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The Owen value for differential cooperative games with a coalition structure

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Abstract

This paper researches differential cooperative games with a coalition structure. To show the payoffs of players, the Owen value for traditional case is extended to the new cooperative model, and its existence and uniqueness are discussed. Furthermore, the relationship between the core and the Owen value is shown. In addition, the sub-game consistency of the Owen value is analyzed that maintains the efficiency of the payoff throughout the game. With the defined characteristic function, it is proved that the Owen value is sub-game consistent. Finally, theoretical results of this paper are applied to solve the problem of cost allocation of environmental governance.

Keywords: *differential cooperative game, coalition structure, Owen value, sub-game consistency*

1. Introduction

In practical situations, the external environment usually varies with time, which leads to people's strategic choices being dynamic rather than static. Differential games (usually called continuous-time dynamic games) can cope well with this case, presented by Isaacs [10] according to military tracking problem in the 1940s. Pontryagin [24] studied differential games in an open-loop solution with the maximum principle. Later, Bellman [5] adopted a dynamic programming technique to research the solutions of discrete-time dynamic games (which were multi-stage counterparts of differential games). Since then, research about the theory and application of differential games have rapidly developed in many areas including mathematics, economics, biology, and environment.

Non-cooperative behaviors among players will usually result in a no-Pareto optimal solution. What is worse, highly undesirable outcomes may be obtained (like the prisoner's dilemma) when the players in the game only care about their own interests. Cooperation offers the best promise to alleviate the

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problem and provide a group optimal and individually rational solution [31]. According to the cooperation procedure, cooperative games include static cooperative games and differential cooperative games. Compared with the former, differential games can investigate interactive decision-making over time. Stalford [27] first discussed differential cooperative games on a convex control set and offered two criteria to determine whether a Pareto-optimal control policy belongs to the boundary of the control set or the interior. Reddy and Engwerda [25] offered the necessary and sufficient conditions for the existence of Pareto solutions in infinite horizon cooperative differential games with an open-loop information structure. Petroysan [20] and Jorgensen et al. [11] discussed the time consistency in differential cooperative games to maintain the stability of cooperation. Yeung et al. [32] analyzed sub-game consistent solutions in cooperative stochastic differential games with nontransferable payoffs. The applications about cooperative differential games can be referred to Huang et al. [9] and Li [14]. Filar and Petrosyan [6] defined the characteristic functions of differential cooperative games, which make the theories of static cooperative games extend to differential cooperative games.

In some cases, players usually form a prior union with other players to increase their voice and interests. Then, players in one union cooperate with other players as one player (for instance, trade coalitions, currency blocs, and political and economic unions). People call this type of cooperation a cooperative game with a coalition structure. Many scholars studied static cooperative games with a coalition structure, and a series of results have been obtained [2–4, 12, 17]. Like other cooperative game models, the payoff indices of static cooperative games with a coalition structure are the main research topic. Among them, the Owen value [17] is one of the most important indices that can be seen as the extension of the Shapley value [26]. Many scholars devoted themselves to studying its theory and application [1, 8, 13, 15, 16, 28]. On the other hand, some scholars studied non-cooperative differential games with a coalition structure. For example, Petrosyan and Mamkina [22] explored the properties of multistage games with perfect information and coalition structures and proposed a new imputation value in terms of a PMS vector. Wang et al. [30] investigated the problem of strategic stability of long-range cooperative agreements in differential games with coalition structure, and the conditions of Nash equilibrium and strong Nash equilibrium were obtained.

In some situations,e differential cooperative games cannot cope with them sufficiently. For example, environmental governance is a long-term activity (the Kyoto Protocol specifies emission targets for the period 2008–2012). Countries's strategic choices will change over time. Furthermore, different countries will form a prior union to increase their voice and interests (such as the EU, NATO, BRICS, etc.), which will act as one player. Then, how to distribute the cost of reducing emissions among the countries in the environmental governance? This is a problem of differential cooperative games with a coalition structure, which previous differential cooperative game models cannot solve. Therefore, we present differential cooperative games with a coalition structure to extend the application. According to the characteristic function defined by Gromavo and Petrosyan [7], the specific expression of the Owen value of differential cooperative games with a coalition structure is defined, and its existence and uniqueness are discussed. Furthermore, the relationship between the core and the Owen value is shown. Considering the stability of differential cooperative games, an additional stringent condition on the solution is required: the specific optimality principle must remain optimal at any instant of time throughout the game along the optimal state trajectory chosen at the outset Yeung and Petrosyan [31]. Sub-game consistent solutions are derived to ensure sustainable cooperation. In particular, sub-game consistency guarantees that the optimality principle agreed upon at the outset will remain effective throughout the game. Hence, there is no incentive for any player to deviate from the cooperation scheme. Based on the research for sub-game consistency in differential cooperative games [19, 21, 21, 31, 32] we further give the concept of the sub-game consistent solution for differential cooperative games with a coalition structure. However, the sub-game consistent solution does not always exist. Then, we obtain the Owen value with sub-game consistency by modifying the defined characteristic function, which ensures that the players will not terminate the contract or deviate from the original cooperation plan.

The paper is organized as follows. Section 2 presents the basic concepts of differential cooperative games. Section 3 gives the definitions of characteristic function and differential cooperative games with a coalition structure. Section 4 offers the specific expression of the Owen value and discusses its existence and uniqueness. Section 5 studies the Owen value with sub-game consistency by modifying the characteristic function. Section 6 provides an application in environmental management. Concluding remarks are given in Section 7.

2. Preliminaries

Consider n-player nonzero-sum differential cooperative game $\Gamma(x_0, T – t_0)$ with initial state x_0 and duration $T – t_0$, in which the state dynamics has the form [18]:

$$\dot{x}(t) = f(t, x(t), u_1(t), u_2(t), \ldots, u_n(t)), \quad x(t_0) = x_0 \tag{1}$$

The payoffs of player i is

$$\int_{t_0}^{T} g^i\,(t, x(t), u_1(t), u_2(t), \ldots, u_n(t))dt + q^i(x(T)) \tag{2}$$

where $g^i\,(t, x(t), u_1(t), u_2(t), \ldots, u_n(t)) \geq 0$, $q^i(x(T)) \geq 0$, $x(t) \in X \subset R^n$ denotes the state variable of game, $u_i \in U_i$ is the control of player i, $i \in N = \{1, 2, 3, \ldots, n\}$. In particular, the players' payoffs are transferable. We assume that differential equation (1) satisfies all conditions necessary for the existence, sustainability and uniqueness of the solution for any n-tuple control $u^*(t) = (u_1^*(t), u_2^*(t), \ldots u_n^*(t))$.

Definition 1. Suppose [18] that there are an n-tuple control $u^*(t) = (u_1^*(t), u_2^*(t), \ldots u_n^*(t))$ and a trajectory $x^*(t)$, $t \in [t_0, T]$, such that

$$\begin{aligned} &\max_{u_1(s), \ldots, u_n(s)} \left(\sum_{i=1}^{n} \int_{t_0}^{T} g^i[t, x(t), u_1(t), u_2(t), \ldots, u_n(t)]dt + \sum_{i=1}^{n} q^i(x(T)) \right) \\ &= \sum_{i=1}^{n} \int_{t_0}^{T} g^i[t, x^*(t), u_1^*(t), u_2^*(t), \ldots, u_n^*(t)]dt + \sum_{i=1}^{n} q^i(x^*(T)) \end{aligned} \tag{3}$$

We call a trajectory $\{x^*(t)\}_{t=t_0}^{T}$ satisfying equation (3) an optimal cooperative trajectory. For simplicity, we use $x^*(t)$ and x_t^* interchangeably.

Assume that the players agree to adopt the control $u^*(t) = (u_1^*(t), u_2^*(t), \dots, u_n^*(t))$ and the differential cooperative game $\Gamma(x_0, T\text{–}t_0)$ always develops along the optimal trajectory $x^*(t)$.

Definition 2. The characteristic function of the differential cooperative game $\Gamma(x_0, T\text{–}t_0)$ is defined as [18]:

$$V(x_0, T\text{–}t_0, S) = \begin{cases} \sum_{i=1}^{n} \int_{t_0}^{T} g^i[t, x^*(t), u_1^*(t), u_2^*(t), \dots, u_n^*(t)]dt + \sum_{i=1}^{n} q^i(x^*(T)), & S = N \\ \mathrm{Val}\Gamma_{S,N\setminus S}(x_0, T\text{–}t_0), & S \subset N \\ 0, & S = \varnothing \end{cases}$$

where $\mathrm{Val}\Gamma_{S,N\setminus S}(x_0, T\text{–}t_0)$ is the value of the zero-sum game when the coalition S acts as one player and the coalition $n|S$ acts as the other player. Furthermore, the value of the coalition S equals to

$$\sum_{i\in S} \int_{t_0}^{T} g^i[t, x(t), u_1(t), u_2(s), \dots, u_n(t)]dt + \sum_{i\in S} q^i(x(T))$$

Let $\Gamma(x_0, T\text{–}t_0, V, N)$ be an n-player differential cooperative game with V being the characteristic function, and let $\Psi_N(x_0, T\text{–}t_0)$ be the family of all n-player differential cooperative games with the initial state x_0 and the duration $T\text{–}t_0$. Next, we review the imputation set and core of differential cooperative games.

Definition 3. Let $\Gamma(x_0, T\text{–}t_0, V, N)$ be an n-player differential cooperative game. Then, its imputation set is defined as [29]:

$$L(x_0, T\text{–}t_0, V, N) = \left\{\xi = (\xi_1, \dots, \xi_n) : \sum_{i=1}^{n} \xi_i = V(x_0, T\text{–}t_0, N), \xi_i \geq V(x_0, T\text{–}t_0, i), i \in N\right\}$$

Definition 4. Let $\Gamma(x_0, T\text{–}t_0, V, N)$ be an n-player differential cooperative game. Then, its core is defined as [29]:

$$C(x_0, T\text{–}t_0, V, N) = \left\{\xi = (\xi_1, \xi_2, \dots, \xi_n) : \sum_{i\in S} \xi_i \geq V(x_0, T\text{–}t_0, S), \forall S \subset N, \sum_{i\in N} \xi_i = V(x_0, T\text{–}t_0, N)\right\}$$

To study the sub-game consistency, we consider the family of sub-games of the differential cooperative game $\Gamma(x_0, T\text{–}t_0, V, N)$ for the optimal trajectory $\Gamma(x^*(t), T - t, V, N)$, i.e., the family of differential cooperative games with the initial position $x^*(t)$ that defines on the time interval $[t, T]$, $t \in [t_0, T]$.

The characteristic function of $\Gamma(x^*(t), T - t, V, N)$ is defined as:

$$V(x^*(t), T-t, S) = \begin{cases} \sum_{i=1}^{n} \int_{t}^{T} g^i[t, x^*(t), u_1^*(t), u_2^*(s), \ldots, u_n^*(t)]dt + \sum_{i=1}^{n} q^i(x^*(T)), & S = N \\ \mathrm{Val}\Gamma_{S,N\setminus S}(x^*(t), T-t), & S \subset N \\ 0, & S = \varnothing \end{cases}$$

Notations are as in Definition 2.

Accordingly, we can define the imputation set $L(x^*(t), T–t, V, N)$ and the core $C(x^*(t), T–t, V, N)$ for the sub-game $\Gamma(x^*(t), T–t, V, N)$. The key to deducing the cooperative solution of sub-games is to establish an imputation distribution procedure (IDP), which is defined as:

Definition 5. Let $\xi^* = (\xi_1^*, \xi_2^*, \ldots, \xi_n^*) \in L(x_0, T–t_0, V, N)$. $\beta(t) = \{\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\}$ is called an IDP if [19]

$$\xi_i^* = \int_{t_0}^{T} \beta_i(t)dt, \quad \beta_i(t) \geq 0$$

for all $i \in N$.

Let $\bar{\xi}_i(t) = \int_{t_0}^{t} \beta_i(t)dt$, $i = 1, 2, \ldots, n$, and $OP(x_0, T–t_0, V, N) \subseteq L(x_0, T–t_0, V, N)$ be any of the known classical optimal principles from the cooperative game theory (core, Shapley value, Banzhaf value or any other optimality principle). Consider $C(x_0, T–t_0, V, N)$ as an optimality principle for $\Gamma(x_0, T–t_0, V, N)$. Similarly, define $C(x^*(t), T–t, V, N)$ as an optimality principle for the game $\Gamma(x^*(t), T-t, V, N)$, $t \in [t_0, T]$. Then, we give the definition of sub-game consistency for differential cooperative games as follows.

Definition 6. The optimality principle (OP) $C(x_0, T–t_0, V, N)$ is sub-game consistent for the differential cooperative game $\Gamma(x_0, T–t_0, V, N)$ if there exists an IDP $\beta(t) = \{\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\}$ such that [19]

$$\xi^* - \bar{\xi}(t) \in C(x^*(t), T–t, V, N)$$

for all $t \in [t_0, T]$. $C(x_0, T–t_0, V, N)$ is called strongly sub-game consistent if there exists an IDP $\beta(t) = \{\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\}$ such that

$$\bar{\xi}(t) \oplus C(x^*(t), T–t, V, N) \subseteq C(x_0, T–t_0, V, N)$$

for all $t \in [t_0, T]$, where $a \oplus B$ is $\{a + b | \, a \in R^n, b \in B, B \subseteq R^n\}$.

3. Differential cooperative games with a coalition structure

In this section, we introduce the definition of differential cooperative games with a coalition structure and set up standard terminologies and notations.

Let $P = \{S_1, S_2, \ldots, S_m\}$ be a partition of the player set N, i.e., $\bigcup_{k=1}^{m} S_k = N$ and $S_k \bigcap S_l = \varnothing$, for any $k \neq l$, where $k, l \in M = \{1, 2, \ldots, m\}$. A coalition structure in n is denoted by (n, P). For any S, $S \in F(n, P)$ is called a feasible coalition, where $F(n, P)$ denotes the set of all feasible coalitions in (n, P), i.e.,

$$F(n, P) = \left\{ S \subseteq N | \, S = T \bigcup Q, \forall T \subseteq S_k, k \in M, \forall Q = \bigcup_{l \in H} B_l, H \subseteq M \backslash k \right\}$$

Example 1. Llet $N = \{1, 2, 3, 4\}$ and $P = \{S_1, S_2\}$, where $S_1 = 1, 2$ and $S_2 = 3, 4$, then $F(n, P) = \{\varnothing, \{i\}(i \in N), \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N\}$.

Based on the above, we define differential cooperative games with a coalition structure as follows.

Definition 7. Let $\Gamma(x_0, T\text{–}t_0, V, N) \in \Psi_N(x_0, T\text{–}t_0)$, and $P = \{S_1, S_2, \ldots, S_m\}$ be a partition of the player set n. Player i takes equation (1) and equation (2) as the state dynamics and the payment function, respectively. Let $u_S = \{u_i, i \in S\}$ and $x_S = \{x_i, i \in S\}$ be the control strategy and trajectory of the feasible coalition S, $S \in F(n, P)$, respectively. Define

$$\sum_{i \in S} \int_{t_0}^{T} g^i \left(t, x(t), u_1(t), u_2(t), \ldots, u_n(t)\right) dt + \sum_{i \in S} q^i(x(T))$$

as the payment function of feasible coalition S. We call the above game differential cooperative games with a coalition structure denoted by $\Gamma(x_0, T\text{–}t_0, V^P, n, P)$, and we denote $\Psi_N(x_0, T\text{–}t_0, P)$ as the family of all n-player differential cooperative games with a coalition structure with the initial state x_0 and duration $T\text{–}t_0$.

Two essential properties that a cooperative scheme has to satisfy are group optimality and individual rationality. To satisfy group optimality, the players will maximize their joint payoff by solving the dynamic optimization problem which maximizes

$$\begin{aligned} &W(x_0, t_0, N) \\ &= \max_{\substack{u_i(t) \\ i \in N}} \left(\sum_{S_k \in P} \left(\sum_{i \in S_k} \int_{t_0}^{T} g^i[t, x(t), u_1(t), u_2(t), \ldots, u_n(t)] dt \right) + \sum_{S_k \in P} \left(\sum_{i \in S_k} q^i(x(T)) \right) \right) \end{aligned} \tag{4}$$

We can obtain the optimal control $u^*(t) = (u_1^*(t), u_2^*(t), \ldots, u_n^*(t))$ satisfying equation (4) by Bellman's dynamic programming theory. Denote $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))$ as the optimal trajectory of equation (4).We use it to denote the optimal cooperative trajectory.

Definition 8. Let $\Gamma(x_0, T\text{–}t_0, V^P, n, P) \in \Psi_N(x_0, T\text{–}t_0, P)$, if there exists an n-tuple control $u^*(t) = (u_1^*(t), u_2^*(t), \ldots u_n^*(t))$ and a trajectory $x^*(t)$, $t \in [t_0, T]$ such that

$$\max_{\substack{u_i(t)\\ i\in N}} \left(\sum_{S_k\in P} \left(\sum_{i\in S_k} \int_{t_0}^{T} g^i\big(t, x(t), u_1(t), u_2(t), \ldots, u_n(t)\big)dt \right) + \sum_{S_k\in P} \left(\sum_{i\in S_k} q^i(x(T)) \right) \right)$$
$$= \sum_{S_k\in P} \sum_{i\in S_k} \int_{t_0}^{T} g^i\big(t, x^*(t), u_1^*(t), u_2^*(t), \ldots, u_n^*(t)\big)dt + \sum_{S_k\in P} \sum_{i\in S_k} q^i(x^*(T)) \tag{5}$$

We call a trajectory $\{x^*(t)\}_{t=t_0}^{T}$ satisfying equation (5) an optimal cooperative trajectory for the game $\Gamma(x_0,\, T\text{–}t_0, V^P, n,\, P)$. For simplicity, we use $x^*(t)$ and x_t^* interchangeably.

Then, similar to Definition 2 we define the characteristic function for $\Gamma(x_0,\, T\text{–}t_0, V^P, n,\, P)$ as follows.

Definition 9. Let $\Gamma(x_0,\, T\text{–}t_0, V^P, n,\, P) \in \Psi_N(x_0,\, T\text{–}t_0, P)$, the characteristic function of $\Gamma(x_0,\, T\text{–}t_0, V^P, n,\, P)$ is defined as:

$$V^P(x_0,\, T\text{–}t_0, S) = \begin{cases} W(x_0, t_0, N), & S = N \\ W(x_0, t_0, S), & S \in F(n,\, P) \\ 0, & S = \varnothing \end{cases} \tag{6}$$

where the payoff $W(x_0, t_0, S)$ of coalition S equals

$$W(x_0, t_0, S) = \min_{\substack{u_i(t)=u_i^*(t), i\in S\\ u_j(t), j\in N\setminus S}} \left(\sum_{i\in S} \int_{t_0}^{T} g^i\big(t, x(t), u_1(t), u_2(t), \ldots, u_n(t)\big)dt + \sum_{i\in S} q^i(x(T)) \right)$$

$u_i^*(t)$,$i \in S$ are the optimal controls of equation (5) for the players who belong to coalition S.

Definition 9 is based on the fact that players from S use the control $u_S^*(t) = u_i^*(t), i \in S$ from the optimal n-tuple $u^*(t)$, while the other players, those from the set $n \setminus S$, minimize the payoff of the coalition S. The characteristic function equation (6) has the following advantages. First, the characteristic function is superadditive [7]. Second, it can be computed in two stages using the expression of optimal control, which greatly simplifies the computation process compared with Definition 2. Furthermore, the new characteristic function can be used for the differential cooperative games with a coalition structure.

Based on above discussion for differential cooperative games with a coalition structure, we give the following basic concepts for game $\Gamma(x_0,\, T\text{–}t_0, V^P, n,\, P)$.

Definition 10. Let $\Gamma(x_0,\, T\text{–}t_0, V^P, n,\, P) \in \Psi_N(x_0,\, T\text{–}t_0, P)$, if $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ satisfies

$$\begin{cases} \sum_{i=1}^{n} \xi_i = V^P(x_0,\, T\text{–}t_0, N) \\ \xi_i \geq V^P(x_0,\, T\text{–}t_0, i),\; i \in N \end{cases}$$

then ξ is called an imputation in $\Gamma(x_0,\, T\text{–}t_0, V^P, n,\, P)$, where ξ_i represents the payoff of player *i*. The set of all imputations in $\Gamma(x_0,\, T\text{–}t_0, V^P, n,\, P)$ is denoted by $L(x_0,\, T\text{–}t_0, V^P, n,\, P)$, i.e.,

$$L(x_0,\, T–t_0, V^P, n,\, P) \\ = \left\{ \xi = (\xi_1,\, \dots,\, \xi_n) : \sum_{i=1}^{n} \xi_i = V^P(x_0,\, T–t_0, N), \xi_i \geq V^P(x_0,\, T–t_0, i), i \in N \right\}$$

Condition $\xi_i \geq V(x_0,\, T–t_0, i)$ $(i = 1, 2, 3,\, \dots,\, n)$ shows that the payoff received by the players under cooperation is not less than the payoff under non-cooperation (i.e., individual rationality), and $\sum_{i=1}^{n} \xi_i = V^P(x_0,\, T–t_0, N)$ represents that the sum of all the players' payoff is equal to the grand coalition's payoff (i.e., efficiency). Therefore, similar to static cooperative games, imputation set in the game $\Gamma(x_0,\, T–t_0, V^P, n,\, P)$ refers to the set of all n-dimensional vectors ξ that satisfies individual rationality and efficiency.

Definition 11. Let $\Gamma(x_0,\, T–t_0, V^P, n,\, P) \in \Psi_N(x_0,\, T–t_0, P)$, define $C(x_0,\, T–t_0, V^P, n,\, P)$ as the core of $\Gamma(x_0,\, T–t_0, V^P, n,\, P)$:

$$C(x_0, XT–t_0, V^P, n, P) = \left\{ \xi = (\xi_1, \xi_2, \dots, \xi_n) : \sum_{i \in S} \xi_i \geq V^P(x_0, T\text{S}–t_0, S), \\ \forall S \in F(n, P), \sum_{i \in N} \xi_i = V^P(x_0, T–t_0, N) \right\}$$

For differential cooperative games, the core refers to a set of imputations in which no player can make efforts to improve his payoff. If $P = \{\{1\}, \{2\},\, \dots,\, \{n\}\}$, then the core of differential cooperative games with a coalition structure degenerates into the core of traditional differential cooperative games.

We consider the family of sub-games of the game $\Gamma(x_0,\, T–t_0, V^P, n,\, P)$ for the optimal trajectory $\Gamma(x^*(t), T–t, V^P, n,\, P)$, i.e., the family of differential cooperative games with a coalition structure from the initial position $x^*(t)$ that defines on the time interval $[t, T]$ $(t \in [t_0, T])$. We denote $V^P(x^*(t), T–t, S)$ as the characteristic function of $\Gamma(x^*(t), T–t, V^P, n,\, P)$. Similarly, we can defi the imputation set $L(x^*(t), T–t, V^P, n,\, P)$ and the core $C(x^*(t), T–t, V^P, n,\, P)$ for the sub-game $\Gamma(x^*(t), T–t, V^P, n,\, P)$.

Assumed that the core $C(x_0,\, T–t_0, V^P, n,\, P)$ is the optimality principle for $\Gamma(x_0,\, T–t_0, V^P, n,\, P)$. Let $\xi_i^* = \int_{t_0}^{T} \beta_i(t)dt$ and $\bar{\xi}_i(t) = \int_{t_0}^{t} \beta_i(t)dt$, $i \in N$, where $\xi^* = (\xi_1^*, \xi_2^*,\, \dots,\, \xi_n^*) \in C(x_0,\, T–t_0, V^P, n,\, P)$. Then, we can give the following definition of sub-game consistency for differential cooperative games with a coalition structure.

Definition 12. The optimality principle (OP) $C(x_0,\, T–t_0, V^P, n,\, P)$ is sub-game consistent for the game $\Gamma(x_0,\, T–t_0, V^P, n,\, P)$ if there exists an IDP $\beta(t) = \{\beta_1(t), \beta_2(t),\, \dots,\, \beta_n(t)\}$ such that

$$\xi^* - \bar{\xi}(t) \in C(x^*(t), T–\text{t}, V^P, n,\, P)$$

for all $t \in [t_0, T]$. $C(x_0,\, T–t_0, V^P, n,\, P)$ is called strongly sub-game consistent if there exists an IDP $\beta(t) = \{\beta_1(t), \beta_2(t),\, \dots,\, \beta_n(t)\}$ such that

$$\bar{\xi}(t) \oplus C(x^*(t), \text{T–t}, V^P, n,\ P) \subseteq C(x_0,\ T\text{–}t_0, V^P, n,\ P) \tag{7}$$

for all $t \in [t_0, T]$, where $a \oplus B$ is $\{a + b|\ a \in R^n, b \in B, B \subseteq R^n\}$.

If $P = \{\{1\}, \{2\}, \ldots, \{n\}\}$, we can easily find that Definition 12 is equivalent to Definition 6.

4. The Owen value

In this section, we give the definition of the Owen value for differential cooperative games with a coalition structure, and provide its explicit form.

The Owen value is computed in two steps. First, the Shapley value is used to distribute the payoff of the grand coalition among the prior unions. At this stage, each prior union is regarded as one player. Secondly, each prior union uses the Shapley value again to distribute the payoff obtained in the first stage among its members. The specific steps are as follows:

Step 1. Allocate the total payoff among the prior unions as the Shapley value of induced game played by the unions, When we calculate the payoff of the prior union S_k, the prior union S_k in *P* should be regarded as one player to participate in differential cooperative game. Then, we give the definition of differential quotient game as follows:

Definition 13. Let $\Gamma(x_0,\ T\text{–}t_0, V^P, n,\ P) \in \Psi_N(x_0,\ T\text{–}t_0, P)$ and $P = \{S_1, S_2, \ldots, S_m\}$, if we have $V^M(x_0,\ T\text{–}t_0, R) = V^P(x_0,\ T\text{–}t_0, \bigcup\limits_{\substack{i \in S_k \\ k \in R}} i)$ for any $R \subseteq M$, and take equation (1) and

$$\sum_{i \in S_k} \int_{t_0}^{T} g^i[t, x(t), u_1(t), u_2(t), \ldots, u_n(t)]dt + \sum_{i \in S_k} q^i(x(T))$$

as the state dynamics and the payment function of player *k*, $k \in M$, then we call the above game differential quotient game denoted by $\Gamma(x_0,\ T\text{–}t_0, V^M, M)$.

Then, according to Definition 7, we find that $u_{S_k} = \{u_i, i \in S_k\}$ and $x_{S_k} = \{x_i, i \in S_k\}$ are the control strategy and trajectory of player *k* in $\Gamma(x_0,\ T\text{–}t_0, V^M, M)$, respectively.

It can be known from the definition of differential quotient game that the payoff obtained by prior union S_k is the Shapley value of player *k* in differential quotient game $\Gamma(x_0, T\text{–}t_0, V^M, M)$, i.e., $Sh_k(x_0,\ T\text{–}t_0, V^M, M)$.

Step2. Distribute the payoff within each prior union by means of Shapley value. Based on the Shapley value of the prior union S_k obtained in Step 1, this step discusses how to distribute the total payoff of the prior union $Sh_k(x_0,\ T\text{–}t_0, V^M, M)$ among its members.

Let $P(S) = \{S_1, S_2, \ldots, S_{k-1}, S, S_{k-1}, \ldots, S_m\}$, for any $S \subseteq S_k \backslash \varnothing$, and define $\Gamma(x_0,\ T\text{–}t_0, V^{P(S)}, M)$ as the differential quotient game on $P(S)$, in which the characteristic function $V^{P(S)}(x_0,\ T\text{–}t_0, \text{C})$, $\text{C} \subseteq M$ is defined as:

$$V^{P(S)}(x_0,\ T\text{–}t_0, \text{C}) = V^P(x_0,\ T\text{–}t_0, \bigcup_{\substack{l \in \text{C} \\ i \in S_l}} i)$$

$\forall S \subset S_k$, let $\Gamma(x_0, T–t_0, V^{S_k}, S_k)$ denote the differential cooperative games defined on the prior union S_k. The characteristic function for the game $\Gamma(x_0, T–t_0, V^{S_k}, S_k)$ is defined as $V^{S_k}(x_0, T–t_0, S) = Sh_k(x_0, T–t_0, V^{P(S)}, M)$, where $Sh_k(x_0, T–t_0, V^{P(S)}, M)$ represents the Shapley value of player *k* in differential quotient game $\Gamma(x_0, T–t_0, V^{P(S)}, M)$.

From the above two steps, we can define the Owen value in differential cooperative games with a coalition structure as follows:

Definition 14. Let $\Gamma(x_0, T–t_0, V^P, n, P) \in \Psi_N(x_0, T–t_0, P)$. Define the Owen value as $Ow(x_0, T–t_0, V^P, n, P) = (Ow_1(x_0, T–t_0, V^P, n, P), Ow_2(x_0, T–t_0, V^P, n, P), \ldots, Ow_n(x_0, T–t_0, V^P, n, P))$, where $Ow_i(x_0, T–t_0, V^P, n, P)$ represents the Owen value of player *i*, and

$$Ow_i(x_0, T–t_0, V^P, n, P) = Sh_i(x_0, T–t_0, V^{S_k}, S_k) \tag{8}$$

Then, we can calculate the specific expression of the Owen value by the equation (8)as follows:

$$\begin{aligned} Ow_i(x_0, T–t_0, V^P, n, P) = & \sum_{\substack{R\subseteq M \\ k\notin R}} \sum_{i\in S\subseteq S_k} \frac{|R|!(|M|-|R|-1)!}{|M|!} \frac{(|S|-1)!(|S_k|-|S|)!}{|S_k|!} \\ & \times \left(V^P(x_0, T–t_0, \bigcup_{l\in R} S_l \bigcup S) - V^P(x_0, T–t_0, \bigcup_{l\in R} S_l \bigcup (S\backslash i)) \right) \end{aligned} \tag{9}$$

Similarly, the Owen value for the sub-game $\Gamma(x^*(t), T-t, V^P, n, P)$ ($t \in [t_0, T]$)is defined as follows.

$$\begin{aligned} Ow_i(x(t), T-t, V^P, n, P) = & \sum_{\substack{R\subseteq M \\ k\notin R}} \sum_{i\in S\subseteq S_k} \frac{|R|!(|M|-|R|-1)!}{|M|!} \frac{(|S|-1)!(|S_k|-|S|)!}{|S_k|!} \\ & \times \left(V^P(x(t), T-t, \bigcup_{l\in R} S_l \bigcup S) - V^P(x(t), T-t, \bigcup_{l\in R} S_l \bigcup (S\backslash i)) \right) \end{aligned}$$

Note 1. Since $g^i[t, x(t), u_1(t), u_2(t), \ldots, u_n(t)] \geq 0$, $q^i(x(T)) \geq 0$, the characteristic function $V^P \geq 0$. In addition, the characteristic function is superadditive, for any $S, T \in S(n, P)$, we have

$$V^P(x_0, T–t_0, S) + V^P(x_0, T–t_0, U) \leq V^P(x_0, T–t_0, S\bigcup U) + V^P(x_0, T–t_0, S\bigcap U)$$

where $S\bigcup U, S\bigcap U \in S(n, P)$. Thus, $\Gamma^P(x_0, T–t_0, V^P, N)$ is called differential convex cooperative game. We can know from the relationship between the Owen value and the core in static cooperative games that $Ow_i(x_0, T–t_0, V^P, n, P) \in C^P(x_0, T–t_0, V^P, N)$. Therefore, there is no player can make his own payoff greater than the Owen value without reducing other players' payoff.

Note 2. The Owen value considers the possibility of cooperation among coalitions, which is more in line with the actual situation. It greatly expands the application scope of differential cooperative games in practical situations. In addition, the results of this section can also be applied to differential cooperative games with infinite-horizon. According to the property of Owen value in static games, the uniqueness of the Owen value in differential cooperative games is also characterized by Efficiency, Intra-coalitional symmetry, Coalitional symmetry, Additivity and Null. Since Additivity is not easy to satisfy, we give a new characterization on the Owen value in differential cooperative games with a coalition structure based on the definition of Intra-coalitional balanced contributions for the Shapley value [15].

Denote $w(x_0,\, T–t_0, V^P, n,\, P) = \left(w_i(x_0,\, T–t_0, V^P, n,\, P)\right)_{i\in N}$ as an imputation vector or coalitional value for differential cooperative games with a coalition structure $\Gamma(x_0,\, T–t_0, V^P, n,\, P)$.

In static cooperative games, unanimity game is the important tool to prove the uniqueness of imputation value. Therefore, we give the following definition of differential unanimity game:

Definition 15. Let $\Gamma(x_0,\, T–t_0, V^P, n,\, P) \in \Psi_N(x_0,\, T–t_0, P)$, for any $H \subseteq N\backslash\varnothing$, if $H \subseteq S$, then $u^H(x_0,\, T–t_0, S) = 1$, or $u^H(x_0,\, T–t_0, S) = 0$, we call $\Gamma(x_0,\, T–t_0, u^H, N)$ the differential unanimity game.

Since the family $\left\{\Gamma(x_0,\, T–t_0, u^H, N)\right\}_{H\in 2^N\backslash\varnothing}$ is a basis for $\Psi_N(x_0,\, T–t_0)$, for any $\Gamma(x_0,\, T–t_0, V^P, N) \in \Psi_N(x_0,\, T–t_0)$, there are unique coefficients $\{c_H\}_{H\in 2^N\backslash\varnothing}$ such that

$$V(x_0,\, T–t_0, S) = \sum_{H\in 2^N\backslash\varnothing} c_H u^H(x_0,\, T–t_0, S)$$

where $c_H = \sum_{S\subseteq H}(-1)^{h-s}V(x_0,\, T–t_0, S)$. Moreover, $Sh_i(x_0,\, T–t_0, u^H, N) = \dfrac{1}{h}$, where $h = |H|$ and $s = |S|$.

Similar to the description of the axioms in static cooperative games, we introduce the following axioms used to characterize the Owen value for the game $\Gamma(x_0,\, T–t_0, V^P, n,\, P)$.

Efficiency (EFF). Let $\Gamma(x_0,\, T–t_0, V^P, n,\, P) \in \Psi_N(x_0,\, T–t_0, P)$, we have

$$\sum_{i\in N} w_i(x_0,\, T–t_0, V^P, n,\, P) = V(x_0,\, T–t_0, n,\, P)$$

Coalitional symmetry (CSY): Let $\Gamma(x_0,\, T–t_0, V^P, n,\, P) \in \Psi_N(x_0,\, T–t_0, P)$, for any $S_k, S_l \in P$, which satisfy $V^M(x_0,\, T–t_0, R\bigcup k) = V^M(x_0,\, T–t_0, R\bigcup l)$, for all $R \subseteq M\backslash\{k, l\}$, then

$$\sum_{i\in S_k} w_i(x_0,\, T–t_0, V^P, n,\, P) = \sum_{i\in S_l} w_i(x_0,\, T–t_0, V^P, n,\, P)$$

Coalitional marginality (CMA). Let $\Gamma(x_0,\, T–t_0, \tilde{V}^P, n,\, P)$, $\Gamma(x_0,\, T–t_0, V^P, n,\, P) \in \Psi_N(x_0,\, T–t_0, P)$, if $V^P(x_0,\, T–t_0, S\bigcup S_k) - V^P(x_0,\, T–t_0, S) = \tilde{V}^P(x_0,\, T–t_0, S\bigcup S_k) - \tilde{V}^P(x_0,\, T–t_0, S)$, for all $S \subseteq N\backslash S_k$, then

$$\sum_{i\in S_k} w_i(x_0,\, T–t_0, V^P, n,\, P) = \sum_{i\in S_k} w_i(x_0,\, T–t_0, \tilde{V}^P, n,\, P)$$

Intra-coalitional balanced contributions (IBC). Let $\Gamma(x_0,\, T–t_0, V^P, n,\, P) \in \Psi_N(x_0,\, T–t_0, P)$, for all $i, j \in S_k \in P$ with $i \neq j$,

$$\begin{aligned} &w_i(x_0,\, T–t_0, V^P, n,\, P) - w_i(x_0,\, T–t_0, V^P, N\backslash j, P_{N\backslash j}) \\ &= w_j(x_0,\, T–t_0, V^P, n,\, P) - w_j(x_0,\, T–t_0, V^P, N\backslash i, P_{N\backslash i}) \end{aligned}$$

where $P_{N\backslash j}=\{S_l\in P: l\neq k\}\bigcup\{S_k\backslash j\}$.

Lemma 1. Let $\Gamma(x_0,\,T\text{–}t_0,V^P,n,\,P)\in\Psi_N(x_0,\,T\text{–}t_0,P)$, for all $i\in S_k$ with $S_k\in P$, a coalitional value w satisfies IBC if and only if [15]

$$w_i(x_0,\,T\text{–}t_0,V^P,n,\,P)=Sh_i(x_0,\,T\text{–}t_0,S_k,V^{S_k})$$

where $V^{S_k}(x_0,\,T\text{–}t_0,S)=\sum_{i\in S}w_i\left(x_0,\,T\text{–}t_0,V^P,(N\backslash S_k)\bigcup S,P_{(N\backslash S_k)\bigcup S}\right)$, for all $S\subseteq S_k$.

Theorem 1. Let $\Gamma(x_0,\,T\text{–}t_0,\,V,\,n,\,P)\in\Psi_N(x_0,\,T\text{–}t_0,P)$, the coalitional value *w* in $\Gamma(x_0,\,T\text{–}t_0,\,V,\,n,\,P)$ is called the Owen value if and only if *w* satisfies EFF, IBC, CSY and CMA.

Proof. It can be known easily from the equation (9) that the Owen value satisfies the four axioms listed, and the existence is established. The following proof is unique.

First, we prove that if the coalitional value *w* satisfies EFF, CSY and CMA, then

$$\sum_{i\in S_k}w_i(x_0,\,T\text{–}t_0,V^P,n,\,P)=Sh_k(x_0,\,T\text{–}t_0,M,V^M)$$

Given $\Gamma(x_0,\,T\text{–}t_0,V^P,n,\,P)\in\Psi_N(x_0,\,T\text{–}t_0,P)$, we know that

$$V^P(x_0,\,T\text{–}t_0)=\sum_{H\in 2^N\backslash\varnothing}c_Hu^H(x_0,\,T\text{–}t_0)$$

Then, for all $H\subseteq N$, with $H\neq\varnothing$, let $\Gamma^M(x_0,\,T\text{–}t_0,V^M,M)$ us denote $H_k=S_k\bigcap H$, and $M_H=\{k\in M: H_k\neq\varnothing\}$. The differential quotient game can also be expressed in terms of differential unanimity games as $V^M(x_0,\,T\text{–}t_0)=\sum_{H\in 2^N\backslash\varnothing}c_Hu^{M_H}(x_0,\,T\text{–}t_0)$ since, for all $R\subseteq M$

$$V^M(x_0,\,T\text{–}t_0,R)=V^P(x_0,\,T\text{–}t_0,\bigcup_{k\in R}S_k)$$

$$=\sum_{H\in 2^N\backslash\varnothing}c_Hu^H(x_0,\,T\text{–}t_0,\bigcup_{k\in R}S_k)=\sum_{H\in 2^N\backslash\varnothing}c_Hu^{M_H}(x_0,\,T\text{–}t_0,R)$$

Thus, since the Shapley value satisfies the axiom of ADD, the proof is finished if we prove that

$$\sum_{i\in S_k}w_i(x_0,\,T\text{–}t_0,V^P,n,\,P)=Sh_k(x_0,\,T\text{–}t_0,\sum_{H\in 2^N\backslash\varnothing}c_Hu^{M_H},M)$$

$$=\sum_{H\in 2^N\backslash\varnothing:k\in M_H}\frac{c_H}{|M_H|}$$

For any differential cooperative games with a coalition structure $\Gamma(x_0,\,T\text{–}t_0,V^P,n,\,P)$, let us denote *I* as the minimum number of unanimity games necessary to represent V^P in terms of these differential unanimity games. Also in both cases, the proof will be done by induction on the cardinality of *I*.

If $I = 0$, then $V^P(x_0, T–t_0, S) = 0$, for all $S \subseteq N$. According to EFF,

$$\sum_{i\in N} w_i(x_0, T–t_0, V^P, n, P) = \sum_{S_k\in P}\sum_{i\in S_k} w_i(x_0, T–t_0, V^P, n, P) = 0$$

Since any pair of unions in this game are symmetric, by CSY we deduce that

$$\sum_{i\in S_k} w_i(x_0, T–t_0, V^P, n, P) = Sh_k(x_0, T–t_0, M, V^M) = 0, \forall S_k \in P$$

Let us now assume that $\sum_{i\in S_k} w_i(x_0, T–t_0, V^P, n, P) = Sh_k(x_0, T–t_0, V^M, M)$ for $I \leq q$, where q is a non-negative integer.

Next, consider the case $I = q + 1$, i.e., $V^P(x_0, T–t_0) = \sum_{l=1}^{q+1} c_{H_l} u^{H_l}(x_0, T–t_0)$, and denote $H = \bigcap_{l=1}^{q+1} H^l$. For each $S_k \in P$, we distinguish two possibilities:

Case 1. $k \notin M_H$, define the differential cooperative games with a coalition structure $\Gamma(x_0, T–t_0, V^k, n, P)$, where $V^k(x_0, T–t_0) = \sum_{l\in\{1,2,\dots,q+1\}:k\in M_{H^l}} c_{H^l} u^{H^l}(x_0, T–t_0)$. Obviously, $I \leq q$. By induction hypothesis

$$\sum_{i\in S_k} w_i(x_0, T–t_0, V^k, n, P) = \sum_{l\in\{1,2,\dots,q+1\}:k\in M_{H^l}} \frac{c_{H^l}}{|M_{H^l}|}$$

For any $S \subseteq N\backslash S_k$,

$$V^k(x_0, T–t_0, S\bigcup S_k) - V^k(x_0, T–t_0, S) = \sum_{l\in\{1,2,\dots,q+1\}:k\in M_{H^l}} c_{H^l} u^{H^l}\left(x_0, T–t_0, S\bigcup S_k\right)$$

$$= V^P(x_0, T–t_0, S\bigcup S_k) - V^P(x_0, T–t_0, S)$$

Thus, by CMA

$$\sum_{i\in S_k} w_i(x_0, T–t_0, V^P, n, P) = \sum_{i\in S_k} w_i(x_0, T–t_0, V^k, n, P) = \sum_{l\in\{1,2,\dots,q+1\}:k\in M_{H^l}} \frac{c_{H^l}}{|M_{H^l}|}$$

Case 2: $k \in M_H$, By EFF

$$\sum_{k\in M_H}\sum_{i\in S_k} w_i(x_0, T–t_0, V^P, n, P) = V^P(x_0, T–t_0, N) - \sum_{k\notin M_H}\sum_{i\in S_k} w_i(x_0, T–t_0, V^P, n, P)$$

$$= \sum_{l=1}^{q+1} c_{H^l} - \sum_{l=1}^{q+1} (|M_{H^l}| - |M_H|) \frac{c_{H^l}}{|M_{H^l}|} = |M_H| \sum_{l=1}^{q+1} \frac{c_{H^l}}{|M_{H^l}|}$$

When $|M_H| \geq 2$, for all $r \in M_H$, $V^P(x_0, T–t_0, S\bigcup S_k) = V^P(x_0, T–t_0, S\bigcup S_r)$, and CSY implies that $\sum_{i\in S_k} w_i(x_0, T–t_0, V^P, n, P) = \sum_{l=1}^{q+1} \frac{c_{H^l}}{|M_{H^l}|}$.

Thus, we have $\sum_{i\in S_k} w_i(x_0,\, T\text{–}t_0, V^P, n,\; P) = Sh_k(x_0,\, T\text{–}t_0, V^M, M)$.

Because the coalitional value w satisfies IBC, by Lemma 1 we have that for all $i \in S_k \in P$,

$$w_i(x_0,\, T\text{–}t_0, V^P, n,\; P) = Sh_i(x_0,\, T\text{–}t_0, S_k, V^{S_k})$$

where for all $S \subseteq S_k$

$$V^{S_k}\,(x_0,\, T\text{–}t_0, S) = \sum_{i\in S} w_i\left(x_0,\, T\text{–}t_0, V^P, (N\backslash S_k)\bigcup S, P_{(N\backslash S_k)\bigcup S}\right)$$

$$= Sh_k\left(x_0,\, T\text{–}t_0, V^{P(S)}, M\right)$$

It means that w is uniquely determined and leads us to deduce that for all $\Gamma(x_0,\, T\text{–}t_0, V^P, n,\; P) \in \Psi_N(x_0,\, T\text{–}t_0, P)$ and all $i \in S_k \in P$

$$w_i(x_0,\, T\text{–}t_0, V^P, n,\; P) = Ow_i(x_0,\, T\text{–}t_0, V^P, n,\; P)$$

□

It should be noted that, different from static cooperative games, due to the particularity of characteristic functions for differential cooperative games (which change with time), the axioms given in this paper are also based on the dynamic perspective.

5. Sub-game consistency

As described in the introduction, when the game proceeds along the "optimal" trajectory, we should guarantee that the initially agreed optimality principle is still optimal for all players. Therefore, in this section, we will consider the sub-game consistency of the Owen value.

From the definition of sub-game consistency for the differential cooperative games with a coalition structure, it can be known that the optimal solution ξ needs to satisfy

$$\bar{\xi}(t) \oplus C(x^*(t), T-t, V^P, n,\; P) \subseteq C(x_0,\, T\text{–}t_0, V^P, n,\; P), t \in [t_0, T]$$

Suppose that the set $C(x_0,\, T\text{–}t_0, V^P, n,\; P)$ consists of the unique imputation: the Owen value. In this case from sub-game consistency the strong sub-game consistency follows immediately. Condition $\bar{\xi}(t) \oplus C(x^*(t), T-t, V^P, n,\; P) \subseteq C(x_0,\, T\text{–}t_0, V^P, n,\; P)$ can be rewritten in the form

$$Ow(x_0,\, T\text{–}t_0, V^P, n,\; P) = \int_{t_0}^{t} \beta(\tau)\mathrm{d}\tau + Ow(x^*(t), T-t, V^P, n,\; P)$$

(here $Ow(x^*(t), T-t, V^P, n,\; P)$ represents the Owen value for the sub-game $\Gamma(x^*(t), T-t, V^P, n,\; P)$), which give us the expression for $\beta(t)$

$$\beta(t) = -Ow'(x^*(t), T-t, V^P, n,\; P)$$

If we suppose the differentiability of $V^P(x^*(t), T-t, S)$, $t \in [t_0, T]$ along $x^*(t)$, then

$$\beta_i(t) = -Ow'_i(x^*(t), T-t, V^P, n,\ P)$$
$$= -\sum_{\substack{R\subseteq P\\ k\notin R}} \sum_{i\in S\subseteq S_k} \frac{|R|!(|M|-|R|-1)!}{|M|!} \frac{(|S|-1)!(|S_k|-|S|)!}{|S_k|!}$$
$$\times \left[(V^P)'(x^*(t), T-t, \bigcup_{l\in R} S_l \bigcup S) - (V^P)'(x^*(t), T-t, \bigcup_{l\in R} S_l \bigcup (S\backslash i)) \right]$$

The above expression shows that condition $\beta_i(t) \geq 0$, $i \in \{1, 2, \ldots, n\}$ may not take place, since the differences in brackets may take negative values. Thus $\beta_i(t)$ may not be an IDP, which means that the Owen value for characteristic function $V^P(x^*(t), T-t, S)$, $t \in [t_0, T]$ may be not sub-game consistent. To this end, we construct a new characteristic function $\bar{V}^P(x_0,\ T–t_0, S)$, $S \subseteq N$ by the formula

$$\bar{V}^P(x_0,\ T–t_0, S) = -\int_{t_0}^{T} V^P(x^*(t), T-t, S) \frac{(V^P)'(x^*(t), T-t, N)}{V^P(x^*(t), T-t, N)} \mathrm{d}t$$

In the same way, for $t \in [t_0, T]$

$$\bar{V}^P(x^*(t), T-t, S) = -\int_{t}^{T} V^P(x^*(t), T-t, S) \frac{(V^P)'(x^*(t), T-t, N)}{V^P(x^*(t), T-t, N)} \mathrm{d}t \tag{10}$$

We can find easily that the characteristic function $\bar{V}^P(x_0,\ T–t_0, S)$, $S \subseteq N$ is superadditive for the superadditivity of $V^P(x_0,\ T–t_0, S)$, $S \subseteq N$, which is proofed in Theorem 1.

Theorem 2. Let $\Gamma(x_0,\ T–t_0, V^P, n,\ P) \in \Psi_N(x_0,\ T–t_0, P)$, the characteristic function $\bar{V}^P(x_0,\ T–t_0, S)$, $S \subseteq N$ defined by equation (10) is superadditive for $\Gamma(x_0,\ T–t_0, V^P, n,\ P)$.

Proof.

$$\bar{V}^P(x_0,\ T–t_0, S_1 \bigcup S_2) = -\int_{t_0}^{T} V^P(x^*(t), T-t, S_1 \bigcup S_2) \frac{(V^P)'(x^*(t), T-t, N)}{V^P(x^*(t), T-t, N)} \mathrm{d}t$$
$$\geq -\int_{t_0}^{T} V^P(x^*(t), T-t, S_1) \frac{(V^P)'(x^*(t), T-t, N)}{V^P(x^*(t), T-t, N)} \mathrm{d}t$$
$$-\int_{t_0}^{T} V^P(x^*(t), T-t, S_2) \frac{(V^P)'(x^*(t), T-t, N)}{V^P(x^*(t), T-t, N)} \mathrm{d}t$$
$$= \bar{V}^P(x_0,\ T–t_0, S_1) + \bar{V}^P(x_0,\ T–t_0, S_2)$$

for any $S_1 \subset N$, $S_2 \subset N$, $S_1 \bigcap S_2 = \varnothing$. □

Theorem 3. Let $\Gamma(x_0,\, T\text{–}t_0, V^P, n,\ P) \in \Psi_N(x_0,\, T\text{–}t_0, P)$, the Owen value defined for the "refined" characteristic function $\bar{V}^P(x_0,\, T\text{–}t_0, S)$, $S \subseteq N$ is sub-game consistent for $\Gamma(x_0,\, T\text{–}t_0, V^P, n,\ P)$.

Proof. For any $\xi(t) \in C(x^*(t), T-t, V^P, n,\ P)$, define the IDP $\beta(t)$, $t \in [t_0, T]$ by the formula

$$\beta_i(t) = \frac{\xi_i(t) \sum\limits_{S_k \in P} \sum\limits_{i \in S_k} (g^i[t, x^*(t), u_1^*(t), u_2^*(t),\ \ldots,\ u_n^*(t)] + q^i(x^*(T))}{V^P(x^*(t), T-t, N)}$$

$$= -\frac{\xi_i(t)(V^P)'(x^*(t), T-t, N)}{V^P(x^*(t), T-t, N)} \geq 0$$

If the optimal principle $C(x^*(t), T-t, V^P, n,\ P)$ consists of the unique imputation, the Owen value, i.e.,

$$C(x^*(t), T-t, V^P, n,\ P) = Ow(x^*(t), T-t, V^P, n,\ P) = \xi(t)$$

where

$$Ow_i(x^*(t), T-t, V^P, n,\ P) = \sum_{\substack{R \subseteq M \\ k \notin R}} \sum_{i \in S \subseteq S_k} \frac{|R|!(|M|-|R|-1)!}{|M|!} \frac{(|S|-1)!(|S_k|-|S|)!}{|S_k|!}$$

$$\times \left[V^P(x_0,\, T\text{–}t_0, \bigcup_{l \in R} S_l \bigcup S) - V^P(x_0,\, T\text{–}t_0, \bigcup_{l \in R} S_l \bigcup (S \backslash i)) \right]$$

the formula for $\beta_i(t)$ gives us

$$\beta_i(t) = -\frac{\xi_i(t)(V^P)'(x^*(t), T-t, N)}{V^P(x^*(t), T-t, N)}$$

$$= -\sum_{\substack{R \subseteq M \\ k \notin R}} \sum_{i \in S \subseteq S_k} \frac{|R|!(|M|-|R|-1)!}{|M|!} \frac{(|S|-1)!(|S_k|-|S|)!}{|S_k|!}$$

$$\times \left[V^P(x^*(t), T-t, \bigcup_{l \in R} S_l \bigcup S) - V^P(x^*(t), T-t, \bigcup_{l \in R} S_l \bigcup (S \backslash i)) \right] \frac{(V^P)'(x^*(t), T-t, N)}{V^P(x^*(t), T-t, N)}$$

$$= -\sum_{\substack{R \subseteq M \\ k \notin R}} \sum_{i \in S \subseteq S_k} \frac{|R|!(|M|-|R|-1)!}{|M|!} \frac{(|S|-1)!(|S_k|-|S|)!}{|S_k|!}$$

$$\times \left[V^P(x^*(t), T-t, \bigcup_{l \in R} S_l \bigcup S) \frac{(V^P)'(x^*(t), T-t, N)}{V^P(x^*(t), T-t, N)} \right.$$

$$\left. - V^P(x^*(t), T-t, \bigcup_{l \in R} S_l \bigcup (S \backslash i)) \frac{(V^P)'(x^*(t), T-t, N)}{V^P(x^*(t), T-t, N)} \right]$$

At the same time, we have

$$\bar{V}^P(x^*(t), T-t, S) = -\int_t^T V^P(x^*(t), T-t, S)\frac{(V^P)'(x^*(t), T-t, N)}{V^P(x^*(t), T-t, N)}\mathrm{d}t$$

The Owen value computed for this characteristic function $\bar{V}^P(x^*(t), T-t, S)$ in every sub-game $\Gamma(x^*(t), T-t, V^P, n,\ P)$, $t \in [t_0, T]$ is equal to

$$Ow_i(x^*(t), T-t, V^P, n,\ P) = \int_t^T \beta_i(t)\mathrm{d}t$$

and trivially we have

$$Ow(x_0,\ T\text{–}t_0, V^P, n,\ P) = \int_{t_0}^t \beta(\tau)\mathrm{d}\tau + Ow(x^*(t), T-t, V^P, n,\ P)$$

which is equivalent to equation (7), i.e.,

$$Ow(x_0,\ T\text{–}t_0, V^P, n,\ P) = C(x_0,\ T\text{–}t_0, V^P, n,\ P) = \xi$$

$$\int_t^T \beta_i(t)\mathrm{d}t = \xi(t), Ow_i(x^*(t), T-t, V^P, n,\ P) = C(x^*(t), T-t, V^P, n,\ P)$$

which means the sub-game consistency of the Owen value for the refined characteristic function $\bar{V}^P(x_0,\ T\text{–}t_0, S), \subseteq N$. The theorem is proved. □

In the case under consideration, the IDP $\beta_i(t) \geq 0,\ i \in \{1, 2,\ \ldots,\ n\}$ has a natural interpretation as an Owen value in the instantaneous game (small game) with the characteristic function equal to $(\bar{V}^P)'(x^*(t), T-t, S)$, $S \subset N$. At the same time, $\beta_i(t)$, $\forall i \in \{1, 2,\ \ldots,\ n\}$ divides the instantaneous common payoff

$$\sum_{S_k \in P}\sum_{i \in S_k} g^i[t, x^*(t), u_1^*(t), u_2^*(t),\ \ldots,\ u_n^*(t)] = -(V^P)'(x^*(t), T-t, N)$$

proportional to the Owen value for the sub-game $\Gamma(x^*(t), T-t, V^P, n,\ P)$ starting from $x^*(t)$, and with the duration $T-t$, and characteristic function $V^P(x^*(t), T-t, S)$. Thus the "refined" characteristic function and the corresponding Owen value may be considered as differential optimality principles in differential cooperative games [18]. It is easily seen that the Banzhaf–Owen index [7] will also be sub-game consistent for the refined characteristic function $\bar{V}^P(x_0,\ T\text{–}t_0, S)$.

In differential cooperative games with a coalition structure, not only optimal imputation sets (core, stable set), or imputations (Owen value, Banzhaf–Owen value) have to be found, but also the additional imputation distribution procedures (IDP) $\beta(t)$, to define the earnings of the players on the time interval $[t_0, T]$. To follow the optimal trajectory $x^*(t)$, the players must be sure that the future earnings on the time interval $[t, T]$ remain optimal in the sense they were in the initial game $\Gamma(x_0,\ T\text{–}t_0, V^P, n,\ P)$. This is the sub-game consistent condition. If we do not require $\beta(t) \geq 0$, the sub-game consistency problem can be easily solved, as it was in the case of the Owen value, by putting $\beta(t) = -Ow'(x^*(t), T-t, V^P, n, P)$. But negative $\beta(t)$ does not have much sense, since no player would like to give back his earnings.

6. Application in environmental governance

In this section, we provide an application of differential cooperative games with a coalition structure in environmental governance. Take the ecological economics model discussed by Petrosyan and Zaccour [23]. Denote n as the set of countries involved in the game of emission reduction. Emission of player $i \in \{1, 2, \ldots, n\} = N$ at time $t \in [0, +\infty)$ is denoted as $m_i(t)$. Let $x(t)$ denote the stock of accumulated pollution by time t. The evolution of this stock is governed by the following differential equation:

$$\dot{x}(t) = \sum_{i \in N} m_i(t) - \delta x(t), \quad x(0) = x_0 \tag{11}$$

where δ denotes the natural rate of pollution absorption.

The emission reduction cost of country $i \in \{1, 2, \ldots, n\}$ equals

$$J^i(m, x) = \int_0^\infty \mathrm{e}^{-rt} \left(C_i(m_i(t)) + D_i(x(t))\right) \mathrm{d}t \tag{12}$$

where $m = \{m_1, m_2, \ldots, m_n\}$, r is the common social discount rate, $C_i(m_i)$ denotes the emission reduction cost incurred by country i when limiting its emissions to level m_i, and $D_i(x)$ denotes its damage cost. We assume that $\mathrm{e}^{-rt} \{C_i(m_i(t)) + D_i(x(t))\} \geq 0$, and both functions $C_i(m_i)$ and $D_i(x)$ are continuously differentiable and convex, with $C'_i(m_i) < 0$, $D'_i(x) > 0$.

Let $N = \{1, 2, 3, 4\}$, and the coalition structure $P = \{\{1, 2, 3\}, \{4\}\}$. Thus, $F(n, P) = \{\varnothing, \{4\}, \{1, 4\}, \{2, 4\}\{3, 4\}, \{1, 2, 3, 4\}\}$. The emission and damage cost functions are as follows:

$$C_i(m_i) = \frac{\gamma}{2}(m_i - \bar{m}_i)^2, \quad 0 \leq m_i \leq \bar{m}_i, \quad \gamma > 0, \quad i \in \{1, 2, 3, 4\}$$

$$D_i(x) = \eta x(t), \eta > 0, i \in \{1, 2, 3, 4\}$$

From the definition of the game $\Gamma(x_0, T\text{–}t_0, V^P, n, P)$, the emission reduction cost of the feasible coalition S is

$$J^S(m, x) = \sum_{i \in S} J^i(m, x) = \sum_{i \in S} \int_0^\infty \mathrm{e}^{-rt} \left\{\frac{\gamma}{2}(m_i - \bar{m}_i)^2 + \eta x(t)\right\} \mathrm{d}t$$

However, we will not adopt the method of Definition 9 to construct the characteristic function in the context of environmental problems. It is unlikely that if a subset of players forms a coalition to tackle an environmental problem, then the remaining players would form an antagonistic anti-coalition. Therefore, we still adopt the method in Petrosyan and Zaccour [23] to construct the characteristic function, which assumes that the remaining players stick to their feedback Nash strategies. Then we have the following definition of the characteristic function:

$$V^P(x_0,\, T\text{–}t_0, S) = \begin{cases} W(x_0, t_0, S), & S \in F(n,\, P) \\ W(x_0, t_0, \{i\}), & i \in N \\ 0, & S = \varnothing \end{cases} \tag{13}$$

where

$$W(x_0, t_0, \{i\}) = \min_{u_i(t)} \left(\int_{t_0}^{T} g^i[t, x(t), u_1(t), u_2(t),\, \ldots,\, u_n(t)]dt + q^i(x(T)) \right), \quad i \in N$$

$$W(x_0, t_0, S) = \min_{\substack{u_j(t)=\bar{u}_j(t), j\in N\setminus S \\ u_i(t), i\in S}} \left(\sum_{i\in S} \int_{t_0}^{T} g^i[t, x(t), u_1(t), u_2(t),\, \ldots,\, u_n(t)]dt + \sum_{i\in S} q^i(x(T)) \right)$$

$\bar{u}_j(t)$, $i \in N\backslash S$ are the feedback Nash equilibrium optimal strategies of the players who belong to coalition $N\backslash S$.

Then, we can obtain the characteristic function of the differential cooperative game (11), (12) as follows:

$$V^P(x(t), T-t, S) = \begin{cases} \dfrac{s\eta}{r(r+\delta)} \left(\displaystyle\sum_{i=1}^{3} \bar{m}_i - \frac{s^2\eta}{2\gamma(r+\delta)} - \frac{(n-s)\eta}{\gamma(r+\delta)} + rx^*(t) \right), & S \in F(n,\, P) \\ \dfrac{\eta}{r(r+\delta)} \left(\dfrac{\eta}{2\gamma(r+\delta)} + \displaystyle\sum_{i=1}^{3} \bar{m}_i - \frac{n\eta}{\gamma(r+\delta)} + rx^*(t) \right), & i \in N \\ 0, \quad S = \varnothing \end{cases}$$

where $s = |S|$,

$$x^*(t) = \mathrm{e}^{-rt}x(0) + \frac{1}{\delta} \left\{ \left(\sum_{i=1}^{3} m_i^* \right) \left(1 - \mathrm{e}^{-rt}\right) \right\}$$

$$m_i^* = \bar{m}_i - \frac{3\eta}{\gamma(r+\delta)}, i \in \{1, 2, 3, 4\}$$

The detailed calculation process can be referred to Petrosyan and Zaccour [23]. Therefore, the Owen value can be obtained by the equation (9) as follows

$$\begin{aligned} Ow_i(x^*(t), T-t, V^P, n,\, P) = &\sum_{\substack{R\subseteq M \\ k\notin R}} \sum_{i\in S\subseteq S_k} \frac{|R|!(|M|-|R|-1)!}{|M|!} \frac{(|S|-1)!(|S_k|-|S|)!}{|S_k|!} \\ &\times \left(V^P(x^*(t), T-t, \bigcup_{l\in R} S_l \bigcup S) - V^P(x^*(t), T-t, \bigcup_{l\in R} S_l \bigcup (S\backslash i)) \right) \end{aligned} \tag{14}$$

$$= \frac{1}{2}\left(\frac{1}{6}V^P(1) + \frac{1}{3}(V^P(1,2) - V^P(2)) + \frac{1}{3}(V^P(1,3) - v(3)) + \frac{1}{6}(V^P(1,2,3) - V^P(2,3))\right)$$

$$+ \frac{1}{2}\left(\frac{1}{6}V^P(1,4) + \frac{1}{3}\left(V^P(1,2,4) - V^P(2,4)\right) + \frac{1}{3}\left(V^P(1,3,4)\right.\right.$$

$$\left.\left.-V^P(3,4)\right) + \frac{1}{6}\left(V^P(1,2,3,4) - V^P(2,3,4)\right)\right)$$

$$= \frac{1}{12}[V^P(1,2,3,4) + 4V^P(1,2,3) - 3V^P(1)]$$

$$= \frac{4\eta}{r(r+\delta)}\left(\sum_{i=1}^{3}\bar{m}_i + rx^*(t)\right) - \frac{29\eta^2}{6r\gamma(r+\delta)^2}, \quad \forall i \in \{1,2,3\}$$

$$\begin{aligned} Ow_4(x^*(t), T-t, V^P, n,\ P) &= \frac{1}{12}\left(V^P(1,2,3,4) - 4V^P(1,2,4) + 3V^P(3)\right) \\ &= \frac{4\eta}{r(r+\delta)}\left(\left(\sum_{i=1}^{3}\bar{m}_i + rx^*(t)\right)\right) - \frac{7\eta^2}{r\gamma(r+\delta)^2} \end{aligned} \tag{15}$$

If we adopt the Shapley value, then

$$Sh_i(x^*(t), T-t, V^P, N)$$

$$= \sum_{i\in S\subseteq N} \frac{|S|!(|N|-|S|)!}{|N|!}\left[V^P(x^*(t), T-t, S) - V^P(x^*(t), T-t, S\backslash i)\right]$$

$$= \frac{6}{24}v(1) + \frac{4}{24}\left(v(1,2) - v(2) + v(1,3) - v(3) + v(1,4) - v(4)\right)$$

$$+ \frac{6}{24}\left(v(1,2,3) - v(2,3) + v(1,3,4)\right)$$

$$- v(3,4) + v(1,2,4) - v(2,4)) + \frac{4}{24}(v(1,2,3,4) - v(2,3,4))$$

$$= \frac{4\eta}{r(r+\delta)}\left(\left(\sum_{i=1}^{3}\bar{m}_i + rx^*(t)\right)\right) - \frac{127\eta^2}{3\gamma(r+\delta)^2}, \quad \forall i \in \{1,2,3,4\}$$

By comparing the calculation results in this paper (Owen value) with the results calculated by using the Shapley value, we can find that

$$Ow_i(x^*(t), T-t, V^P, n,\ P) < Sh_i(x^*(t), T-t,\ V,\ N), i = \{1,2,3\}$$

$$Ow_4(x^*(t), T-t, V^P, n,\ P) > Sh_4(x^*(t), T-t,\ V,\ N)$$

It shows that when countries 1, 2, and 3 cooperate to form a prior union before the game starts, environmental governance costs are lower than without prior cooperation. However, the cost of environmental governance of country 4 is relatively higher, which is consistent with the practical significance. Although this result is somehow expected, our approach permits us to compute the actual savings resulting from prior cooperation. In addition, $\Gamma(x_0,\ T\text{–}t_0, V^P, n,\ P)$ is a differential convex game [23], so the Owen value belong to the core $C(x_0,\ T\text{–}t_0, V^P, n,\ P)$. That is, players cannot make the management cost lower than the Owen value through their own efforts. This means that all countries prefer to choose the Owen value as the payoff distribution mechanism.

Next, we provide a time-dependent allocation over time of $Ow_i(x_0,\ T\text{–}t_0, V^P, n,\ P)$. As natural discount rate r is considered in the emission reduction model we studied, the IDP function $\beta(t)$ is given by

$$Ow_i(x_0,\ T\text{–}t_0, V^P, n,\ P) = \int\limits_{t_0}^{t} \mathrm{e}^{-r\tau}\beta_i(\tau)\mathrm{d}\tau + Ow_i(x^*(t), T-t, V^P, n,\ P)$$

Straightforward calculations lead to

$$\beta_i(t) = rOw_i(x^*(t), T-t, V^P, n,\ P) - Ow_i{}'(x^*(t), T-t, V^P, n,\ P)$$

Substituting the Owen value (equations (14) and (15)) into the above equation leads to

$$\beta_i(t) = \eta x^*(t) + \frac{29\eta^2}{6\gamma(r+\delta)^2},\quad i=\{1,2\}\,,\quad \beta_3(t) = \eta x^*(t) + \frac{7\eta^2}{\gamma(r+\delta)^2}$$

Obviously, $\beta_i(t) \geq 0,\ i \in \{1,2,3,4\}$. Therefore, the Owen value calculated according to the characteristic function defined by the equation (13) is sub-game consistent. So we don't have to modify the characteristic function by the equation (10). Obviously, the allocation mechanism depends on the accumulated pollution and cost parameters as it should be expected.

7. Conclusion

This paper presents a new class of differential cooperative games in which players form prior unions to increase their payoff, i.e., the differential cooperative games with a coalition structure. By defining its characteristic function and calculating the specific expression of the Owen value, the method of calculating the income distribution of each player in this kind of game model is obtained. However, since the Owen value cannot ensure IDP $\beta(t) \geq 0$, we propose a "refinement" of the characteristic function based on the idea of local optimality which is also a characteristic function. It is proved that the Owen value defined for the "refined" characteristic function is sub-game consistent. The differential cooperative games with a coalition structure extends the application of differential cooperative games to a wider spectrum of real-life scenarios. For instance, the redistribution of interests among the countries for Brexit, the cost distribution among the countries in environmental governance, the formation of coalitions in supply chain management and the threat of nuclear war, etc, which can be solved by the model presented in this paper. Therefore, the calculation results of this paper have important practical significance.

This is the first time that differential cooperative games with a coalition structure are formulated and further research along this line is expected. We supposed that the player participates in the coalition throughout the game, and once the coalition is formed, it will not change during the game. Well, in fact, in order to maximize their own interests, players may participate in different unions in the differential cooperative games, which results in the change of the coalition structure. Therefore, further research on the game is to allow the changes of coalition structure in differential cooperative games.

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