Vol. 34, No. 4 (2024) | DOI: 10.37190/ord240405

OPEN ACCESS

Operations Research and Decisions

www.ord.pwr.edu.pl

OPERATIONS RESEARCH AND DECISIONS QUARTERLY

On non-Pareto efficiency

Ignacy Kaliszewski[1, 2]

[1] *Systems Research Institute, Polish Academy of Sciences, Warsaw, Poland*
[2] *Warsaw School of Information Technology, Warsaw, Poland*
Email address: ignacy.kaliszewski@ibspan.waw.pl

Abstract

We investigate assessment functions, i.e., functions that aggregate numerical attribute values into single numbers. All assessment functions in the current use share the same limitation: they do not explicitly account for the attribute values balance. Here, we present assessment functions that provide for that. However, those functions are at odds with the well-established paradigm of Pareto efficiency. As an example, the relevance of assessment functions to rankings is discussed.

Keywords: *multiple attributes, attribute values balance, assessment functions, Leontief function, Pareto dominance--consistency*

1. Introduction

We appraise objects (or ideas, e.g., plans) in any domain of human activity. Whenever possible, this is done by evaluating objects quantitatively and aggregating their *numerical* attribute values by (multiattribute) *assessment* functions. Assessment functions are at the root of decision theories, like the social choice theory, the production theory, the utility theory, or the multiple criteria decision making theory. They all share a common framework: object attributes become arguments of an assessment function, and the assessment function value constitutes the object appraisal.

The standard mechanism of assessment functions is that larger attribute values[1], with no attribute value smaller, result in a higher assessment function value; it is then said that assessment functions are increasing. Thus, for two objects with the same assessment function value, increases in some attribute values have to be compensated by a decrease in values of some other attributes. This is the principle of Pareto efficiency. But what if one goes beyond that mechanism, by admitting cases when larger attribute

[1] To simplify the presentation, it is assumed that all attributes are of the type "the more, the better" or are transformed to that type by multiplication their values by -1.

Received 6 October 2023, accepted 1 October 2024, published online 19 December 2024
ISSN 2391-6060 (Online)/© 2024 Authors

The costs of publishing this issue have been co-financed by the Department of Operations Research and Business Intelligence at the Faculty of Management, Wrocław University of Science and Technology, Wrocław, Poland

values do not necessarily result in higher assessment function values? We present this idea formally and then show its relevance to decision making on the example of rankings.

In the production theory, the utility theory, the decision theory, and the multiple criteria decision making theory, the assessment functions are based on the generalized mean functions (GMFs) (it is commonly assumed in the first three domains that attribute values are nonnegative; below we stick to this assumption):

$$f(x) = \sqrt[p]{\frac{1}{k}\sum_{l=1}^{k}(x_l)^p}, \quad p \le 1 \tag{1}$$

where x_l is the lth attribute of an object. To align with the assumption that all attributes are of the type "the more, the better", these functions have to be maximized. For $p \to 0,\ f(x) = \sqrt[k]{\prod_{l=1}^{k} x_l}$ (the geometric mean), and for $p \to -\infty$, $f(x) = \min_{1 \le l \le k} x_l$ (the Leontief function), cf. Figure 1.

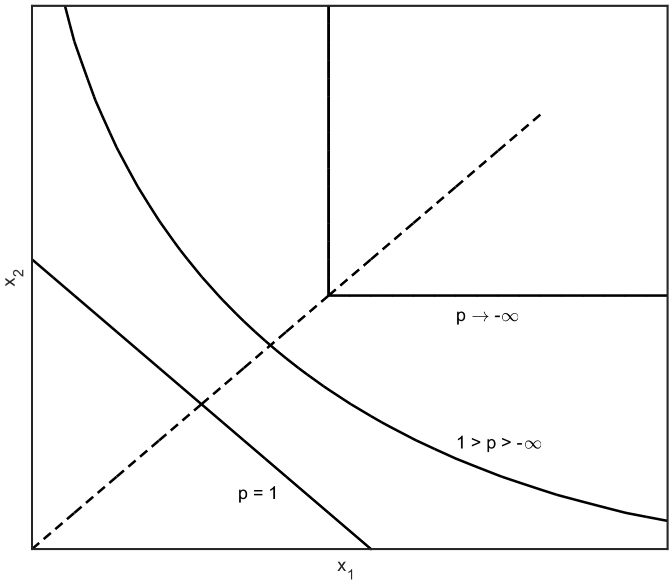

Figure 1. Contours of the generalized mean functions for selected p (solid) and the CHL (dashed)

In the production theory, GMFs with $p < 1$ are known as CES functions ([12]). For $p = 1$, function (1) is the linear function. Here, to simplify the exposition, we use the unweighted form of assessment functions, absorbing weights by substituting weighted attributes x_l by unweighted x'_l, namely:

$$w_l x_l = x'_l \tag{2}$$

where $w_l > 0,\ \ l = 1, \ldots, k$, are weights.

2. Assessment functions inconsistent with Pareto efficiency

Assessment functions in the current use are consistent with Pareto efficiency. Larger attribute values (and no one smaller) incur higher assessment function values (this immediately follows from the definition of

GMFs), and that is the contemporary dogma in all mentioned domains. But can larger attribute values incur lower assessment function values? It all depends on how assessment function values are interpreted.

We say that the attribute values are perfectly balanced, if $x_l/x_{l'} = 1$ for all $l,\ l' = 1, \ldots, k,\ l \neq l'$, as represented by the dashed half line in Figure 1 (the compromise half line, CHL, the term introduced in [6], cf. also [9, 10])[2]. We define the attribute values balance as the distance (in the sense of a selected distance measure) between the collection (vector) of k attribute values and the CHL. Here the lower balance, the better, thus one can interpret so defined attribute values balance as *unbalance*. With that, objects can be appraised by their attribute values balances. Attribute values collections that are located on the CHL are perfectly balanced, their balance is zero (cf. the definition of "extremeness aversion" discussed in the next section).

For $k = 2$, equidistant collections of objects' attribute values are located on lines parallel to the CHL, and for $k = 3$ on surfaces of convex bodies with the CHL as the axis.

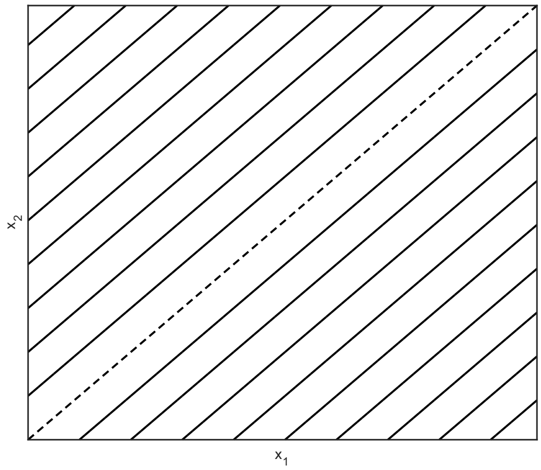

Figure 2. Contours of the generalized Leontief function (solid and dashed), $\rho = -\frac{1}{k}$, $k = 2$. Each line contains points (x_1, x_2) equidistant to the CHL (dashed)

Apprising objects according to their attribute values balances, as outlined above, focus on the positions of attribute values collections relative to the CHL, rather than, as in the classic methods of objects' apprising, on attribute values collections relative to one another. A novel object appraising approach is needed that would compromise that two perspectives.

3. Related works

In behavioral decision theory the issue of attribute values balance has been of research interests since ninethies of the previous century. In [16] (cf. also [15]), the authors come out with a hypothesis that the attractiveness of an object is enhanced if it is an intermediate object in the choice set and is diminished if it is an extreme object. The authors term that effect the *extremeness aversion*. According to this hypothesis, the extremeness aversion is a function of the relational properties of choice objects and the middle object, defined such that its attribute values are between the values of the other objects, is viewed as the least extreme, compromise object. The hypothesis is further elaborated in [1, 2] where it is argued

[2] In the context of this paper, the term "the attribute values balance line", as in [1, 2], would be perhaps more adequate but here we stick to the former term for consistency with our previous works.

that an object with equal attribute values will be perceived as the compromise object even when it is not the middle object. The hypothesis is supported by data studies. However, no formal assessment framework is offered.

Whereas the behavioral decision theory tends to explain the decision maker (such as an individual consumer) choices ex post, the utility theory (cf. e.g., [4, 13]), the production theory (cf. e.g., [14, 18]), and the multiple criteria decision making theory (cf. e.g., [3, 5, 10, 11, 17, 19]) offer more normative approaches. Works in these domains tend to rely on assessment functions (named, respectively, utility functions, production functions, scalarizing functions, with the generic term value functions referring to all of them) that ex ante specify properties of attributes of objects delivered by the assessment function maximization.

In the first two named domaines, as a rule, researches confine themselves to GMFs that are, with the exception for $p \to +\infty$ resulting in the Leontief function, smooth functions and admit a variety of analyses based on infinitesimal calculus (cf. e.g., [12]). The attribute values balancing effect of different curvatures of GMFs has been noticed, resulting in occasional propensity to make use of, e.g., the geometric mean function instead of the linear function. However, the issue of attribute value balance (and extremeness aversion) has been not exploited, at least explicitly. The Leontief function, that is not smooth, is the limit of possible GMFs curvatures that until now no research ventured to cross beyond.

In contrast, the multiple criteria decision making theory is more open to the use of the Leontief funcion and the family of generalized Leontief functions steaming from it (see the next section). This can be related to the fact that contours of the Leontief function are shifted nonnegative orthants R^k_+, where $R^k_+ = \{x \in \mathbf{R}^k \,|\, x_l \geq 0,\ l = 1, \ldots, k\}$. R^k_+ serves for an alternative definition of Pareto efficiency: object $\bar{x}$ (variant, alternative) is Pareto efficient in a set of objects X iff $(\bar{x} + R^k_+) \cap X = \bar{x}$. That form of the Pareto efficiency definition offers an obvious but elegant, geometrical interpretation of that notion. The Leontief function (as its generalization), constructed on collections of linear functions (hyperplanes) has favorable numerical properties that are exploited in algorithms for deriving Pareto efficient objects, notably in large-scale multiple criteria decision problems [17]. In [7] and [8], the generalized Leontief functions (see the next section) serve there as the base for a method to limit the derivation of Pareto efficient objects to those with limited trade-offs, in line with, but not in a relation to, the findings of [16]. Again, the issue of attribute values balance (and extremeness aversion) is not exploited in those works, at least explicitly.

4. “Beyond the wall” to reconcile the extremes

Let us observe that GMFs with $p \leq 1$ are sensitive to the attribute values balance in appraisals they provide, in the following manner. Suppose that for $p \leq 1$, two objects are equally appraised by the corresponding GMF but their attribute values collection distances to the CHL are different. For any p', $p' < p$, an object with the attribute values collection closer to the CHL is apprised higher than the other object with the attribute values collection further from the CHL. This is because the contours of GMFs are symmetric to the CHL. So, with p becoming smaller and smaller, more and more objects with attribute values collections distant from the CHL are penalized by lower appraisals based on GMFs. This is illustrated in Figure 3. The Leontief function ($p \to -\infty$) is the limit, the wall. Is there anything beyond this “wall”? To see, we first equip ourselves with the (family of) generalized Leontief functions:

$$f(x) = \min_{1 \le l \le k} \left(x_l + \rho \sum_{l=1}^{k} x_l \right), \quad -\frac{1}{k} \le \rho \le +\infty \tag{3}$$

To align with the assumption that all attributes are of the type "the more, the better" these functions have to be maximized. The contours of the GLFs (3) are presented in Figure 4.

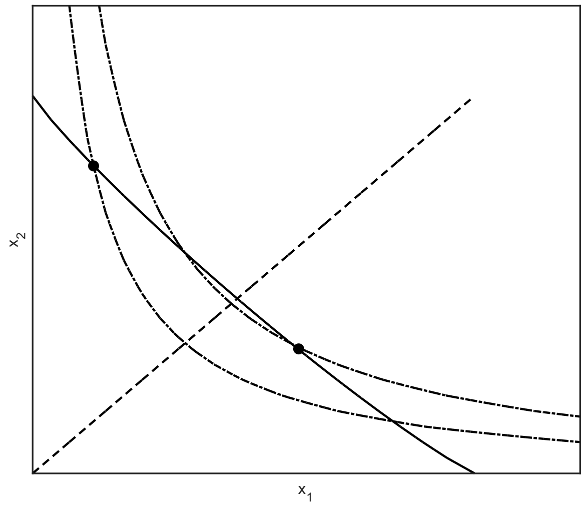

Figure 3. From two objects equally appraised by generalized mean function for some $p \le 1$ (solid contour), for any $p' < p$ the object with attribute values less distant from the CHL (dashed) is appraised higher than the other object (dash-dotted contours)

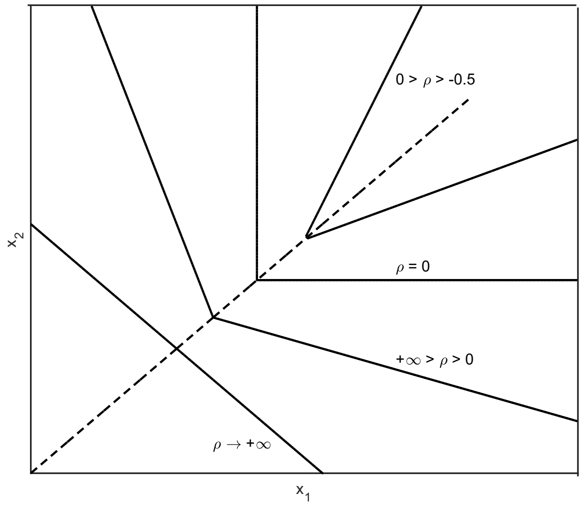

Figure 4. Contours of the Generalized Leontief Functions for selected ρ (solid) and the CHL (dashed)

For $\rho = 0$, GLF is the standard Leontief function. For $\rho = -\frac{1}{k}, k = 2$, contours of the GLF are lines parallel to the CHL, as in Figure 2. GLFs with $0 \le \rho \le +\infty$ are widely used in the domain of multiobjective optimization, the underlying formalism for multiple criteria decision making theory (cf. e.g., [3, 5, 10, 11, 17, 19]). For $-\frac{1}{k} \le \rho < 0$, GLFs are inconsistent with Pareto efficiency as they no longer are increasing functions. And in contrast to GMFs, GLFs are not smooth. But like GMFs, irrespective of the sign of ρ, GLFs penalize high attribute values in the following manner. Suppose that for $\rho > -\frac{1}{k}$ two objects are equally appraised by the corresponding GLF, but their attribute values collection distances to the CHL are different. For any ρ', $\rho' < \rho$, the object with the attribute values collection closer to the CHL is appraised higher than the other object with its attribute values collection farther from the CHL. This is because the contours of GMFs are symmetric to the CHL. So, with ρ becoming smaller and smaller (but not smaller than $-\frac{1}{k}$), more and more objects with attribute values collections distant from the CHL are penalized by lower appraisals based on GLFs.

5. How does this work in practice?

We illustrate the working of the GLFs on the Human Development Index (HDI) 2022 ranking data[3] for 20 top positions in the ranking (Table 1).

[3] https://hdr.undp.org/data-center/human-development-index#/indicies/HDI.

Table 1. Data for top twenty *HDI 2022* ranked countries.

No.	Country	Life	Education	Wealth
1	Switzerland	0.9844	0.9203	0.9828
2	Norway	0.9728	0.9335	0.9776
3	Iceland	0.9643	0.9589	0.9553
4	Hong Kong	1.0000	0.8875	0.9727
5	Australia	0.9927	0.9242	0.9364
6	Denmark	0.9442	0.9320	0.9672
7	Sweden	0.9690	0.9203	0.9517
8	Ireland	0.9538	0.8861	1.0000
9	Germany	0.9328	0.9422	0.9519
10	Netherlands	0.9490	0.9194	0.9558
11	Finland	0.9544	0.9291	0.9371
12	Singapore	0.9655	0.8565	1.0000
13	Belgium	0.9608	0.9315	0.9196
13	New Zealand	0.9520	0.9125	0.9455
15	Canada	0.9639	0.9166	0.9288
16	Liechtenstein	0.9732	0.8397	1.0000
17	Luxembourg	0.9635	0.8338	1.0000
18	UK	0.9345	0.9277	0.9236
19	Japan/Korea	0.9800	0.8760	0.9212
20	Japan/Korea	0.9967	0.8684	0.9134

The HDI is a summary measure of average achievement in key dimensions of human development: a long and healthy life (“Life”), being knowledgeable (“Education”), and having a decent standard of living (“Wealth”). The HDI is the geometric mean of the normalized indicators for each dimension. The health dimension is assessed by the life expectancy at birth.

The education dimension is measured by the mean of years of schooling for adults aged 25 years and more, and expected years of schooling for children of school-entering age. The standard of living dimension is measured by gross national income (GNI) per capita. The HDI uses the logarithm of income to reflect the diminishing importance of income with the increasing GNI. Refer to the HDI website[4] for more details.

Table 2 illustrates ”the climb” of countries with balanced indicators (attributes) towards higher and higher positions in rankings as the value of parameter ρ decreases. A good example is New Zealand. However, the climb is not always monotonous, as is the case of Denmark which for $\rho = -0.1$ climbs to the second position and is pushed down the rankings for $\rho = -0.2$ and for $\rho = -1/3$ by the climb of the United Kingdom and Finland to a higher than Denmark positions. Table 3 contains values of GLFs for all the rankings from Table 2.

6. Discussion

There is no such thing as objective single-number appraisals. Mathematically, objects with two or more attributes are generally partially ordered, whereas objects in single-number appraisals are (have to be) linearly ordered, and the latter is the special case of the former. When a partial order is not a linear order by itself, it can be reduced to a linear order only by brute force in a subjective (even if expert)

[4] https://hdr.undp.org/sites/default/files/2021-22_HDR/hdr2021-22_technical_notes.pdf.

Table 2. Top twenty *HDI 2022* ranked countries
in the rankings defined by the Generalized Leontief Functions for different ρ.

No.	$\rho \to \infty$	HDI	$\rho = 10$	$\rho = 1$	$\rho = 0$	$\rho = -0.1$	$\rho = -0.2$	$\rho = -1/3$
1	Switzerland	Switzerland	Switzerland	Iceland	Iceland	Iceland	Iceland	Iceland
2	Norway	Norway	Norway	Norway	Norway	Germany	Germany	UK
3	Iceland	Iceland	Iceland	Switzerland	Germany	Denmark	UK	Germany
4	Hong Kong	Hong Kong	Hong Kong	Australia	Denmark	Finland	Finland	Finland
5	Australia	Australia	Australia	Denmark	Finland	Norway	Denmark	Denmark
6	Denmark	Denmark	Denmark	Sweden	Australia	UK	New Zealand	New Zealand
7	Sweden	Sweden	Sweden	Germany	UK	Australia	Norway	Canada
8	Ireland	Ireland	Ireland	Finland	Switzerland	New Zealand	Canada	Netherlands
9	Germany	Germany	Germany	Hong Kong	Sweden	Netherlands	Netherlands	Belgium
10	Netherlands	Netherlands	Netherlands	Netherlands	New Zealand	Sweden	Australia	Sweden
11	Singapore	Finland	Finland	New Zealand	Netherlands	Canada	Sweden	Australia
12	Finland	Singapore	Singapore	Ireland	Canada	Switzerland	Belgium	Norway
13	Liechtenstein	Belgium	New Zealand	Canada	Belgium	Belgium	Switzerland	Switzerland
13	New Zealand	New Zealand	Belgium	Belgium	Hong Kong	Ireland	Korea	Korea
15	Belgium	Canada	Canada	UK	Ireland	Hong Kong	Ireland	Japan
16	Canada	Liechtenstein	Liechtenstein	Singapore	Korea	Korea	Hong Kong	Ireland
17	Luxembourg	Luxembourg	Luxembourg	Korea	Japan	Japan	Japan	Hong Kong
18	UK	UK	UK	Liechtenstein	Singapore	Singapore	Singapore	Singapore
19	Japan	Japan/Korea	Japan	Japan	Liechtenstein	Liechtenstein	Liechtenstein	Liechtenstein
20	Korea	Japan/Korea	Korea	Luxembourg	Luxembourg	Luxembourg	Luxembourg	Luxembourg

UK – United Kingdom.

Table 3. Assessment function values for rankings from Table 2

No.	$\rho \to +\infty$	HDI	$\rho = 10$	$\rho = 1$	$\rho = 0$	$\rho =$ -0.1	$\rho =$ -0.2	$\rho =$ -1/3
1	2.89	0.96	29.80	3.83	0.96	0.67	0.38	0.00
2	2.88	0.96	29.77	3.82	0.93	0.65	0.37	-0.01
3	2.88	0.96	29.74	3.81	0.93	0.65	0.37	-0.01
4	2.86	0.95	29.49	3.78	0.93	0.65	0.36	-0.01
5	2.85	0.95	29.46	3.78	0.93	0.65	0.36	-0.02
6	2.84	0.95	29.37	3.76	0.92	0.65	0.36	-0.02
7	2.84	0.95	29.33	3.76	0.92	0.64	0.36	-0.02
8	2.84	0.95	29.28	3.75	0.92	0.64	0.35	-0.02
9	2.83	0.94	29.20	3.75	0.92	0.64	0.35	-0.02
10	2.82	0.94	29.16	3.74	0.92	0.64	0.35	-0.03
11	2.82	0.94	29.14	3.73	0.92	0.64	0.35	-0.03
12	2.82	0.94	29.08	3.73	0.92	0.63	0.35	-0.03
13	2.81	0.94	29.04	3.73	0.91	0.63	0.34	-0.04
13	2.81	0.94	29.01	3.72	0.89	0.60	0.32	-0.05
15	2.81	0.94	29.01	3.71	0.89	0.60	0.32	-0.06
16	2.81	0.93	28.97	3.68	0.88	0.60	0.32	-0.06
17	2.80	0.93	28.81	3.65	0.87	0.59	0.31	-0.07
18	2.79	0.93	28.78	3.65	0.86	0.57	0.29	-0.08
19	2.78	0.92	28.65	3.65	0.84	0.56	0.28	-0.10
20	2.78	0.92	28.65	3.63	0.83	0.55	0.27	-0.10

Due to number rounding, for the exact position of a country in the ranking see Table 2.

manner; that is precisely how single–number appraisals (rankings!) are made. The choice of an instance of assessment function as well as the choice of attribute weights are subjective. Hereby, acknowledging the inherent subjectivity so understood, we offer an extension of assessment functions currently in use by stepping beyond the Pareto efficiency principle. By this, one can appraise objects in a manner that accounts for *and* reflect the attribute values balance.

References

[1] Chernev, A. Extremeness aversion and attribute-balance effects in choice. *Journal of Consumer Research 31*, 2 (2004), 249–263.

[2] Chernev, A. Context effects without a context: Attribute balance as a reason for choice. *Journal of Consumer Research 32*, 2 (2005), 213–223.

[3] Ehrgott, M. *Multicriteria Optimization*. Springer, 2005.

[4] Fishburn, P. C. Utility theory and decision theory. In *Utility and Probability* (London, 1990), J. Eatwell, M. Milgate, and P. Newman, Eds., Palgrave Macmillan, pp. 303–312.

[5] Kaliszewski, I. *Quantitative Pareto Analysis by Cone Separation Technique*. Kluwer Academic Publishers, 1994.

[6] Kaliszewski, I. Multiple criteria decision making: selecting variants along compromise lines. *Techniki Komputerowe (Computer Aided Methodologies) 1* (2006), 49–66.

[7] Kaliszewski, I., and Michalowski, W. Efficient solutions and bounds on trade-offs. *Journal of Optimization Theory and Applications 94*, 2 (1997), 381–394.

[8] Kaliszewski, I., and Michalowski, W. Searching for psychologically stable solutions of multiple criteria decision problems. *European Journal of Operational Research 118*, 3 (1999), 549–562.

[9] Kaliszewski, I., Miroforidis, J., and Podkopaev, D. Interactive multiple criteria decision making based on preference driven evolutionary multiobjective optimization with controllable accuracy. *European Journal of Operational Research 216*, 1 (2012), 188-199.

[10] Kaliszewski, I., Miroforidis, J., and Podkopaev, D. *Multiple Criteria Decision Making by Multiobjective Optimization - A Toolbox*. Springer, 2016.

[11] Miettinen, K. *Nonlinear Multiobjective Optimization*. Springer, 1999.

[12] Miller, E. *An Assessment of CES and Cobb-Douglas Production Functions* Working Paper No. 19992, Congressional Budget Office, 2008.

[13] Moscati, I. *The History and Methodology of Expected Utility*. Series: Elements in Decision Theory and Philosophy. Cambridge University Press, 2023.

[14] Ray, S. C., Chambers, R. G., and Kumbhakar, S. C., *Handbook of Production Economics*. Springer, 2023.

[15] Simonson, I. Choice based on reasons: The case of attraction and compromise effects. *Journal of Consumer Research 16*, 2 (1989), 158–174.

[16] Simonson, I., and Tverski, A. Choice in context: Tradeoff contrast and extremeness aversion. *Journal of Marketing Research 29*, 3 (1992), 281–295.

[17] Steuer, R. E. *Multiple Criteria Optimization: Theory, Computation and Application*. John Wiley & Sons, 1986.

[18] Truett, D., and Roberts, B. Classical production functions, technical optimality, and scale adjustments of the firm. *The American Economic Review 63*, 5 (1973), 975–982.

[19] Wierzbicki, A. P. Reference point approaches. In *Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory and Applications* (New York, NY, 1999), T. Gal, T. J. Stewart, and T. Hanne, Eds., vol. 21 of *International Series in Operations Research & Management Science*, Springer, pp. 237–275.