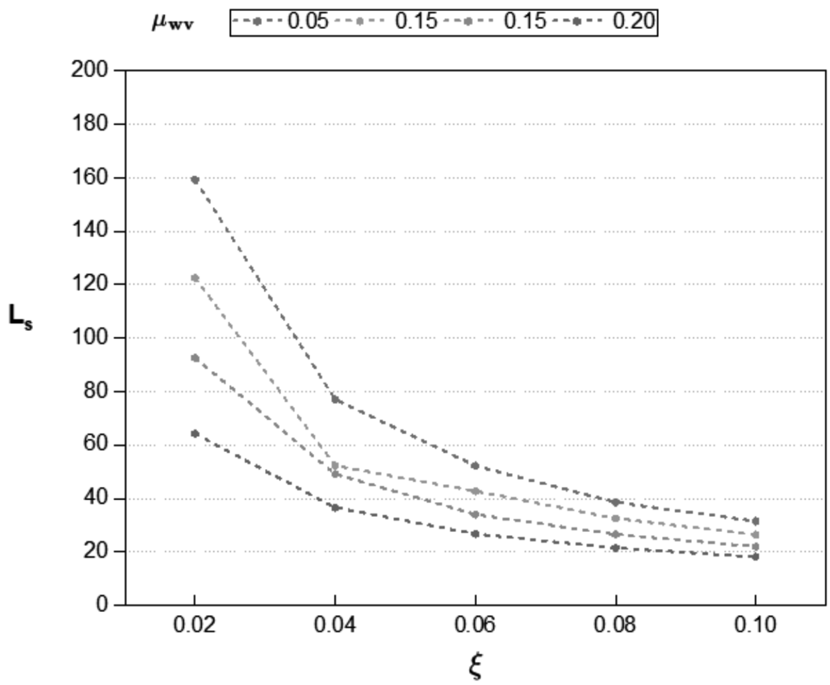

Figure 1. Expected length of the System
of heterogeneous multiple working vacations with types of breakdown

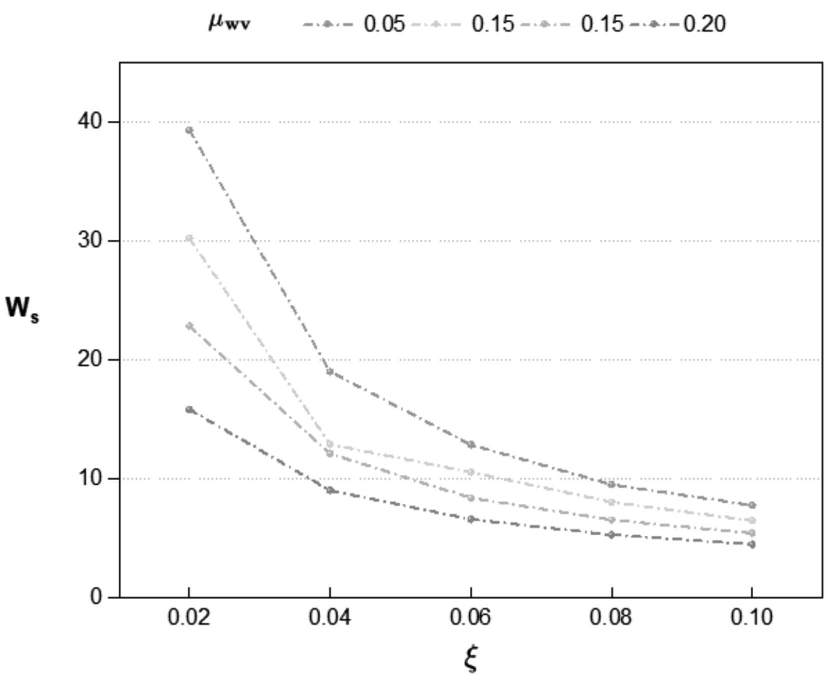

Figure 2. Expected waiting time of the system
of Heterogeneous multiple working vacations with types of breakdown

From Figures 1 and 2, the vacation parameter x_i increases the length of the system, and the waiting time of the system decreases step by step. When $\mu_{wv} = 0.05$ and $\xi = 0.02$, the length and waiting time of the system reached the maximum values of 159.24 and 39.319. Similarly, when $\mu_{wv} = 0.05$ and $\xi = 0.10$, the length and waiting time reached the minimum values of 31.537 and 7.7869.

Table 2 represents different values for arrival rate λ_{bv}, μ_{wv} and $\mu_{rb} = 0.9$, $\xi = 0.01$ and $\beta_{v2} = 0.002$ are calculated and Figure 3 shows that arrival of the customers increases the length of the system also increases.

Table 2. Expected queue length and expected system length of heterogeneous multiple working vacations with types of breakdown with respect to λ_{bv}

μ_{wv}	r_v	λ_{bv}	L_q	L_s	μ_{wv}	r_v	λ_{bv}	L_q	L_s
	0.9877	4.05	79.088	79.388		0.9809	4.05	50.307	50.607
	0.9846	4.04	62.992	63.292		0.9769	4.04	41.417	41.717
0.05	0.9822	4.03	54.39	54.69	0.15	0.973	4.03	35.297	35.597
	0.9763	4.02	40.673	40.973		0.9659	4.02	27.772	28.072
	0.9714	4.01	33.601	33.901		0.9593	4.01	23.151	23.451
	0.982	4.05	53.613	53.913		0.9746	4.05	37.417	37.717
	0.9815	4.04	52.138	52.438		0.9701	4.04	31.627	31.927
0.10	0.9784	4.03	44.527	44.827	0.20	0.9659	4.03	27.612	27.912
	0.9721	4.02	34.293	34.593		0.9581	4.02	22.311	22.611
	0.9659	4.01	27.934	28.234		0.951	4.01	18.972	19.272

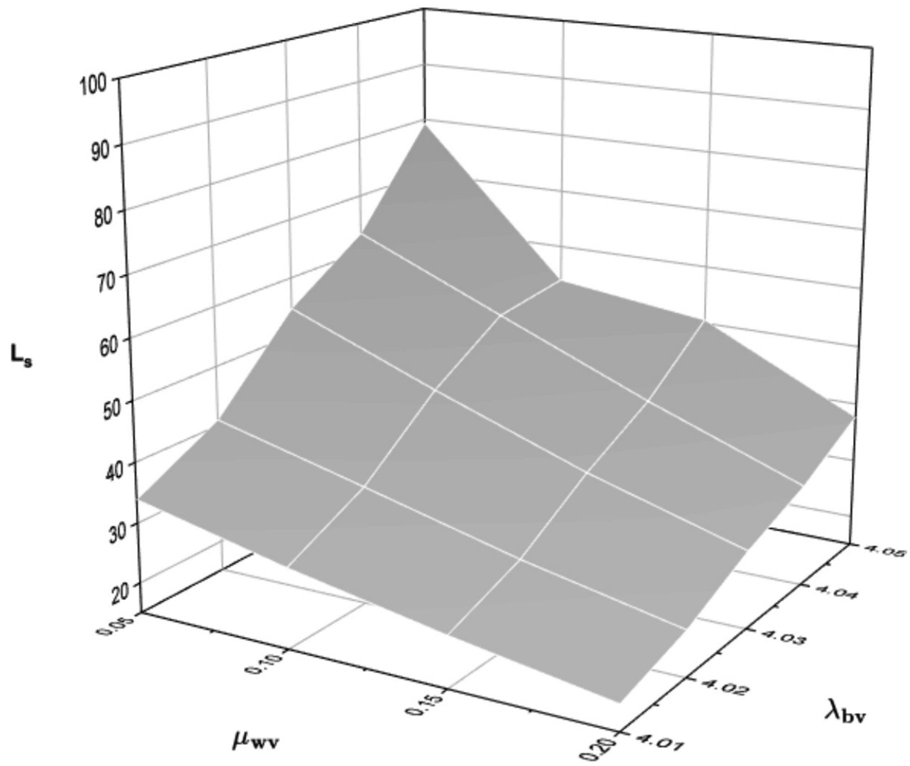

Figure 3. Expected length of the system of heterogeneous multiple working vacations with types of breakdown

Table 3. Mean queue length of multiple working vacations with breakdown and heterogeneous multiple working vacations with breakdown

μ_{wv}	r_v	β_v	L_{MWVB}	L_{HMWVB}	μ_{wv}	r_v	β_v	L_{MWVB}	L_{HMWVB}
0.05	0.9846	0.004	63.011	62.9929	0.15	0.9769	0.004	41.44	41.4178
	0.9822	0.003	54.407	54.3919		0.973	0.003	35.317	35.2983
	0.9763	0.002	40.685	40.6749		0.9659	0.002	27.788	27.7732
	0.9714	0.001	33.609	33.6024		0.9593	0.001	23.164	23.152
0.10	0.9815	0.004	52.159	52.1389	0.20	0.9701	0.004	31.65	31.6275
	0.9784	0.003	44.546	44.5285		0.9659	0.003	27.633	27.6128
	0.9721	0.002	34.307	34.2944		0.9581	0.002	22.329	22.3127
	0.9659	0.001	27.944	27.9355		0.951	0.001	18.987	18.9738

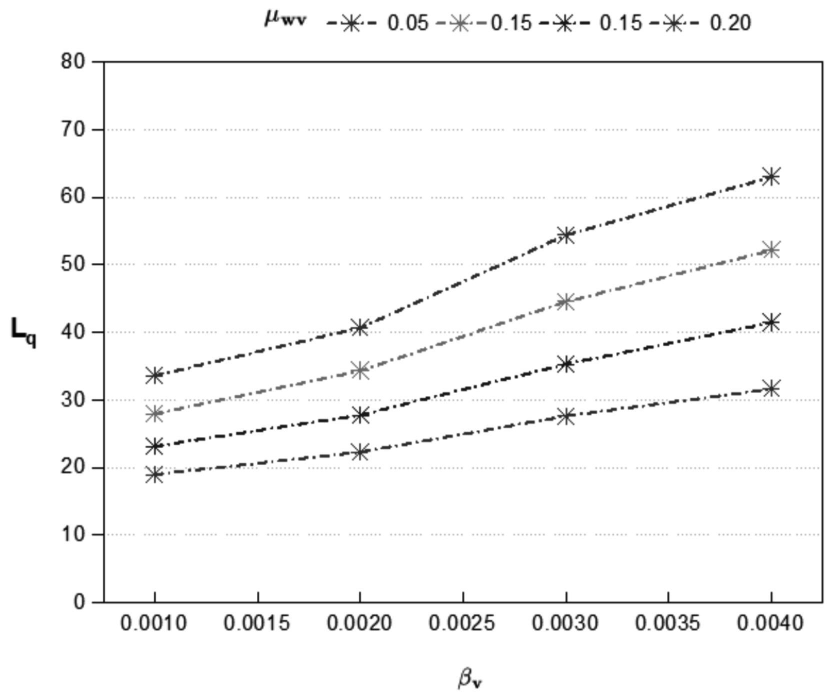

Figure 4. Expected length of the queue of heterogeneous multiple working vacations with types of breakdown

The table value 4 and the graphical representation 3 show that the β_v and μ_{wv} increase the mean queue size decreases both heterogeneous arrival and types of breakdowns under the MWV (HMWVB) model and homogeneous arrival and types of breakdowns under the MWV (MWVB) model. Also, notice that $\lambda_{iv} = \lambda_{wv} = \lambda_{bv} = \lambda_v$, the queue size of the HMWVB model approaches the queue size of the MWVB model. Also, notice that L_q decreases notably with respect to heterogeneous multiple working vacations with types of breakdowns queuing models.

In this paper, heterogeneous arrivals and types of breakdowns under MWV (HMWVB) are discussed. When breakdown happens in a queuing system, it can have several impacts and consequences on the system's operation. In Table 4, notice that the queue length increases in MWVB compared to HMWVB. Heterogeneous arrivals of customers can reduce waiting times, facilitate faster processing, and improve the overall performance compared to the homogeneous arrival of customers.

A breakdown can lead to delays in serving customers in the queue, it can increase the waiting times for customers and reduced the overall performance. In our model, customers enter the system in heterogeneous process, it may reduce the waiting time for customers.

The probability for various states are $Pr_{(\text{idle})}$, $Pr_{(wv)}$ and $Pr_{(\text{busy})}$ are calculated and tabulated in (Table 4) by using equations (22) to (24). The table value shows the effect of the parameters on the performance measures. The chosen parameters are$\lambda_{iv} = 3.9$, $\lambda_{wv} = 4$, $\lambda_{bv} = 4.05$, and $\mu_{rb} = 0.9$ and by varying μ_{wv} and ξ, notice that $Pr_{(\text{busy})}, Pr_{(\text{idle})}$ increase and $Pr_{(wv)}$ decreases as the vacation parameter ξ increases for a particular value of μ_{wv}.

Table 4. Probability of idle, working vacation and busy period

μ_{wv}	ξ	r_v	$Pr_{(\text{busy})}$	$Pr_{(wv)}$	$Pr_{(\text{idle})}$	μ_{wv}	ξ	r_v	$Pr_{(\text{busy})}$	$Pr_{(wv)}$	$Pr_{(\text{idle})}$
	0.01	0.9969	0.26405	0.72954	0.00640		0.01	0.9939	0.16091	0.82939	0.00968
	0.02	0.9939	0.27606	0.71145	0.01247		0.02	0.9896	0.19302	0.78938	0.01758
0.05	0.03	0.9908	0.28377	0.69787	0.01835	0.20	0.03	0.9851	0.20997	0.76482	0.02520
	0.04	0.9877	0.29164	0.68437	0.02397		0.04	0.9809	0.22533	0.74243	0.03223
	0.05	0.9846	0.29938	0.67123	0.02937		0.05	0.9769	0.23921	0.72198	0.03879
	0.01	0.9958	0.21251	0.77984	0.00763		0.01	0.9915	0.12505	0.86231	0.01263
	0.02	0.9921	0.23258	0.75282	0.01458		0.02	0.9851	0.15118	0.82603	0.02278
0.10	0.03	0.9884	0.24571	0.73307	0.02120	0.20	0.03	0.9795	0.17216	0.79611	0.03172
	0.04	0.9820	0.23373	0.73718	0.02907		0.04	0.9746	0.19111	0.76925	0.03963
	0.05	0.9815	0.27042	0.69629	0.03327		0.05	0.9701	0.20798	0.74516	0.04684

The above model can be applicable whenever there's heterogeneous arrival and single-server batch service in the system. For example, teaching, fast-food restaurants, automated machine packing, printing service and metropolitan area network (MAN) etc.

9. Conclusion

The heterogeneous arrivals and types of breakdowns under MWV are analyzed. In this model, customers enter with various arrival rates, and GBSR is followed. The breakdown occurs in the working during the vacation and busy period. The steady-state solution, the performance of measures for the system, and particular cases are calculated. Finally, in the form of tables and graphs, numerical results have been evaluated.

Queuing systems with breakdowns leads to several impacts on the system's operation. Due to breakdowns, server delays in serving customers in the queue may increase the queue length. The queue length increases the number of customers waiting in the queue for a longer time, which may lead to customers dissatisfaction. This can affect relationships, and customers may lose trust in the system. Heterogeneous arrivals reduced the waiting time for customers. Understanding the behavior of customers in breakdown situations may help the system perform and provide the best service for the customers. In the future, the model may be extended to the heterogeneous arrival of multiple working vacations queues with balking under breakdowns.

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