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Minimum cost flow problems in generalized fuzzy environments. Credibilistic CVaR minimization approach

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Abstract

The paper focuses on the problems of linear programming (LP) with generalized fuzzy numbers (GFNs) as coefficients of the objective function. It is necessary to characterize consistent arithmetic operations to lower the error and information loss compared to the minimum operator usage and normalization in cases where experts are not completely certain of their subjective opinions. The uncertainty is eliminated using the total cost as a loss function and credibilistic conditional value at risk (CVaR) minimization. To crispify and generate a GFN, we utilize a ranking function that allows us to consider risky realizations. By solving many deterministic problems with LP solvers, projections of the error in the objective function can be presented. To describe and implement our methodology, we mainly focus on network optimization problems, especially generalized fuzzy transportation, assignment, and shortest path problems.

Keywords: *conditional value at risk, credibility distribution function, generalized fuzzy numbers, minimum cost flow problems, ranking functions*

1. Introduction

Optimization is the process of mathematical modeling and finding the optimum solution under certain conditions. Therefore, it is the most important part of the decision-making process. But in most cases, some uncertain situations may arise in the decision variables as well as the parameters of the objectives and/or constraints. A technique employed in such circumstances is fuzzy optimization. Uncertainty is a common feature of real-world situations; hence, fuzzy optimization is important in this context.

LP is an optimization model with a single linear objective function, continuous variables, and linear equations or inequalities as constraints. Fuzzy LP (FLP), however, deals with problems where some or all of the objective function constants, right-hand side, or constraint coefficients are fuzzy. Therefore, FLP offers a more flexible and effective approach.

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Normal fuzzy numbers (FNs) are used to represent the parameters in the majority of methods that have been proposed in the literature to solve LP problems in a fuzzy environment. However, in many cases, it is not possible to limit the membership function to the normal form. Experts who only partially rely on their judgments often need to provide data for real-world decision-making problems. For this reason, the necessity of the concept of GFN has emerged. While there are few studies using GFNs, most of these articles use GFNs by transforming them into normal FNs through normalization. Although this procedure is mathematically correct, it reduces the amount of information contained in the original data.

In this context, FLP is used in many fields, such as transportation, assignment, shortest path problems, etc. The studies carried out in these application areas can be summarized as follows: For the fuzzy transportation problem (FTP) utilizing generalized trapezoidal fuzzy numbers (GTFNs), Maheswari et al. [17] suggested a modified approach for determining an initial basic feasible solution. By assuming that the decision-maker was only uncertain about the precise values of transportation costs and there was no uncertainty regarding the supply and demand of the product, Kaur and Kumar [13] proposed an algorithm as a direct extension of the conventional approach for solving a specific type of FTP. Islam and Roy [11] tackled the multi-objective FTP, aiming for minimum transportation costs and a maximum amount of entropy, and solved it with primal geometric programming. Kumar et al. [14] presented a technique using the harmonic mean method to solve FTPs. Samuel and Raja [25] used the zero division method for solving a special type of FTP, see also [29].

Based on a ranking method, Thorani and Shankar [28] introduced a methods for finding the optimal solutions to assignment problems with fuzzy costs. See also [7]. Rostam and Haydar [24] aimed to determine an FLP method to reach a production decision that contributed to maximizing profits. Vincent et al. [31] proposed a robust method for ranking GFNs and a fuzzy multi-criteria decision-making approach that did not require the normalization process, i.e., avoided information loss. Ebrahimnejad [9] suggested a simpler and computationally more efficient approach based on the ranking function for solving FTP by assuming that the values of transportation costs were represented by GTFNs. Mathur and Srivastava [19] developed an innovative process for optimizing the generalized fuzzy trapezoidal transport problem using classical ranking techniques and reducing the computational complexity of existing methods.

Mahmoodirad and Sanei [18] developed the best approximation method with a representation of both the transportation cost and the fixed cost of the GTFNs. This method obtained lower and upper bounds on the fuzzy optimal value of the fixed-charge FTP, which can be easily obtained by using the approximate solution. Singh and Singh [26] solved the FTP by applying a particle swarm optimization (PSO) algorithm that has been modified to incorporate additional modules as well as the transportation costs represented by GTFNs, whereas the supply and demand levels were crisp numbers. Moreover, the proposed algorithm worked efficiently to obtain optimal solutions and removed the barricades of traditional solution techniques.

In the generalized sense, Anusuya and Kavitha [3] proposed the roulette ant wheel selection, which combined the traits of ants and the roulette wheel selection algorithm, and solved the fuzzy shortest path problem, which has numerous applications in robotics, communication, transportation, scheduling, routing, and mapping. Valdes et al. [30] dealt with the problem of searching for the shortest path (or more efficient path, in a more general sense) between two nodes in a communication network, but the cost of each link was modeled as a triangular GFN. Moreover, they defined different fuzzy cost allocation functions and fuzzy optimization strategies and applied them to the search for the shortest path between two nodes. Gupta et al. [10] addressed the limitations of existing methods and presented a way of solving generalized fuzzy assignment and traveling salesman problems.

In intuitionistic or picture fuzzy environments, credibilistic value-at-risk (VaR) and CVaR minimization in assignment and transportation models were considered in [2], assuming the normality condition holds. In addition, simulation and standardization of non-standard fuzzy variables and simultaneous modeling of risk-averse behaviors have been taken into consideration.

For other related works addressing modeling, robust optimization, uncertainty handling, and risk management, the reader can refer to [4, 12, 20, 27].

In this paper, we mention the extensions of the definitions of the credibility distribution function (CDF), VaR, and CVaR for FNs to the generalized setting. By making use of the existing fuzzy credibility theory, these VaR measures can also help in the management of generalized fuzzy risk. In this way, we can treat optimization problems involving both normalized and non-normalized cost coefficients. We prevent information loss by not requiring the normalization procedure, which converts GFNs into normal form. We introduce not only shape-preserving, error-reducing arithmetic operations but also redesign a way to defuzzify GFNs that assess risk aversion by considering whether the fuzzy variable is a cost or benefit type. Additionally, generalized fuzzy sample generation is also carried out from a pessimistic perspective. We particularly consider minimum-cost flow types of problems with trapezoidal objective coefficients when applying our methodology. The sole source of the uncertainty is assumed to be generalized fuzzy costs. Assuming the total cost serves as the loss function, CVaR minimization models are used to identify the optimal values of the decision variables that result in lower risk, and error analyses are further performed using sample generation.

This study attempts to fill several research gaps, which can be summed up as follows:

- Adaptation of arithmetic operations to reduce the effects of downsides such as error increases or inconsistencies resulting from the employment of the minimum operator in arithmetic operations or from the direct normalization of generalized fuzzy parameters in uncertain optimization problems.
- To examine risk-averse solutions to several LP problems with generalized fuzzy parameters in objective function coefficients in cases where the experts are not completely sure of their opinions.
- To construct the concepts of VaR and CVaR for generalized fuzzy losses,
- To propose a pessimistic generalized fuzzy sample generation technique for use in error analysis in uncertain environments where the normality requirement is not fulfilled.
- To obtain a framework strategy for dealing with similar problems whose parameters satisfy the normality condition.

The paper is organized as follows: The following section presents some introductory information on GFNs. Section 3 contains not only the descriptions of the related VaR and CVaR measures but also the simulation technique to generate trapezoidal-type generalized fuzzy samples considering risk attitudes. Numerical examples are given to support the proposed approach in Section 4. The paper concludes with some remarks in Section 5.

2. Preliminaries

In this section, we provide some important concepts in a generalized context that will be used in our discussion.

Definition 1 ([5]). Let X be an arbitrary non-empty set of the universe or domain . A fuzzy set $\widetilde{A}$ in X is characterized by a membership function $\mu_{\widetilde{A}}(x) : X \to [0,1]$, which assigns to each object $x \in X$ a real number in the interval $[0,1]$, so as $\mu_{\widetilde{A}}(x)$ represents the grade of belonging of the element x to the fuzzy set $\widetilde{A}$. Then the fuzzy set takes the following form:

$$\widetilde{A} = \left\{ \left\langle x, \mu_{\widetilde{A}}(x) \right\rangle \middle| \, x \in X \right\}$$

Definition 2 (GFN [6]). Let $a, b, c, d \in \mathbb{R}, a \leq b \leq c \leq d, h \in (0,1]$. A GFN $\widetilde{A}$ is a fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_{\widetilde{A}}$ satisfies the following conditions:

(i) $\mu_{\widetilde{A}} : \mathbb{R} \to [0,h]$ is continuous,
(ii) $\mu_{\widetilde{A}}(x) = 0$ for $x < a$ or $x > d$,
(iii) $\mu_{\widetilde{A}}(x)$ is strictly increasing on $[a,b]$ and strictly decreasing on $[c,d]$,
(iv) $\mu_{\widetilde{A}}(x) = h$, for $x \in [b,c]$.

If normality holds, i.e., if $h = 1$, then the GFN $\widetilde{A}$ is referred to as an FN. The adoption of a more generic framework is made possible by the use of the parameter h, which establishes the height of the GFN and stands for the degree of confidence in expert judgments. $\widetilde{A}$ is subnormal, with the greatest value $h \in (0,1)$.

Definition 3 (GTFN). A GTFN $\widetilde{A}$ on real line $\mathbb{R}$ is characterized by its linear membership function:

$$\mu_{\widetilde{A}}(x) = \max\left\{ \min\left\{ \frac{h(x-a)}{b-a}, h, \frac{h(d-x)}{d-c} \right\}, 0 \right\} \tag{1}$$

where $a < b \leq c < d$ and $0 < h \leq 1$. We denote $\widetilde{A} = \langle (a,b,c,d)\,; h\rangle$. Here, h is called the maximal degree of membership.

Definition 4 (the expected value). Without loss of generality, let us consider a GTFN:

$$\widetilde{A} = \langle (a,b,c,d)\,; h\rangle$$

Then, the inverse left-hand side and right-hand side membership functions are respectively given as:

$$\widetilde{A}_L(\alpha) = a + \frac{\alpha}{h}(b-a)\,, \widetilde{A}_R(\alpha) = d - \frac{\alpha}{h}(d-c)\,, \alpha \in [0,h] \tag{2}$$

The expected value of $\widetilde{A}$ is defined as follows:

$$EV(\widetilde{A}) = \frac{1}{2}\int_0^h \left(\widetilde{A}_L(\alpha) + \widetilde{A}_R(\alpha)\right) d\alpha = h\left(\frac{a+b+c+d}{4}\right) \tag{3}$$

2.1. Arithmetic operations

The following arithmetic operations were proposed for intuitionistic FNs in [32]. We adapt them to GTFNs to reduce error and information loss in comparison to the use of the minimum operator and normalization. Thus, an outcome with average reliability is produced when a highly reliable non-negative value is added to a somewhat less reliable non-negative value.

Definition 5. Let $\widetilde{A} = \langle (a_1, b_1, c_1, d_1)\,; h_1 \rangle$, $\widetilde{B} = \langle (a_2, b_2, c_2, d_2)\,; h_2 \rangle$ be two non-negative GTFNs, that is, $a_1, a_2 \geq 0$, and $\lambda \in \mathbb{R}$, then we have the following shape-preserving operations:

- Addition of two GTFNs

$$\widetilde{A} + \widetilde{B} = \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)\,; h \rangle \tag{4}$$

where

$$h = \frac{EV(\widetilde{A}_N)h_1 + EV(\widetilde{B}_N)h_2}{EV(\widetilde{A}_N) + EV(\widetilde{B}_N)}, \quad \widetilde{A}_N = \langle (a_1, b_1, c_1, d_1)\,; 1 \rangle\,, \quad \widetilde{B}_N = \langle (a_2, b_2, c_2, d_2)\,; 1 \rangle$$

- Scalar multiplication

$$\lambda \widetilde{A} = \begin{cases} \langle (\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1); h_1 \rangle\,, & \text{if } \lambda \geq 0 \\ \langle (\lambda d_1, \lambda c_1, \lambda b_1, \lambda a_1); h_1 \rangle\,, & \text{if } \lambda < 0 \end{cases}$$

- Multiplication of two GTFNs

$$\widetilde{A} \times \widetilde{B} = \langle (a, b, c, d)\,; h_1 h_2 \rangle$$

where

$$a = \min\{a_1a_2, a_1d_2, d_1a_2, d_1d_2\}\,, \quad b = \min\{b_1b_2, b_1c_2, c_1b_2, c_1c_2\}$$
$$c = \max\{b_1b_2, b_1c_2, c_1b_2, c_1c_2\}\,, \quad d = \max\{a_1a_2, a_1d_2, d_1a_2, d_1d_2\}$$

Theorem 1. Let $\lambda_k \in \mathbb{R}^+$ and $\widetilde{A_k}$ be independent GTFNs with finite expected values for $k \in \{1, 2, \ldots, K\}$. The expected value operator is linear, i.e.,

$$EV\left(\sum_{k=1}^{K} \lambda_k \widetilde{A_k}\right) = \sum_{k=1}^{K} \lambda_k EV(\widetilde{A_k})$$

See [32] for proof.

2.2. Ranking functions

Since GFNs are represented by membership functions that may overlap, it is challenging to determine which GFN is greater than the other. One way to order GFNs is through the use of ranking functions, which pair each GFN with a number from the real line, where a natural order exists. To support our findings and order GTFNs, we selected a few well-known ranking functions from the literature, which we give below. For the centroid-based formulations, we occasionally have to assume that $\widetilde{A} = \langle (a, b, c, d)\, ; h \rangle$ is a standardized GTFN where $a \geq 0$ and $d \leq 1$.

Definition 6 (defuzzification of a GTFN). [16] Let $\widetilde{A} = \langle (a, b, c, d)\, ; h \rangle$ be a GTFN. The ranking function can be given as follows:

$$R(\widetilde{A}) = (1-\omega)h(\frac{a+b}{2}) + \omega h(\frac{c+d}{2}) \tag{5}$$

where $\omega \in [0, 1]$. A higher value of an ω indicates a higher degree of optimism. The total integral value $R(\widetilde{A})$ is a convex combination of the left and right integral values through an optimism index ω.

Definition 7 (centroid-based distance method [33]). Centroid points and distance index are respectively given as:

$$\begin{aligned} x_0 &= \frac{1}{3}\left(a+b+c+d-\frac{dc-ab}{d+c-a-b}\right) \\ y_0 &= \frac{h}{3}\left(1+\frac{c-b}{d+c-a-b}\right), \quad R(\widetilde{A}) = \sqrt{x_0^2+y_0^2} \end{aligned} \tag{6}$$

for a standardized GTFN $\widetilde{A} = \langle (a, b, c, d)\, ; h \rangle$.

Definition 8 (incentre of centroids [23]). For a standardized GTFN $\widetilde{A} = \langle (a, b, c, d)\, ; h \rangle$, incentre of centroid points and distance index are respectively given as:

$$\begin{aligned} x_0 &= \frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma} \\ y_0 &= \frac{\alpha\left(\frac{h}{3}\right)+\beta\left(\frac{h}{2}\right)+\gamma\left(\frac{h}{3}\right)}{\alpha+\beta+\gamma}, \quad R(\widetilde{A}) = \sqrt{x_0^2+y_0^2} \end{aligned} \tag{7}$$

where

$$\alpha = \frac{\sqrt{(c-3b+2d)^2+h^2}}{6}, \quad \beta = \frac{2c-a-2b+d}{3}, \quad \gamma = \frac{\sqrt{(3c-2a-b)^2+h^2}}{6}$$

Definition 9 (area between the centroid of the centroids [22]). Centroid of the centroid points and area index are respectively given as:

$$x_0 = \frac{2a+7b+7c+2d}{18}, \quad y_0 = \frac{7h}{18}, R(\widetilde{A}) = x_0 y_0. \tag{8}$$

for an arbitrary GTFN $\widetilde{A} = \langle (a, b, c, d)\, ; h \rangle$

3. Simulation, credibilistic VaR, and CVaR measures for GTFNs

In sample generation, we alter the ranking function of (5) to represent the GTFN with a crisp number. In order to defuzzify a generalized fuzzy parameter, we make the assumption that $\omega \in [0, 0.5)$ or $\omega \in (0.5, 1]$ depending on whether $\widetilde{A} = \langle (a, b, c, d)\,; h\rangle$ is respectively a benefit or cost type GTFN with $a \geq 0$. If the variable is of the cost (benefit) type, it will be overestimated (underestimated) compared to the expected value. This assumption means that a pessimistic point of view is always adopted. An optimistic perspective is obtained when this scale is considered in reverse for ω. However, only risk-averse decision-making behavior is considered in this study. Notice that it corresponds to risk-neutral behavior when $\omega = 0.5$ and thereby $R(\widetilde{A}) = EV(\widetilde{A})$ in (5). For example, the expected value of the generated cost type values will be $h(a + b + 3c + 3d)/8$, since $\omega \sim U\,(0.5, 1)$ and the expected value of ω is 0.75. We also assume that the GTFNs are mutually independent. See also [1] for an effort to simulate non-standard fuzzy numbers while considering risk.

Now, we extend the existing definitions of the CDF, VaR, and CVaR for FNs to the generalized setting. The following definitions are reduced to classical forms for $h = 1$. Since we focus on trapezoidal-shaped costs, linearity holds true.

Definition 10 (credibility distribution function [8, 15]). Let r be a real number, $\widetilde{A}$ be a GFN with the membership $\mu_{\widetilde{A}}$ and maximal membership h. The CDF is defined as follows:

$$F_{\widetilde{A}}(r) = Cr\{\widetilde{A} \leq r\} = \frac{1}{2}\left(\sup_{x \leq r}\mu_{\widetilde{A}}(x) + h - \sup_{x > r}\mu_{\widetilde{A}}(x)\right) \tag{9}$$

Remark 1. From Identity (9), it can easily be obtained that

$$Cr\{\widetilde{A} \leq r\} = \begin{cases} 0, & \text{if } r < a \\ \dfrac{h}{2}\left(\dfrac{r-a}{b-a}\right), & \text{if } a \leq r < b \\ \dfrac{h}{2} & \text{if } b \leq r < c \\ \dfrac{h}{2}\left(\dfrac{r+d-2c}{d-c}\right), & \text{if } c \leq r < d \\ h, & \text{if } d \leq r \end{cases} \tag{10}$$

where $\widetilde{A} = \langle (a, b, c, d)\,; h\rangle$ is a GTFN.

VaR was initially defined as a statistical risk measure that estimates the maximum loss that may be experienced on an investment with a certain level of confidence but does not predict how much an investor will lose in the most unlikely situations. To tackle risky scenarios associated with fuzzy market conditions due to volatilities, fuzzy VaR has been interpreted just as in a stochastic context. A monetary value $\widetilde{A}$ is defined by a generalized fuzzy variable representing possible losses. In this paper, a positive value of $\widetilde{A}$ denotes a loss. The following two definitions are constructed using Definition 10. Peng [21] introduced VaR and CVaR measures for normal fuzzy variables; on the other hand, we adapt these definitions for generalized fuzzy losses as follows:

Definition 11 (generalized fuzzy VaR (GFVaR)). GFVaR is the greatest loss which is not exceeded with a given high confidence level $\rho \in (h/2, h)$, so:

$$GFVaR_\rho(\widetilde{A}) = \inf\{r \in \mathbb{R} \,\Big|\, Cr\{\widetilde{A} \leq r\} \geq \rho\} = F_{\widetilde{A}}^{-1}(\rho). \tag{11}$$

Remark 2. Let GTFN $\widetilde{A} = \langle (a, b, c, d)\,; h \rangle$ be a loss, then $GFVaR_\rho(\widetilde{A}) = c + \dfrac{2\rho - h}{h}\,(d - c)$

Definition 12 (generalized fuzzy CVaR (GFCVaR)). GFCVaR is a risk measure that addresses the question what is the expected loss incurred in the worst $(h - \rho)$ losses.

If $F_{\widetilde{A}}$ is continuous, GFCVaR equal the conditional expectation of loss when the GFVaR is exceeded, that is:

$$GFCVaR_\rho(\widetilde{A}) = \frac{1}{h - \rho} \int_\rho^h F_{\widetilde{A}}^{-1}(y)dy \tag{12}$$

Remark 3. For a generalized trapezoidal fuzzy loss $\widetilde{A} = \langle (a, b, c, d)\,; h \rangle$, $GFCVaR_\rho(\widetilde{A}) = c + \dfrac{\rho}{h}\,(d - c)$. For GFCVaR minimization formulations, higher values of ρ close to h lead to more pessimistic solutions, which have lower variability for the total costs.

4. Computational tests

This section presents and solves numerical instances involving generalized fuzzy environments to demonstrate our approach. However, we only examine the impact of uncertain costs for the sake of simplicity.

In the first stage, we solve GFCVaR minimization models. To analyze errors, in the second stage, generalized fuzzy costs are generated pessimistically via the defuzzification function (5), in which $\omega \sim U\,(0.5, 1)$, and the resulting crisp problems are solved to compare optimal values of objective functions. This process is repeated too many times to yield meaningful results. Furthermore, the variabilities of total costs are measured in order to elaborate from a risk-averse point of view. On a computer running MS Windows 10 Pro and equipped with an Intel Core i5-7400 CPU (3.00 GHz) and 4 GB of RAM, all computational tests were carried out using MATLAB R2018a. We particularly use the following built-in functions: "rand", "linprog", "intlinprog", and "shortestpath".

When we substitute the optimal values of the decision variables obtained from the experimental models into the model, it is assumed that we obtain the real value of the objective. If different optimal decision variable values are obtained from the GFCVaR minimization model, we then face an error in the objective. The error is assumed to be the percentage error, which is the absolute error divided by the best objective value and multiplied by 100%. The error is calculated as follows:

$$\text{Error} = \frac{\text{Objective value} - \text{The best objective value}}{\text{The best objective value}} \times 100\%, \tag{13}$$

where objective value refers to the value of the objective function determined by the sum of multiplications of the generated objective coefficients with the optimal solutions of the GFCVaR minimization model. Similarly, the best objective value is the minimum objective value for the original problem's objective function when the parameters are simulated.

4.1. Generalized fuzzy transportation problem

For the following illustrative example, which we call test problem 1, the transportation costs are considered GTFNs, representing the uncertainty therein, whereas the supplies and demands are precise numbers. Refer to [9, 13, 26].

Consider a transportation problem with four sources and six destinations in which each unit cost of transporting the product from ith source to jth destination $\widetilde{c_{ij}}, i \in \{1,2,3,4\}, j \in \{1,2,3,4,5,6\}$ is a GTFN with a known CDF. Refer to Table 1 for the cost parameters of the problem. Let the total availabilities at the sources be crisp values equal to $124, 120, 150$, and 170, respectively. Similarly, let the demands at the destinations have crisp values equal to $112, 90, 84, 92, 106$, and 80, respectively. The transportation problem is said to be balanced since the total supply is equal to the total demand. It is required to deliver the product from these sources to destinations by satisfying the demand without exceeding the capacity with the minimum total transportation cost.

Table 1. Generalized transportation costs of test problem 1

Source	Destination		
	D_1	D_2	D_3
S_1	$\langle(20,27,35,41);0.7\rangle$	$\langle(9,11,12,14);0.6\rangle$	$\langle(10,15,18,20);0.7\rangle$
S_2	$\langle(20,25,35,41);0.7\rangle$	$\langle(9,11,12,16);0.8\rangle$	$\langle(9,11,12,14);0.6\rangle$
S_3	$\langle(9,10,12,16);0.7\rangle$	$\langle(65,70,74,76);0.8\rangle$	$\langle(20,25,35,41);0.8\rangle$
S_4	$\langle(9,11,12,14);0.6\rangle$	$\langle(10,15,21,24);0.7\rangle$	$\langle(20,25,35,41);0.6\rangle$
	$D4$	$D5$	$D6$
$S1$	$\langle(15,20,22,24);0.7\rangle$	$\langle(10,15,18,20);0.8\rangle$	$\langle(9,12,15,18);0.7\rangle$
$S2$	$\langle(10,15,21,23);0.8\rangle$	$\langle(10,15,18,23);0.6\rangle$	$\langle(9,12,15,18);0.8\rangle$
$S3$	$\langle(12,15,22,24);0.7\rangle$	$\langle(10,12,14,18);0.6\rangle$	$\langle(15,20,26,28);0.8\rangle$
$S4$	$\langle(10,15,18,20);0.6\rangle$	$\langle(8,10,12,14);0.8\rangle$	$\langle(15,20,25,28);0.7\rangle$

Kaur and Kumar [13] provided the following optimal solution:

$$X_1 = \left(x_{ij}^*\right) = \begin{pmatrix} 0 & 90 & 0 & 0 & 0 & 34 \\ 0 & 0 & 84 & 36 & 0 & 0 \\ 112 & 0 & 0 & 38 & 0 & 0 \\ 0 & 0 & 0 & 18 & 106 & 46 \end{pmatrix}.$$

In [9], the optimal solution was presented as follows:

$$X_2 = \begin{pmatrix} 0 & 80 & 0 & 0 & 0 & 44 \\ 0 & 0 & 84 & 0 & 0 & 36 \\ 112 & 0 & 0 & 38 & 0 & 0 \\ 0 & 10 & 0 & 54 & 106 & 0 \end{pmatrix}.$$

Also using different PSO parameters, three following alternative solutions were provided in [26]:

$$X_3 = \begin{pmatrix} 0 & 81 & 0 & 0 & 0 & 43 \\ 0 & 0 & 84 & 0 & 0 & 36 \\ 99 & 0 & 0 & 50 & 0 & 1 \\ 13 & 9 & 0 & 42 & 106 & 0 \end{pmatrix}$$

$$X_4 = \begin{pmatrix} 0 & 80 & 0 & 0 & 0 & 44 \\ 0 & 0 & 84 & 0 & 0 & 36 \\ 112 & 0 & 0 & 33 & 5 & 0 \\ 0 & 10 & 0 & 59 & 101 & 0 \end{pmatrix}$$

$$X_5 = \begin{pmatrix} 0 & 80 & 0 & 0 & 0 & 44 \\ 0 & 0 & 84 & 0 & 0 & 36 \\ 112 & 0 & 0 & 29 & 9 & 0 \\ 0 & 10 & 0 & 63 & 97 & 0 \end{pmatrix}$$

Our objective function is as follows:

$$GFCVaR_\rho(\widetilde{TC}) = c + \frac{\rho}{h}(d - c) \tag{14}$$

where $\widetilde{TC} = \sum_{i=1}^{4}\sum_{j=1}^{6}\widetilde{c_{ij}}x_{ij} = \langle (a,b,c,d)\,;h\rangle$ is the generalized total cost function, which is computed using the shape-preserving arithmetic operations of addition and scalar multiplication given in Definition 5.

We ask the decision-maker to specify a constant degree of $\frac{\rho}{h} \in (0.5, 1)$ and, if he wishes, a lower and/or upper bound for the overall maximal membership degree of h. Fixing $\frac{\rho}{h}$ allows us to derive a linear objective function, which further simplifies the model and reduces it to the traditional fuzzy CVaR minimization formulation. However, we should consider a nonlinear constraint regarding h in addition to linear supply and demand constraints. For test problem 1, we suppose that $\frac{\rho}{h} = 0.8$ and $h \geq 0.65$. Notice that

$$\min_{i,j} h_{ij} = 0.6 < 0.65 < 0.8 = \max_{i,j} h_{ij}$$

Here, if we assume that $h = 0.6$ using the minimum operator, we do not consider the fact that trapezoidal costs not having the maximal membership of 0.6 can be represented by narrower intervals compared to $0.6-$cuts.

We reach the following optimal solution:

$$X_6 = \begin{pmatrix} 0 & 90 & 0 & 0 & 0 & 34 \\ 0 & 0 & 84 & 0 & 0 & 36 \\ 112 & 0 & 0 & 0 & 28 & 10 \\ 0 & 0 & 0 & 92 & 78 & 0 \end{pmatrix}$$

where

$$h^* = \frac{\sum_{i=1}^{4}\sum_{j=1}^{6} EV((\widetilde{c_{ij}})_N)x_{ij}^* h_{ij}}{\sum_{i=1}^{4}\sum_{j=1}^{6} EV((\widetilde{c_{ij}})_N)x_{ij}^*} = 0.6685$$

Thus, $\rho = 0.5348$ is chosen. Using the same formula for calculating maximal memberships, the optimal generalized fuzzy total transportation costs are obtained as follows:

$$\widetilde{TC}_1 = \langle (5414, 6802, 8280, 9712)\,; 0.6934 \rangle$$

$$\widetilde{TC}_2 = \langle (5148, 6474, 7802, 9244)\,; 0.6849 \rangle$$

$$\widetilde{TC}_3 = \langle (5177, 6491, 7852, 9266)\,; 0.6856 \rangle$$

$$\widetilde{TC}_4 = \langle (5148, 6484, 7792, 9244)\,; 0.6821 \rangle$$

$$\widetilde{TC}_5 = \langle (5148, 6492, 7784, 9244)\,; 0.6798 \rangle$$

$$\widetilde{TC}_6 = \langle (5178, 6570, 7726, 9204)\,; 0.6685 \rangle$$

corresponding to the optimal solutions of X_1, X_2, X_3, X_4, X_5, and X_6, respectively. To compare the different optimal total costs obtained with the aforementioned techniques, we first standardize the minimum total costs in the following way: For any value, subtract 5148, the minimum value of the existing values, and then divide the result by the length of the range, that is, maximum minus minimum, or $(9712 - 5148)$. Thus, all available values and ranking indices will be within the range of $[0, 1]$. According to Table 2, $\widetilde{TC}_1 \succ \widetilde{TC}_3 \succ \widetilde{TC}_2 \succ \widetilde{TC}_4 \succ \widetilde{TC}_5 \succ \widetilde{TC}_6$ for any ranking function of (5), (6), (7), and (8). In generalized fuzzy total costs, the crispified value of the total cost rises as the height, or overall maximum confidence level, increases. It can be interpreted that the decision-maker incurs an additional cost to ensure greater certainty of the information.

Table 2. Ranking indices for standardized total transportation costs of test problem 1

Method	$R(\widetilde{TC}_1)$	$R(\widetilde{TC}_2)$	$R(\widetilde{TC}_3)$	$R(\widetilde{TC}_4)$	$R(\widetilde{TC}_5)$	$R(\widetilde{TC}_6)$
(5) with $\omega = 0.75$	0.3091	0.2488	0.2537	0.2480	0.2473	0.2463
(6)	0.6018	0.5267	0.5328	0.5257	0.5250	0.5208
(7)	0.5976	0.5199	0.5262	0.5193	0.5188	0.5180
(8)	0.1417	0.1169	0.1189	0.1164	0.1160	0.1145

Now let us try to calculate the error and variability that will occur in the objective function for these solutions in the presence of pessimistic states of nature. 2000 problems with simulated parameters are solved, and optimal values of the decision variables and objective functions are obtained. Simultaneously, in the case of the realization of optimal values for decision variables, the errors in the objective functions are stored. Standard deviations are also calculated to measure the variability in the total costs. The whole above process is repeated ten times; the standard deviations and the averages of the percentage errors are presented in Tables 3, 4, and 5. Let us state that in order to compare the errors in the objective function for different solutions, the error values realized in each iteration are calculated using the same generated objective coefficients, that is, within the same loop and using the same state of nature or scenario. For each repetition, our solution X_6 results in the lowest average percentage error. Considering risk aversion, the solution X_1 is the least preferable to other solutions due to its highest variability and error.

Table 3. Errors and standard deviations of test problem 1

Repetition#	Solution X_1		Solution X_2	
	Average % error	Standard deviation	Average % error	Standard deviation
1	14.4810	106.2823	7.1614	97.9132
2	14.3828	104.5653	7.0820	96.9940
3	14.3379	108.5029	7.0170	99.8848
4	14.4361	103.7110	7.1538	95.8319
5	14.3467	105.7828	7.0256	99.0896
6	14.2814	103.9160	7.0317	97.5708
7	14.2878	101.5947	7.0252	95.2042
8	14.3794	106.8572	7.0622	95.6856
9	14.3585	107.7984	7.0963	99.0678
10	14.5470	106.1179	7.1886	97.7343

Table 4. Errors and standard deviations of test problem 1, continued

Repetition#	Solution X_3		Solution X_4	
	Average % error	Standard deviation	Average % error	Standard deviation
1	7.7475	98.0851	6.6804	96.0206
2	7.6591	96.7766	6.6071	95.3290
3	7.5961	99.8322	6.5451	97.8508
4	7.7373	95.9921	6.6741	93.9507
5	7.6104	99.0251	6.5513	97.1861
6	7.6000	97.8741	6.5622	95.5880
7	7.6021	95.7649	6.5534	93.1794
8	7.6411	95.9011	6.5879	93.7554
9	7.6726	98.8525	6.6211	97.2127
10	7.7684	98.0127	6.7082	95.8154

Table 5. Errors and standard deviations of test problem 1, continued

Repetition#	Solution X_5		Solution X_6	
	Average % error	Standard deviation	Average % error	Standard deviation
1	6.2955	94.7937	4.1708	94.2166
2	6.2272	94.2845	4.1301	95.4248
3	6.1676	96.4931	4.0839	94.8240
4	6.2903	92.7423	4.1631	93.2070
5	6.1718	95.9623	4.0775	96.3862
6	6.1865	94.2789	4.1303	92.8062
7	6.1759	91.8571	4.1045	91.2020
8	6.2085	92.5115	4.1003	91.8331
9	6.2410	96.0034	4.1359	95.9166
10	6.3238	94.5789	4.1990	94.2763

4.2. Assignment problem with generalized fuzzy costs

For the following illustrative example [7], which we refer to as test problem 2, the assignment costs are considered as GTFNs (Table 6). Suppose the aim is to allocate four different jobs to four people with a minimum total assignment cost. Each individual can only work on one task at a time, and each job can only be given to one person.

Dinagar and Kamalanathan [7] provided the following two optimal solutions corresponding to two different criteria:

$$X_1 = \left(x_{ij}^*\right) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Table 6. Generalized assignment costs of test problem 2

Job/Person	P_1	P_2	P_3	P_4
J_1	$\langle(15,35,70,80)\,;0.1\rangle$	$\langle(60,75,90,115)\,;0.2\rangle$	$\langle(25,35,40,75)\,;0.4\rangle$	$\langle(15,30,45,60)\,;0.2\rangle$
J_2	$\langle(6,21,45,62)\,;0.2\rangle$	$\langle(15,35,60,90)\,;0.1\rangle$	$\langle(50,60,75,95)\,;0.3\rangle$	$\langle(11,18,38,51)\,;0.2\rangle$
J_3	$\langle(20,30,40,50)\,;0.2\rangle$	$\langle(10,25,45,60)\,;0.2\rangle$	$\langle(20,30,40,50)\,;0.5\rangle$	$\langle(2,6,12,31)\,;0.4\rangle$
J_4	$\langle(10,40,65,85)\,;0.2\rangle$	$\langle(20,40,60,80)\,;0.1\rangle$	$\langle(40,60,80,100)\,;0.2\rangle$	$\langle(25,30,40,45)\,;0.2\rangle$

For test problem 2, we suppose that $\dfrac{\rho}{h} = 0.75$ and $0.10 \leq h \leq 0.17$. Using the GFCVaR value as the objective function, we reach the following optimal solution:

$$X_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The respective optimal generalized fuzzy total assignment costs are given as follows:

$$\widetilde{TC}_1 = \langle(90, 155, 225, 300)\,; 0.1740\rangle$$

$$\widetilde{TC}_2 = \langle(53, 102, 157, 248)\,; 0.2450\rangle$$

$$\widetilde{TC}_3 = \langle(72, 136, 222, 301)\,; 0.1592\rangle$$

Accordingly, using the expected value (3), $\widetilde{TC}_2 \succ \widetilde{TC}_1 \succ \widetilde{TC}_3$ where $EV(\widetilde{TC}_1) = 33.4950$, $EV(\widetilde{TC}_2) = 34.3000$, and $EV(\widetilde{TC}_3) = 29.0938$. The same ordering is yielded with pessimistic rankings via the ranking function (5). The orderings are $\widetilde{TC}_1 \succ \widetilde{TC}_3 \succ \widetilde{TC}_2$, with the ranking functions (6), (7), and (8) when not considering risk aversion. In this manner, ranking functions fail to help us decide since they produce inconsistent outcomes.

Now let us try to calculate the error and variability that will occur in the objective function for these solutions in the presence of pessimistic realizations. 1000 problems with simulated costs are solved, and optimal values of the decision variables and objective functions are obtained. Simultaneously, in the case of the realization of optimal values for decision variables, the errors in the objective functions are stored. Standard deviations, interquartile ranges, and ranges are calculated to measure the variability in the total costs. The whole above process is repeated ten times; the results are presented in Tables 7, 8, and 9. For each repetition, our solution X_3 results in the lowest mean percentage error. Considering risk aversion, the solution X_2 is the least preferable to other solutions due to its highest variability and error.

Table 7. Errors and variabilities of the solution X_1 for test problem 2

Repetition#	Average % error	Standard deviation	Interquartile ranges	Ranges
1	11.1343	1.7122	2.2957	9.6384
2	11.4603	1.6970	2.4345	9.4236
3	11.1476	1.7101	2.4775	9.2778
4	11.1040	1.7636	2.6578	9.8888
5	11.2699	1.7371	2.2961	9.9469
6	11.2029	1.6804	2.3950	9.1484
7	11.2294	1.7237	2.4991	9.6128
8	11.2058	1.7188	2.3930	9.9775
9	11.4552	1.7031	2.4106	9.2408
10	11.3425	1.6642	2.3543	9.5577

Table 8. Errors and variabilities of the solution X_2 for test problem 2

Repetition#	Average % error	Standard deviation	Interquartile ranges	Ranges
1	18.4228	2.2968	3.3183	12.6328
2	18.9900	2.2730	3.2081	11.7943
3	18.5076	2.3038	3.3348	12.5835
4	18.7952	2.2990	3.2061	12.4334
5	18.4804	2.2524	3.2464	11.8095
6	18.4821	2.4112	3.4195	12.2005
7	18.6004	2.2802	3.1475	12.4616
8	18.7676	2.2717	3.1319	13.0434
9	19.0101	2.3130	3.3344	12.6766
10	18.6124	2.3309	3.3373	13.2245

Table 9. Errors and variabilities of the solution X_3 for test problem 2

Repetition#	Average % error	Standard deviation	Interquartile ranges	Ranges
1	0.1754	1.8460	2.5476	10.9049
2	0.2003	1.7983	2.5505	9.8307
3	0.2031	1.8210	2.7274	10.2015
4	0.1843	1.8497	2.6963	9.9355
5	0.1560	1.7943	2.5767	10.3279
6	0.1852	1.7992	2.5120	9.6249
7	0.1803	1.8517	2.5634	10.0397
8	0.1587	1.8336	2.5770	9.6499
9	0.1915	1.8449	2.5141	9.8850
10	0.1529	1.8441	2.6588	9.7857

4.3. Shortest path problem with generalized fuzzy weights

The distances between the nodes or the lengths of the arcs are considered GTFNs for the following illustrative example [3], which we refer to as test problem 3. For the acyclic directed graph of test problem 3, see Figure 1. The shortest path, $1-3-6-10$, is selected as the best solution via their genetic algorithm-based method in [3].

We suppose that $\frac{\rho}{h}=0.9$ and the decision-maker does not deliver an opinion regarding the aggregated maximal degree of membership h. Using the GFCVaR value as the objective function and integer LP formulation of the shortest path problem, we reach the same optimal solution with a total cost of $\widetilde{TC}=\langle(10,12,15,20)\,;0.2789\rangle$.

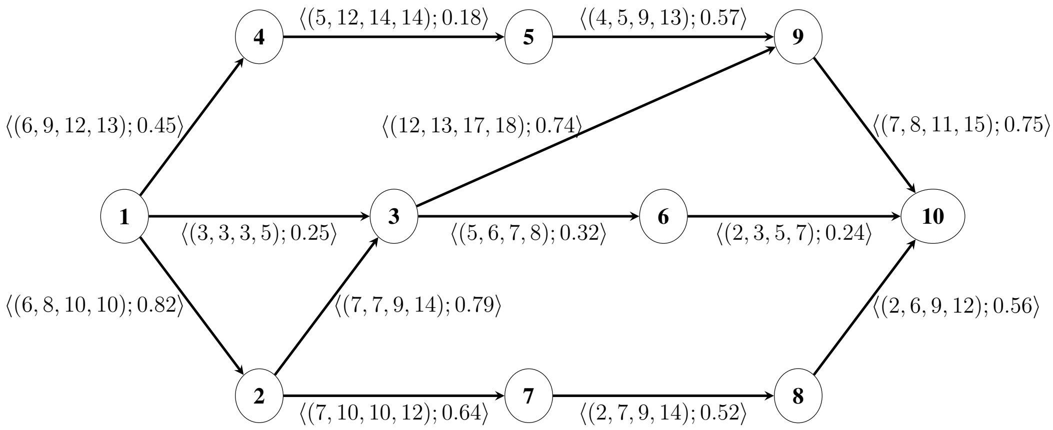

Figure 1. The acyclic directed graph of test problem 3

Now let us try to calculate the error and variability that will occur in the objective function for these solutions in the presence of pessimistic realizations. 1000 problems with simulated distances are solved, and optimal values of the decision variables and objective function are obtained. Simultaneously, in the case of the realization of an optimal path, the error in the objective function is stored. Standard deviations, interquartile ranges, and ranges are calculated to measure the variability in the total distance. The whole above process is repeated ten times; the results are presented in Table 10.

Table 10. Errors and variabilities for test problem 3

Repetition#	Average % error	Standard deviation	Interquartile ranges	Ranges
1	$5.4265e-15$	0.1572	0.2337	0.7910
2	$5.6707e-15$	0.1570	0.2256	0.7822
3	$5.5867e-15$	0.1564	0.2054	0.7787
4	$5.7360e-15$	0.1580	0.2336	0.7560
5	$5.6578e-15$	0.1591	0.2387	0.7809
6	$5.5753e-15$	0.1626	0.2394	0.8114
7	$5.7787e-15$	0.1594	0.2365	0.7947
8	$5.5887e-15$	0.1561	0.2267	0.7954
9	$5.3810e-15$	0.1617	0.2335	0.7772
10	$5.5850e-15$	0.1517	0.2150	0.8214

5. Conclusions

We can use a GFN, which has a height between 0 and 1, instead of the usual FN, whose height is 1, in cases where experts are not completely certain of their subjective opinions. Using the existing definition of the credibility function, the VaR and CVaR measures are constructed for generalized environments. In the proposed models, GFCVaR values are assumed to be objective functions subjected to the original constraints of the problems. They are concerned with determining the optimal values of the decision variables for achieving the optimal objective value at the minimal generalized fuzzy risk. Assuming that the decision-maker wants to minimize both the overall cost and risk, generalized fuzzy VaR and CVaR measures can also be employed for loss minimization problems under risk-leading generalized fuzzy uncertainty. Instead of using the minimum operator in addition operations, we calculate in a way that provides us with a compensatory value for the overall height value. We ask the decision-maker to provide limits on the overall height and a confidence level-to-height ratio. This method allows the decision-maker to conduct an interactive process by making a trade-off between providing greater certainty of information and cost minimization. The higher the reliability of the information, the higher the cost.

First, from a risk aversion perspective, solutions acquired via various methods, and hence related optimal objective function values, are compared using ranking functions that are well-known in the literature and have proven to be efficient. Furthermore, generalized fuzzy sample generation is performed from a pessimistic point of view. In this way, it is possible to have an idea about the size of the error that will arise in the objective function as well as the variability of the objective function under unfavorable market conditions in optimization problems with uncertain coefficients. In our analysis, we observe that the GFCVaR objective models offer the solutions with the lowest error and/or variability, employing pessimistic sample generation. We conducted our study on trapezoidal GFNs; however, extensions can be made using cut sets and/or trapezoidal approximations.

The future outlook involves the development of robust versions for the trade-off between maximizing the expected return and minimizing the risk under generalized uncertainty, inspired by robust counterparts of CVaR under random uncertainty.

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