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Approximate solutions to the multiple-choice knapsack problem by multiobjectivization and Chebyshev scalarization

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Abstract

The method BISSA, proposed by Bednarczuk, Miroforidis, and Pyzel, provides approximate solutions to the multiple-choice knapsack problem. To fathom the optimality gap that is left by BISSA, we present a method that starts from the BISSA solution and it is able to provide a better approximation and in consequence a tighter optimality gap. Like BISSA, the new method is based on the multiobjectivization of the multiple-choice knapsack problem but instead of the linear scalarization used in BISSA, it makes use of the Chebyshev scalarization. We validate the new method on the same set of problems as the one used to validate BISSA.

Keywords: *multiobjectivization, multiple-choice knapsack problem, Chebyshev scalarization, BISSA algorithm*

1. Introduction and motivation

The knapsack problem (KP) is one of the best studied combinatorial optimization problems (see, e.g., [8, 14]). The multi-dimensional knapsack problem and the multiple-choice knapsack problem (MCKP) are the two most noted generalizations of KP and are applied to model many real-life problems, e.g., in project (investments) portfolio selection [16, 20], capital budgeting [17], advertising [20], component selection in IT systems [12, 18], computer networks management [13], adaptive multimedia systems [9], and other.

The multiple-choice knapsack problem is formulated as follows. Given are m sets (categories) $N_1, N_2, \ldots, N_m$ of items, of cardinality $N_j = n_j,\ j = 1, \ldots, m$. Real-valued nonnegative profit $p_{ij} \geq 0$ and cost $c_{ij} \geq 0,\ j = 1, \ldots, m, i = 1, \ldots, n_j$, is assigned to each item of each set. The problem consists in choosing exactly one item from each set N_j, so that the total cost does not exceed a given $b > 0$ and the total profit is maximized.

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Let $x_{ij},\ j = 1, \ldots, m,\ i = 1, \ldots, n_j$, be defined as

$$x_{ij} = \begin{cases} 1 \text{ if item } i \text{ from set } N_j \text{ is chosen} \\ 0 \text{ otherwise} \end{cases}$$

Note that all x_{ij} form a vector x of length $n = \sum_{j=1}^{m} n_j,\ x \in \{0, 1\}^n$, where
$x := (x_{11}, \ldots, x_{n_1 1}, x_{12}, \ldots, x_{n_2 2}, \ldots, x_{1m}, \ldots, x_{n_m m})^{\mathrm{T}}$.
The multiple-choice knapsack problem takes the form

$$\begin{aligned} &\max \ \sum_{j=1}^{m}\sum_{i=1}^{n_j} p_{ij}x_{ij} \\ &\text{subject to:} \\ &\sum_{j=1}^{m}\sum_{i=1}^{n_j} c_{ij}x_{ij} \le b \\ &x \in X := \{(x_{ij}) \ | \sum_{i=1}^{n_j} x_{ij} = 1 \\ &x_{ij} \in \{0, 1\},\ j = 1, \ldots, m,\ \ i = 1, \ldots, n_j\} \end{aligned} \tag{MCKP}$$

Elements $x \in X$ are feasible solutions to MCKP if they satisfy

$$\sum_{j=1}^{m}\sum_{i=1}^{n_j} c_{ij}x_{ij} \le b$$

and are infeasible solutions, if otherwise.

Various exact and approximate methods for solving MCKP are presented in monographs [8] and [14], as well as in a recent review paper on knapsack problems [3], where the heuristic BISSA algorithm by Bednarczuk et al. [1] is mentioned as one of the methods.

In the BISSA algorithm, an approximate solution to MCKP is sought by the problem multiobjectivization and the linear scalarization. By multiobjectivization (see, e.g., [10]) here we mean the formulation based on MCKP of a bi-objective optimization problem in which the second objective function is the left-hand side of the first constraint of MCKP. In BISSA, an approximate solution to MCKP is derived by providing Pareto optimal solutions to the resulting bi-objective optimization problem and exploiting effectively the structure of set X. Namely, by the multiple-choice constraints $\sum_{i=1}^{n_j} x_{ij} = 1$, the objective function $\sum_{j=1}^{m}\sum_{i=1}^{n_j} p_{ij}x_{ij}$ and the constraint function $\sum_{j=1}^{m}\sum_{i=1}^{n_j} c_{ij}x_{ij}$ can be calculated in each jth category independently. In other words, these two functions are additively separable. Thus, a linear scalarizing function over these two functions is also additively separable. Clearly, this observation generalizes to any number of functions structured analogously. The method BISSA provides an approximate solution to MCKP together with the optimality gap estimation.

A conceivable improvement of the method presented in [1] would be replacing the linear scalarization, that can provide only a subset of Pareto optimal solutions, with the Chebyshev scalarization, that in the case of the bi-objective optimization problem based on MCKP (it is a discrete optimization problem) can provide all Pareto optimal solutions (cf. [4–7, 15]).

The use of the Chebyshev scalarization in the context of harnessing multiobjectivization to solve constrained optimization problems has been advocated in ([10], p. 352–353), namely *(...) approximation methods based on the Chebyshev approach have appeared useful in the multicriteria context, and more interesting results could be expected to be obtained in single criterion optimization by transfer of results obtained in the multicriteria setting.* By following this idea, in the current article, we aim to scan a part of the solution space of the bi-objective optimization problem based on MCKP, as it has been proposed in the concluding section of [1], i.e., we try to cope with *(...) the issue of finding a better solution by a smart 'scanning' of the triangle of uncertainty* ([1], p. 908).

However, the Chebyshev scalarizing function, as it is easy to show by a counterexample (as that one given in Appendix 7, cf. also [2]), is not additively separable. Thus, with the Chebyshev scalarization the structure of set X and the additive separability of functions $\sum_{j=1}^{m}\sum_{i=1}^{n_j} p_{ij}x_{ij}$, $\sum_{j=1}^{m}\sum_{i=1}^{n_j} c_{ij}x_{ij}$ cannot be directly exploited for solving MCKP. Nevertheless, as presented in the next sections, the Chebyshev scalarization may provide better approximate solutions to MCKP than the linear scalarization.

In this work, we exploit the multiobjectivization of MCKP and the Chebyshev scalarization of the resulting bi-objective problem (in the context of the two postulates quoted above, this constitutes our research method), for constructing a method that attempts to find better approximate solutions to MCKP, as compared to those that can be found by the method proposed in [1]. We validate the new method on the same set of problems as the one used to validate BISSA. Despite that BISSA produces approximate solutions within tight optimality gaps, especially for correlated objective function and constraint coefficients, in 20% of instances of test problems our method was able to produce better results than that produced by BISSA.

The main aim of the paper is to propose a Chebyshev scalarization-based procedure which allows one to improve the approximate solution to MCKP and consequently to tighten the optimality gap of MCKP.

The outline of the paper is as follows. In Section 2, we present a bi-objective formulation of MCKP and its Chebyshev scalarization. In Section 3, we discuss Pareto optimality of additively separable functions. We exploit the Chebyshev scalarization in Section 4 where we present a procedure to narrow the optimality gap to MCKP, as compared to the optimality gap provided by [1], together with its effective implementation. Section 5 reports results of numerical experiments. Section 6 concludes.

2. Multiobjectivization of MCKP

Multiobjectivization has been proposed [11] to reduce local optima and facilitate improved optimization in heuristic algorithms to solve constrained singleobjective optimization problems. Applications of multiobjectivization in population-based metaheuristic algorithms to solve this class of problems are discussed in [19]. The use of multiobjectivization and scalarization techniques to solve convex and nonconvex constrained singleobjective optimization problems has been elaborated in [10].

In this section, we show how to apply the multiobjectivization technique for MCKP. Given function $F : X \to \mathbb{R}^k$, $F = (f^1, \ldots, f^k)$, $f^l : X \to \mathbb{R},\ l = 1, \ldots, k$, solution $\bar{x} \in X$ is Pareto optimal on X, if

$$(F(\bar{x}) + \mathbb{R}^+_k) \cap F(X) = F(\bar{x})$$

where $\mathbb{R}^+_k = \{y \in \mathbb{R}^k |\, y_l \geq 0,\ l = 1, \ldots, k\}$.

We multiobjectivize MCKP as follows: we keep maximizing the original MCKP objective while minimizing the new objective originated from the constraint. For the sake of clarity of the presentation, in the sequel we replace minimization of the second objective function by maximization of its negative. In this way, we obtain the following bi-objective (maximization) problem

$$\begin{aligned} &\text{vmax } (f^1(x), f^2(x)) \\ &\text{subject to:} \\ &x \in X \end{aligned} \tag{MO\_MCKP}$$

where $f^1(x) = \sum_{j=1}^{m} \sum_{i=1}^{n_j} p_{ij} x_{ij}$, $f^2(x) = -\sum_{j=1}^{m} \sum_{i=1}^{n_j} c_{ij} x_{ij}$, vmax denotes the operator of deriving Pareto optimal solutions.

In this formulation, the structure of the multiple-choice set X is fully exposed. The advantage of such a formulation is that now the feasible set of MO_MCKP is just X and not the set of $x \in X$ restrained by the additional linear inequality, as it is in the case of the original MCKP problem. For a general reformulation of constrained singleobjective problems as multiobjective problems, see [10].

In the paper by Bednarczuk et al. [1], the search for approximate solutions to MCKP was conducted among solutions to MO_MCKP that are Pareto optimal and derived by the linear scalarization (commonly referred to as supported solutions). Since, as observed above, a linear scalarizing function over two objective functions of MO_MCKP is additively separable, such solutions are Pareto optimal for $F_j(x) = (f^1_j(x), f^2_j(x)) = \big(\sum_{i=1}^{n_j} p_{ij} x_{ij}, -\sum_{i=1}^{n_j} c_{ij} x_{ij}\big)$ on $X_j := \{x_j = (x_{ij}) | \sum_{i=1}^{n_j} x_{ij} = 1,\ x_{ij} \in \{0, 1\}\}$, $j = 1, \ldots, m$, and Pareto optimal for $F(x) = (f^1(x), f^2(x))$ on X. Sets $X_j,\ j = 1, \ldots, m$, are referred to as components of X. The structure of MO_MCKP and the adopted notation are presented in Table 1.

Table 1. The structure of MO_MCKP and the adopted notation

$F(x)$	$F_1(x_1)$	$\ldots$	$F_j(x_j)$	$\cdots$	$F_1(x_m)$
$f^1(x)$	$f^1_1(x_1)$	$\ldots$	$f^1_j(x_j)$	$\cdots$	$f^1_1(x_m)$
$f^2(x)$	$f^2_1(x_1)$	$\ldots$	$f^2_j(x_j)$	$\cdots$	$f^2_1(x_m)$
	$x_1 \in X_1$	$\cdots$	$x_j \in X_j$	$\cdots$	$x_m \in X_m$
$X =$	X_1	$\times \cdots \times$	X_j	$\times \cdots \times$	X_m

The following result was proved by Bednarczuk et al. [1]. Let S_P be the set of all Pareto optimal solutions to MCKP.

Theorem 1. ([1], Theorem 3.1, p. 896). Let $x^* \in X$ be a Pareto optimal solution to MO_MCKP, such that

$$b - c^{\mathrm{T}}x^* = \min_{x \in S_P,\ b - c^{\mathrm{T}}x \geq 0} b - c^{\mathrm{T}}x$$

Then x^* solves MCKP.

In [1], the considerations are limited to a subset of S_P, namely to set $S_{\bar{P}}$ of the so-called supported Pareto optimal solutions, i.e., solutions that can be derived by linear scalarizing functions, a task that in the case of multiple-choice constraints reduces to sorting. In consequence, algorithm BISSA allows us to establish the optimality gap with respect to $S_{\bar{P}}$, that amounts to

$$b - c^{\mathrm{T}}\bar{x}^* = \min_{x \in S_{\bar{P}},\ b - c^{\mathrm{T}}x \geq 0} b - c^{\mathrm{T}}x$$

Since the set of supported Pareto optimal solutions $S_{\bar{P}}$ is in general smaller than set S_P, one can only expect the inequality $b - c^{\mathrm{T}}\bar{x}^* \geq b - c^{\mathrm{T}}x^*$.

Actually, BISSA finds two solutions to MO_MCKP: x^* and $x^{'*}$ that solve the following problem

$$c^{\mathrm{T}}x^* - c^{\mathrm{T}}x^{'*} = \min_{x \in S_{\bar{P}},\ b - c^{\mathrm{T}}x \geq 0,\, x' \in S_{\bar{P}},\ b - c^{\mathrm{T}}x \leq 0} c^{\mathrm{T}}x - c^{\mathrm{T}}x' \tag{1}$$

The idea developed in this work is to narrow the optimality gap defined by (1). In the next section, we propose a procedure which narrows the optimality gap with the help of the Chebyshev scalarization. The idea relies on exploiting $x \in X$, such that $x_j \in X_j$, are Pareto optimal for F_j on X_j but not necessarily supported on X_j, $j = 1, \ldots, k$.

For completeness, we formulate the Chebyshev scalarization for MO_MCKP.

Let $y^{*1} > \max_{x \in X} \sum_{j=1}^{m} \sum_{i=1}^{n_j} p_{ij}x_{ij}$, $y^{*2} > \max_{x \in X} - \sum_{j=1}^{m} \sum_{i=1}^{n_j} c_{ij}x_{ij}$. The Chebyshev scalarization takes the following form

$$\begin{gathered} \min_{x \in X} \max \begin{cases} \lambda_1(y^{*1} - f^1(x)) \\ \lambda_2(y^{*2} - f^2(x)) \end{cases} \\ \text{subject to:} \\ x \in X \end{gathered} \tag{2}$$

As observed, this function is not additively separable.

3. Pareto optimality of additively separable functions

Below, we provide the following general facts concerning additively separable functions. The following theorem states the underlying fact for the procedure presented in Section 4.

Theorem 2. Given function $F : Y \to \mathbb{R}^k$, $F = (f^1, \ldots, f^k)$, defined on $Y = Y_1 \times \cdots \times Y_m$, $Y_j \in \mathbb{R}^{n_j}$, $j = 1, \ldots, m$, each $f^l : Y \to \mathbb{R}$, $l = 1, \ldots, k$, is additively separable on Y, i.e., $f^l := \sum_{j=1}^{m} f_j^l$, $l = 1, \ldots, k$, where $f_j^l : Y_j \to \mathbb{R}$, $j = 1, \ldots, m$.

Then, for every Pareto optimal solution $x^P = (x_1^P, \ldots, x_m^P)$ of F on Y, $x_j^P \in Y_j$ is a Pareto optimal solution of $F_j = (f_j^1, \ldots, f_j^k)$ on Y_j, $j = 1, \ldots, m$.

Proof. Suppose that on the contrary, for given Pareto optimal $x^P = (x_1^P, \ldots, x_m^P)$ there exists index j, $j \in \{1, \ldots, m\}$, such that $x_j^P \in Y_j$ is not Pareto optimal of F_j on Y_j. Then there exists $x_j \in Y_j$ such that $f_j^l(x_j) \geq f_j^l(x_j^P)$, $l = 1, \ldots, k$, and $f_j^l(x_j) > f_j^l(x_j^P)$ for some l. Hence, for $x' = (x_1', \ldots, x_j, \ldots, x_m')$, one has $f^l(x') \geq f^l(x^P)$, $l = 1, \ldots, k$, and for some l, $f^l(x') > f^l(x_j^P)$, which contradicts Pareto optimality of x^P. □

The converse statement does not hold, i.e., it is not true that x composed of Pareto optimal solutions x_j^P of F_j on Y_j, $j = 1, \ldots, m$, $x^P = (x_1^P, \ldots, x_m^P)$, is the Pareto optimal solution of F on Y (see the example in the Appendix 7, cf. also [2]).

By Theorem 2, searching for Pareto optimal solutions of F on Y can be limited to the subset of Y, such that for all $j = 1, \ldots, m$, x_j is Pareto optimal of F_j on Y_j.

Corollary 1. Given function $F : Y \to \mathbb{R}^2$, $F = (f^1, f^2)$, defined on $Y = Y_1 \times \cdots \times Y_m$, each f^i, $i = 1, 2$, is additively separable with respect to Y.

Let $x^A \in Y$ and $x^B \in Y$ be such that $f^1(x^A) < f^1(x^B)$.

If the set $\Phi := \{x \in Y | f^1(x) > f^1(x^A),\ f^2(x) > f^2(x^B)\}$ contains no element $(x_1, \ldots, x_m)$ such that x_j is Pareto optimal of F_j on Y_j for each $j = 1, \ldots, m$, then Φ contains no Pareto optimal solution of F on Y.

Proof. The assertion of this corollary follows immediately from Theorem 2. □

4. A procedure to narrow the optimality gap to MCKP

We now deal with $F = (f^1, f^2)$ where $f^1(x) = \sum_{j=1}^{m} \sum_{i=1}^{n_j} p_{ij} x_{ij}$, $f^2(x) = -\sum_{j=1}^{m} \sum_{i=1}^{n_j} c_{ij} x_{ij}$.

The procedure to narrow the optimality gap defined by (1) is based on Theorem 2.

4.1. The procedure

The procedure starts with two supported Pareto optimal solutions to MO_MCKP that solve problem (1): x^* that is feasible to MCKP (below denoted x^A) and $x^{'*}$ that is infeasible to MCKP (below denoted x^B). This choice of x^A and x^B guarantees that above the line passing through $f(x^A)$ and $f(x^B)$ there is no $f(x)$ such that x is a Pareto optimal solution. Thus, the set that can contain better (in the sense of smaller optimal gap, cf. Theorem 1) feasible solutions to MCKP than x^A is defined by $f_1(x^A) \leq f_1(x^B)$,$f_2(x^A) \geq f_2(x^B)$, $f(x) \leq_{\mathbb{R}^2_+} \lambda f(x^A) + (1 - \lambda) f(x^B)$, $0 < \lambda < 1$.

Let $\varepsilon > 0$. To simplify the presentation, we assume for a while that the Chebyshev scalarization always yields Pareto optimal solutions, whereas in general it provides weakly Pareto optimal solutions.

In the next subsection, we will show how to provide Pareto optimal solutions. With this assumption the procedure is as follows.

Let J^1 be the set of all $j \in \{1, \ldots, m\}$ such that $f_j^1(x_j^A) < f_j^1(x_j^B)$, i.e.,

$$J^1 := \{j \in \{1, \ldots, m\} | f_j^1(x_j^A) < f_j^1(x_j^B),\ f_j^2(x_j^A) > f_j^2(x_j^B)\} \tag{3}$$

Observe that at the starting iteration, $J^1 \neq \emptyset$. Indeed, if J^1 were empty, then for all $j = 1, \ldots, m$,

$$f_j^1(x_j^A) < f_j^1(x_j^B) \quad \text{and} \quad f_j^2(x_j^A) \leq f_j^2(x_j^B)$$

hence x^A would be dominated by x^B, a contradiction to Pareto optimality of x^A.

Observe also that since on the first iteration x^A is a Pareto optimal solution for (f^1, f^2) on X, then, by Theorem 2, each x_j^A is a Pareto optimal solution of F_j on $X_j,\ j = 1, \ldots, m$. Hence, for each $j \in J^1$, we have

$$f_j^2(x_j^A) > f_j^2(x_j^B) \tag{4}$$

This is illustrated in Figure 1.

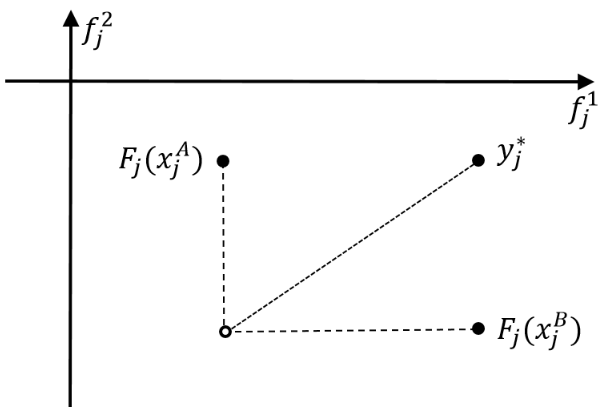

Figure 1. "The layout" at the first iteration of the procedure for some j. Element $\circ$ defines weights of the Chebyshev scalarization

sth iteration of the procedure, $s \in \mathbb{N}^+$ is as follows. Each iteration consists of two steps.

Step 1. Determine J^1. For each $j \in J^1$, determine a Pareto optimal solution x_j^P by solving the Chebyshev problem

$$\min_{x_j \in X_j} \max_{i \in \{1,2\}} \lambda_i^j (y_j^{*i} - f_j^i(x_j))$$

where:

$$y_j^{*1} = \max_{x_j \in X_j} f_j^1(x_j) + \varepsilon, \qquad y_j^{*2} = \max_{x_j \in X_j} f_j^2(x_j) + \varepsilon$$

$$\lambda_1^j = (y_j^{*1} - f_j^1(x_j^A))^{-1}, \qquad \lambda_2^j = (y_j^{*2} - f_j^2(x_j^B))^{-1}$$

Let $J \subseteq J^1$ be such that for $j \in J$, $f_j^1(x_j^A) < f_j^1(x_j^P)$. Observe that for each $j \in J$, we have

$$f^1(x^A) < f^1(x^{A_j}), \text{ where } x^{A_j} = (x_1^A, \ldots, x_j^P, \ldots, x_m^A)$$

Step 2. Let $J^b,\ J^b \subseteq J$, denote the set of indices j such that x^{A_j} are feasible, i.e., $f^2(x^{A_j}) \geq -b$. Choose any $j^* \in J^b$. Set $x^A := x^{A_{j^*}}$ and proceed to the next iteration.

Proposition 4.1. If J is empty and x^A is a Pareto optimal solution to MO_MCKP, then x^A is the optimal solution to MCKP.

Proof. If J is empty, then by Corollary 1 the set

$$\{x \in X | f_j^1(x_j^A) \leq f_j^1(x_j^B),\ f_j^2(x_j^A) \geq f_j^2(x_j^B)\}$$

contains only two Pareto optimal solutions to MO_MCKP, x^A that is feasible to MCKP, and x^B that is infeasible to MCKP, thus x^A is optimal to MCKP. □

Proposition 4.2. If J^b is empty and x^A is a Pareto optimal solution to MO_MCKP, then x^A is the optimal solution to MCKP.

Proof. If J^b is empty, then by Corollary 1 the set

$$\{x \in X | f_j^1(x_j^A) \leq f_j^1(x_j^B),\ f_j^2(x_j^A) \geq f_j^2(x_j^B)\}$$

contains only one Pareto optimal solution to MO_MCKP, namely x^A, that is feasible to MCKP, thus x^A is optimal to MCKP. □

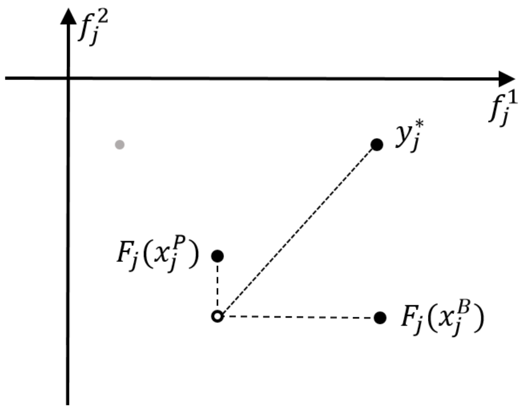

Figure 2. sth iteration of the procedure, $s > 1$, for some j^b in the case the Pareto optimal solution x^A, derived by the Chebyshev scalarization in iteration $s-1$, is feasible; x_j^P, replaces x_j^A from the previous iteration (now in the light gray color); ∘ is the element defining new weights in the Chebyshev scalarization

Observe that starting from the second iteration there is no guarantee that x^A is a Pareto optimal solution for (f^1, f^2) on X. Hence, from the second iteration on, it can happen that

$$f_j^2(x_j^A) < f_j^2(x_j^B) \text{ for some } j \in J^1 \tag{5}$$

Different rules for selecting $j^* \in J^b$ and thus x_j^P can be proposed, for example $\tilde{x}_{j^*}^P = \arg\max_{j \in J^b} f^1(x_j^P)$ (the MAX-PROFIT rule). Figure 2 illustrates operations of one iteration of the procedure for some j^*. The procedure stops when either set J or set J^b becomes empty. The procedure to narrow the optimality gap to MCKP described above we shall call KISSA.

INPUT:

$x^A = (x_1^A, \ldots, x_m^A)$ – a feasible solution to MCKP such that x_j^A is Pareto optimal to F_j on $X_j,\ j = 1, \ldots, m$, provided by BISSA;

$x^B = (x_1^B, \ldots, x_m^B)$ – an infeasible solution to MCKP such that x_j^B is Pareto optimal to F_j on $X_j,\ j = 1, \ldots, m$, provided by BISSA ($f^1(x^A) < f^1(x^B)$).

OUTPUT: a feasible solution to MCKP with a greater than or equal value of the objective function for the input feasible solution x^A.

begin
- Determine ρ.
- **while** ***true*** **do**
  - Determine J^1.
  - $J := \emptyset$.
  - **for** ***each*** $j \in J^1$ **do**
    - Determine $y_j^* = (y_j^{*1}, y_j^{*2})$ by setting:
      - $y_j^{*1} := \max\limits_{x_j \in X_j} f_j^1(x_j) + \varepsilon$,
      - $y_j^{*2} := \max\limits_{x_j \in X_j} f_j^2(x_j) + \varepsilon$.
    - $\lambda_1^j := (y_j^{*1} - f_j^1(x^{A_j}))^{-1}$.
    - $\lambda_2^j := (y_j^{*2} - f_j^2(x^{B_j}))^{-1}$.
    - Solve $\min\limits_{x_j \in X_j} \max\limits_{i \in \{1,2\}} \lambda_j^i (y_j^{*i} - f_j^i(x_j)) + \rho \sum\limits_{i=1}^{m} (y_j^{*i} - f_j^i(x_j))$ to derive x_j^P.
    - **if** $f_j^1(x_j^A) < f_j^1(x_j^P)$ **then**
      - add j to J.
    - **end**
  - **end**
  - **if** $J = \emptyset$ **then**
    - **break**
  - **end**
  - Determine $J^b \subseteq J$.
  - **if** $J^b = \emptyset$ **then**
    - **break**
  - **end**
  - Select $j^* \in J^b$.
  - $x^A := (x_1^A, \ldots, x_{j^*}^P, \ldots, x_m^A)$.
- **end**
- **RETURN** x^A.

end

Algorithm 1. Procedure KISSA to solve MCKP

4.2. The implementation of KISSA

Now, we have to correct for a temporary assumption we have made that solutions to the Chebyshev problem, as above, are Pareto optimal, whereas they are in fact only weakly Pareto optimal. Pareto optimal solutions on $X_j,\ j = 1,\ \dots,\ m$, can be derived by the following result [5], Theorem 4.6, p. 54) adapted to the notation of this work.

Denote

$$\gamma = \{t | x_j^t \in X_j,\ x_j^t \text{ is Pareto optimal on } X_j\}$$

Theorem 3. Let

$$\rho < \min_{t\in\gamma} \left\{ \min_{u\in\gamma\setminus\{t\}} \left\{ \frac{\min_{l,\, f_j^l(x_j^t)-f_j^l(x_j^u)>0} f_j^l(x_j^t) - f_j^l(x_j^u)}{\sum_{l=1}^{k}(f_j^l(x_j^u) - f_j^l(x_j^t))} \middle| \sum_{l=1}^{k}(f_j^l(x_j^u) - f_j^l(x_j^t)) > 0 \right\} \right\} \tag{6}$$

Solution $\bar{x}_j \in X_j$ is Pareto optimal on X_j if and only if there exists a vector $\lambda, \lambda > 0$, such that $\bar{x}_j$ solves

$$\min_{x_j\in X_j} \max_{l\in\{1,\dots,k\}} \lambda_l(y_l^{*j} - f_j^l(x_j)) + \rho \sum_{s=1}^{k}(y_s^{*l} - f_j^s(x_j))$$

where $y_l^{*j} > \max_{x_j\in X_j} f_j^l(x_j),\ l = 1,\ \dots,\ k$.

Let us denote the set of all Pareto optimal solutions by N. To make the formula for ρ, given in Theorem 3, operational, we can use a conservative upper bound on ρ, namely δ, where

$$\delta = \min_{x_j\in X_j} \left\{ \min_{x'_j\in X_j\setminus\{x_j\}} \left\{ \frac{\min\limits_{l\in\{1,2\},\, f_j^l(x_j)-f_j^l(x'_j)>0} \left(f_j^l(x_j) - f_j^l(x'_j)\right)}{\sum_{l=1}^{k}\left(f_j^l(x'_j) - f_j^l(x_j)\right)} \middle| \sum_{l=1}^{k}\left(f_j^l(x_j) - f_j^l(x'_j)\right) > 0 \right\} \right\}$$

In the case of MO_MCKP, function values are just values of single coefficients. Therefore, the above formula takes the form

$$\delta = \min_{i\in\{1,\dots,n_j\}} \left\{ \min_{i'\in\{1,\dots,n_j\}, i'\neq i} \left\{ \frac{\min\limits_{l\in\{1,2\},\, d_{ij}^l - d_{i'j}^l>0} \left(d_{ij}^l - d_{i'j}^l\right)}{\sum_{l=1}^{k}\left(d_{ij}^l - d_{i'j}^l\right)} \middle| \sum_{l=1}^{k}\left(d_{ij}^l - d_{i'j}^l\right) > 0 \right\} \right\}$$

where $d_{ij}^l = p_{ij}^l$ if $l = 1$, and $d_{ij}^l = -c_{ij}^l$ if $l = 2$.

5. Numerical experiments

We conducted numerical experiments on the same set of randomly generated test problems as in Bednarczuk et al. [1]. We considered three sets of 10 problems each with uncorrelated coefficients of the objective function and the constraint, with, respectively, 10 categories and 1000 variables, 100 categories and 100 variables, 1000 categories and 10 variables. We also considered one set of 10 problems with weakly correlated coefficients, notoriously hard to solve for BISSA, with 20 categories and 20 variables. The results are given in Tables 2 and 3.

Table 2. KISSA's results for uncorrelated instances

Instance	EXACT-BISSA GAP [%]	EXACT-KISSA GAP [%]	#KISSA Improv.
Categories: 10, variables: 1000			
1	0.024	0.006	1
2	0.005		
3	0		
4	0		
5	0		
6	0.019	0.012	1
7	0.019	0.002	2
8	0		
9	0		
10	0		
Categories: 100, variables: 100			
1	0.010		
2	0.005		
3	0.011		
4	0.030		
5	0		
6	0.009		
7	0.006		
8	0.005		
9	0.006		
10	0.008		
Categories: 1000, variables: 10			
1	0.019		
2	0.016	0.008	1
3	0.003		
4	0.008		
5	0.009		
6	0.003		
7	0.008		
8	0.016		
9	0.029	0.020	1
10	0.005		

The second column of the tables is the relative gap between the objective function values of the solutions obtained by the exact algorithm used in [1] and BISSA. The third column of the tables is the relative gap between the objective function values of the solutions obtained by the exact algorithm used in [1] and KISSA. The last column of the table shows the number of improvements in the objective function value during KISSA's operation. If empty, no improvement is observed. In the numerical experiments, we used the following parameter values: $\rho = 1\text{E}{-}7$, $\varepsilon = 1\text{E}{-}4$. To select index j^* (line 19 of Algorithm 1), we applied the MAX-PROFIT rule shown in Subsection 4.1.

Table 3. KISSA's results for weakly correlated instances

Instance	EXACT-BISSA GAP [%]	EXACT-KISSA GAP [%]	#KISSA IMPROV.
Categories: 20, variables: 20			
1	0.070		
2	0.011		
3	1.679		
4	2.593	0.836	1
5	3.121	0.279	1
6	0		
7	5.656	1.994	1
8	0.133		
9	0		
10	0.159		

The results show that compared with BISSA, which provides very tight optimality gaps and often optimal solutions, the room for improvements offered by KISSA is limited. Despite that, in 20% of instances of test problems, our method was able to produce better results than that produced by BISSA. Significant tightening of the optimality gaps can be observed for problems with weakly correlated coefficients, where the optimality gaps provided by BISSA are much looser than in the case of uncorrelated coefficients.

As the computation time of KISSA for each of the problems tested was fractions of a second, it can be used as a no-cost plug-in for BISSA.

6. Conclusions

In view of the very good performance of BISSA, the extra effort required by KISSA is worth the trouble in problems where data are precise and the objective value reflects a practical problem of high stakes consequences. In such cases, a quest for approximate solutions tightly close to the optimal one is justified, and KISSA is a viable option. The decision of whether it is the case depends on the nature of practical problems modeled as MCKP.

The advantage of the KISSA method is that, like BISSA, it provides not only an approximate solution but also the optimality gap which is not the case with metaheuristic algorithms, for example. In practical applications, this is an important feature of optimization algorithms. The disadvantage of KISSA is that it must be used together with BISSA.

By applying multiobjectivization and the Chebyshev scalarization to MCKP, we pave the way to use this technique to solve other singleobjective constrained combinatorial problems, where multiobjectivization can uncover the structure of the feasible set, in the sense presented in this work.

In our future work, we want to explore other rules of selecting index j^* (line 19 of Algorithm 1). Developing a version of the algorithm that can work independently of BISSA is also a promising direction for further research.

Declarations

Conflict of interest. The authors declare that they have no conflict of interest.

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7. Appendix

Example 7.1. Assume $k = 2,\ m = 2,\ n_1 = 2,\ n_2 = 2,$

$$\left\{ \begin{array}{r} p^l_{ij} \\ -\ c^l_{ij} \end{array} \right\} = \left\{ \begin{array}{cccc} 2 & 3 & 4 & 2 \\ -1.9 & -3 & -2 & -1 \end{array} \right\}.$$

The values of the objective functions for all four feasible solutions $X = \{1010, 1001, 0110, 0101\}$ are given in Table A1.

Table A1. The values of the objective functions for all four feasible solutions (Example 7.1)

	Pareto optimal			Dominated
X_j	1 0 1 0	1 0 0 1	0 1 1 0	0 1 0 1
$i = 1$	6	4	7	5
$i = 2$	–3.9	–2.9	–5	–4

Now, we derive Pareto optimal solutions in each component separately. In the first component they are $(1\,0)$, $(0\,1)$, and in the second component they are $(1\,0)$, $(0\,1)$, as shown in Table A2.

Table A2. Pareto optimal solutions in each component separately (Example 7.1).

	$j = 1$		$j = 2$	
	Pareto optimal			
X	1 0	0 1	1 0	0 1
$i = 1$	2	3	4	2
$i = 2$	–1.9	–3	–2	–1

The concatenation of the Pareto optimal solution on the component X_1, $(0\,1)$, with the Pareto optimal solution on component X_2, $(0\,1)$, resulting in a feasible solution $(0\,1\,0\,1)$, is not Pareto optimal on $X = X_1 \times X_2$.