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Diverse copulas through Durante's method. Exploring parametric functions

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Abstract

This article unveils the often underestimated potential of a copula methodology introduced by Durante in 2009. It highlights the remarkable ability of the method to generate a broad spectrum of copulas by exploiting various parametric functions. We determine a collection of power-like, exponential-like, trigonometric-like, logarithmic-like, hyperbolic-like and error-like functions, each dependent on one, two, or three parameters, effectively satisfying the necessary assumptions of Durante's method. The proofs provided rely on suitable differentiation, comprehensive factorizations, and judicious application of mathematical inequalities. In the vast repertoire of copulas derived from this methodology, we present three distinct series of eight new copulas, supported by a graphical analysis of their respective densities. This theoretical study expands the understanding of copula generation and also introduces a new perspective on their construction in various contexts.

Keywords: *copula, Durante's method, dependence modeling, copula density, correlation*

1. Introduction

Copulas are mathematical constructs that play a pivotal role in modeling multi-dimensional dependence types among random variables in various fields, including biology, finance, engineering, earth science, and environmental science. They provide a flexible framework for capturing the intricate relationships that exist between these variables, making them invaluable tools for risk assessment and statistical analysis. Over the years, several copula families have been developed to describe different types of dependence structures, ranging from the popular elliptical, extreme-value, and Archimedean copulas to more specialized structures like minimum-maximum-type and perturbed-type copulas. The key references on the basics of copulas are [11, 18, 21]. Current developments are detailed in [4–6, 12, 19, 22–26, 28]. These references collectively provide a solid foundation for further understanding and applying copula theory. While the existing copulas have been crucial in a wide range of applications, researchers have continued to explore innovative ways to extend the repertoire of copula models.

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In retrospect, Durante's 2009 method described in [10, Corollary 3] represents a major step in this direction. It offers a systematic scheme to construct new copulas based on two existing copulas and auxiliary functions. More precisely, based on two (uni-dimensional) functions $f(x)$ and $g(x)$, $x \in [0,1]$, satisfying some precise assumptions (to be specified later) and two (two-dimensional) copulas $A(x, y)$ and $B(x, y)$, $(x, y) \in [0,1]^2$, the method established by Durante guarantees that the following two-dimensional function is a valid copula:

$$C(x, y) = A\left[\frac{x}{f(x)}, \frac{y}{g(y)}\right] B\left[f(x), g(y)\right], \quad (x, y) \in [0,1]^2 \tag{1}$$

Thus, it allows the creation of two-dimensional copulas with diverse structures by introducing functions $f(x)$ and $g(x)$ of varying natures. In particular, these functions can subtly depend on tuning parameters for the sake of flexibility. In [10, page 388], examples based on simple linear, power, or maximum functions are presented. Durante's method has opened up interesting possibilities for copula modeling, as it allows the development of copulas adapted to specific data characteristics and applications. However, most of the related works focus on power functions only, as described in [10, Corollary 4] (see [5, 9, 20]).

In this article, we unveil the versatility of Durante's method by presenting new and sophisticated examples of functions $f(x)$ and $g(x)$, beyond the classical power function scenario. More precisely, we exhibit a collection of power-like, exponential-like, trigonometric-like, logarithmic-like, hyperbolic--like and error-like functions, each dependent on one, two, or three parameters, effectively satisfying the necessary assumptions of Durante's method. The proofs provided rely on suitable differentiation techniques, comprehensive factorizations, and well-chosen mathematical inequalities. On this basis and equation (1), we demonstrate how Durante's method can be exploited to construct a wide range of parametric copulas. A focus is put on three distinct series of eight new copulas. We also perform a graphical study to show the versatility of the associated copula densities by varying the values of the parameters. Thus, the contributions of this study aim to provide a comprehensive overview of Durante's method, making it accessible to a wider audience of researchers and practitioners, and how it can be used to develop custom copulas suitable for their specific modeling needs.

The subsequent sections of this article are organized as follows: In Section 2, we investigate the foundational aspects of copulas and Durante's method within the context of mathematical principles. Section 3 introduces flexible parametric functions $f(x)$ and $g(x)$ that meet the essential criteria for applying this method. Section 4 is dedicated to the derivation of a set of novel copulas. Finally, Section 5 proposes a comprehensive conclusion.

2. Mathematical background

2.1. Copulas

To begin, let us recall the definition of a (two-dimensional) copula.

Definition 1. A (two-dimensional) function $C(x, y)$, $(x, y) \in [0,1]^2$, is said to be a copula if it satisfies the following properties:

P1. $C(x,1) = C(1,x) = x$ for any $x \in [0,1]$.

P2. $C(x,0) = C(0,x) = 0$ for any $x \in [0,1]$.

P3. $C(x_1, y_1) - C(x_1, y_2) - C(x_2, y_1) + C(x_2, y_2) \geq 0$ for any $(x_1, x_2, y_1, y_2) \in [0, 1]^4$ with $x_1 \leq x_2$ and $y_1 \leq y_2$.

In the absolutely continuous case, implying that the mixed derivative of $C(x, y)$ is defined almost everywhere, the property **P3** is equivalent to $\partial^2 C(x, y)/(\partial x \partial y) \geq 0$ for any $(x, y) \in (0, 1)^2$. In this case, the copula density is defined by $c(x, y) = \partial^2 C(x, y)/(\partial x \partial y)$. We may refer to [11, 18, 21]. In fact, copulas may exhibit rich diversity in their shapes and structures. From Archimedean copulas, characterized by their properties of symmetry and tail dependence, to elliptical copulas, capturing multi-dimensional normality, the range of copulas is vast. Table 1 presents some classical and recently introduced copulas, along with the appropriate references.

Table 1. Some selected classical and recently introduced copulas

Name)	Definition	Parameter(s)	Reference
Independence	xy	✗	(classic)
Farlie–Gumbel–Morgenstern (FGM)	$xy[1 + \alpha(1 - x)(1 - y)]$	$\alpha \in [-1, 1]$	[13, 21]
Ali–Mikhail–Haq (AMH)	$\dfrac{xy}{1 - \alpha(1 - x)(1 - y)}$	$\alpha \in [-1, 1]$	[1]
Clayton (C)	$(x^{-\alpha} + y^{-\alpha} - 1)^{-1/\alpha}$	$\alpha > 0$	[7]
Frank (F)	$-\dfrac{1}{\alpha} \log \left[1 + \dfrac{[\exp(-\alpha x) - 1][\exp(-\alpha y) - 1]}{\exp(-\alpha) - 1}\right]$	$\alpha \in \mathbb{R}/\{0\}$	[14]
Gumbel–Hougaard (GH)	$\exp\left[-\{[-\log(x)]^\alpha + [-\log(y)]^\alpha\}^{1/\alpha}\right]$	$\alpha \geq 1$	[16, 17]
Gumbel–Barnett (GB)	$xy \exp[-\alpha \log(x) \log(y)]$	$\alpha \in [0, 1]$	[2, 15]
Celebioglu–Cuadras (CC)	$xy \exp[-\alpha(1 - x)(1 - y)]$	$\alpha \in [-1, 1]$	[3, 8]
Chesneau (C)	$xy \exp\left[-\alpha \left(1 - \dfrac{1}{x}\right)\left(1 - \dfrac{1}{y}\right)\right]$	$\alpha \in [0, 1]$	[6]

These copulas have diverse functional features, depend on one tuning parameter (except the independence copula), and possess their own qualities. It is worth mentioning that some of them are implemented in many software programs to ease their applicability. For reasons explained later, these copulas will be considered as baselines in our investigations of Durante's method.

2.2. Durante's method

To elucidate Durante's method, it is necessary to outline four specific assumptions. They are described by the class denoted as Θ and into the Corollary 3 in [10]. Based on a general function $f(x)$, $x \in [0, 1]$, these assumptions are as follows:

A1. $f(1) = 1$.

A2. $f(0) \neq 1$ and $f(0) \geq 0$.

A3. $f(x)$ is increasing.

A4. $x/f(x)$ is increasing.

They are less stringent than it seems at a first glance, as developed in the first part of the article.

We are now in the position to present Durante's method.

Proposition 1. [10, Corollary 3] Let us consider two functions $f(x)$ and $g(x)$ that satisfy assumptions **A1–A4**. Then, for any copula $A(x, y)$ and $B(x, y)$, the following function is a copula:

$$C(x, y) = A\left[\frac{x}{f(x)}, \frac{y}{g(y)}\right] B\left[f(x), g(y)\right], \quad (x, y) \in [0, 1]^2$$

Thus, based on Durante's method, thanks to the open choices for $f(x)$ and $g(x)$, and also the copulas $A(x, y)$ and $B(x, y)$, we can construct a plethora of new copulas, which can be exchangeable or not. If we focus on $f(x)$ or $g(x)$, the given examples of parametric functions in [10, page 388] are as follows:

- $f(x) = [(b - a)x + a]/b$ for $(a, b) \in (0, 1)^2$ with $a < b$,
- $f(x) = x^{1-c}$ for $c \in [0, 1)$,
- $f(x) = \max(x, 1/d)$ with $d > 1$.

These examples possess the dual merits of simplicity and comprehensiveness in illustrating the applicability of Durante's method for extending and parameterizing various copulas. Notably, in contemporary literature, the significance of [10, Corollary 3] often materializes within a specific context where $f(x) = x^{1-a}$ for $a \in [0, 1)$ and $g(x) = x^{1-b}$ for $b \in [0, 1)$, corresponding to [10, Corollary 4]. As sketched in the introduction, this scenario is exemplified in works such as [5, 9, 20].

This article attempts to elucidate the potential inherent in the aforementioned [10, Corollary 3] and its capacity to generate various copulas beyond those currently established. This exploration is undertaken by presenting original parametric functions that satisfy assumptions **A1–A4**, as outlined in the following section. Through this investigation, we aim to highlight the versatility and wide applicability of Durante's method in modeling copulas.

3. Examples of parametric functions

Several kinds of parametric functions $f(x)$ satisfying assumptions **A1**, **A2**, **A3**, and **A4** are described in this section, namely power-like, exponential-like, trigonometric-like, logarithmic-like, hyperbolic-like and error-like functions.

3.1. Power-like functions

The proposition below exhibits a two-parameter power-like function that satisfies the required assumptions.

Proposition 2. Let $(a, b) \in \mathbb{R}^2$. The following function satisfies assumptions **A1–A4**:

$$f(x) = [a + (1 - a)x^b]^{1/b}, \quad x \in [0, 1]$$

for $a \in (0, 1)$.

Proof. We have $f(1) = [a + (1 - a)]^{1/b} = 1$; **A1** is satisfied. Moreover, since $a \in (0, 1)$, we have $f(0) = a^{1/b} \neq 1$ and $f(0) \geq 0$; **A2** is fulfilled. Let us now investigate **A3**. We have

$$f'(x) = (1 - a)x^{b-1}[a + (1 - a)x^b]^{1/b-1}$$

Since $a \in (0, 1)$, it is immediate that $f'(x) \geq 0$, implying that $f(x)$ is increasing. Therefore, **A3** is satisfied. It remains to prove **A4**. By using standard differentiation rules and some arrangements, we obtain

$$\left(\frac{x}{f(x)}\right)' = a[a + (1-a)x^b]^{-1/b-1}$$

Since $a \in (0, 1)$, it is immediate that $[x/f(x)]' \geq 0$, implying that $x/f(x)$ is increasing. Therefore, **A4** is fulfilled. The proof is completed. □

In Proposition 2, it is important to notice that $b \in \mathbb{R}$ without restriction; we may take b negative in particular. Furthermore, one can prove that $f(x)$ is convex for $b < 1$, concave for $b > 1$ and linear for $b = 1$ (case also covered in [10, page 388]); $f(x)$ has comprehensible properties that makes it interesting to consider as a prime choice into Durante's method.

The result below presents an alternative three-parameter power-like function that successfully meets the necessary assumptions, but under some technical conditions on the parameters.

Proposition 3. Let $(a, b, c) \in \mathbb{R}^3$. The following function satisfies assumptions **A1–A4**:

$$f(x) = 1 + a(1-c)^b - a(1-cx)^b, \quad x \in [0, 1]$$

for $c \leq 1$, $abc > 0$, $a[1-(1-c)^b] \leq 1$, and either

- $a(1-b)b < 0$, or
- $a(1-b)b \geq 0$ and $abc(1-c)^{b-1} \leq 1$.

Proof. It is clear that $f(1) = 1 + a(1-c)^b - a(1-c)^b = 1$; **A1** is fulfilled. Since $a \neq 0$ and $c \neq 0$, we have $f(0) = 1 + a(1-c)^b - a(1-c \times 0)^b = 1 + a[(1-c)^b - 1] \neq 1$ and, since $a[1-(1-c)^b] \leq 1$, we have $f(0) \geq 0$; **A2** is satisfied.

Let us now investigate **A3**. We have

$$f'(x) = abc(1-cx)^{b-1}$$

Since $abc > 0$ and $c \leq 1$, it is immediate that $f'(x) \geq 0$, implying that $f(x)$ is increasing. Therefore, **A3** is fulfilled.

For **A4**, we aim to prove that, under the considered conditions on a, b and c, we have $[x/f(x)]' \geq 0$, implying that $x/f(x)$ is increasing. By using classical differentiation rules and a suitable arrangement, we obtain

$$\left(\frac{x}{f(x)}\right)' = \frac{1 - a[1 - cx(1-b)](1-cx)^{b-1} + a(1-c)^b}{[1 + a(1-c)^b - a(1-cx)^b]^2}$$

It is clear that $[1 + a(1-c)^b - a(1-cx)^b]^2 \geq 0$. As a result, the property $[x/f(x)]' \geq 0$ is established if a, b and c satisfy the following inequality:

$$\sup_{x\in[0,1]} \left\{a[1 - cx(1-b)](1-cx)^{b-1} - a(1-c)^b\right\} \leq 1 \tag{2}$$

Let us now investigate it. By setting $h(x) = a[1 - cx(1-b)](1-cx)^{b-1} - a(1-c)^b$, we have

$$h'(x) = a(1-b)bc^2x(1-cx)^{b-2}$$

Since $c \leq 1$, it is clear that $(1-cx)^{b-2} \geq 0$. Let us now distinguish the cases $a(1-b)b < 0$ and $a(1-b)b \geq 0$.

- For $a(1-b)b < 0$, we have $h'(x) \leq 0$. This implies that $h(x)$ is decreasing, so $\sup_{x\in[0,1]} h(x) = h(0) = a[1-(1-c)^b]$, and the inequality in equation (2) becomes $a[1-(1-c)^b] \leq 1$, which is an initial assumption.
- For $a(1-b)b \geq 0$, we have $h'(x) \geq 0$. This implies that $h(x)$ is increasing, so $\sup_{x\in[0,1]} h(x) = h(1) = a[1-c(1-b)](1-c)^{b-1} - a(1-c)^b = abc(1-c)^{b-1}$, and the inequality in equation (2) becomes $abc(1-c)^{b-1} \leq 1$.

As a result, under the additional conditions: $a(1-b)b < 0$ and $a[1-(1-c)^b] \leq 1$, or $a(1-b)b \geq 0$ and $abc(1-c)^{b-1} \leq 1$, **A4** is satisfied. This ends the proof. □

To highlight the complexity of the conditions on the parameters, let us notice that $abc > 0$ implies that either $a > 0$, $b > 0$ and $c > 0$, or $a > 0$, $b < 0$ and $c < 0$, or $a < 0$, $b > 0$ and $c < 0$, or $a < 0$, $b < 0$ and $c > 0$. Some configurations are excluded based on other conditions, but this must be considered on a case-by-case basis.

In the special case $b = 1$ and $c = 1$, the condition $abc(1-c)^{b-1} \leq 1$ must be read as $a \leq 1$, with the use of the convention $(1-c)^{b-1} = 0^0 = 1$ (see [27]). In this case, we thus have the condition $a \in (0, 1]$.

3.2. Exponential-like functions

The result below exhibits a three-parameter exponential-like function that fulfills the assumptions of Durante's method.

Proposition 4. Let $(a, b, c) \in \mathbb{R}^3$. The following function satisfies assumptions **A1–A4**:

$$f(x) = \exp[a(1-c)^b - a(1-cx)^b], \quad x \in [0, 1]$$

for $c \leq 1$, $abc > 0$, and either

- $bc \leq 1$ and $abc(1-c)^{b-1} \leq 1$, or
- $bc > 1$ and $a(1-1/b)^{b-1} \leq 1$.

Proof. It is clear that $f(1) = \exp[a(1-c)^b - a(1-c)^b] = \exp(0) = 1$; **A1** is fulfilled. Moreover, since $a \neq 0$ and $c \neq 0$, we have $f(0) = \exp\{a[(1-c)^b - 1]\} \neq 1$ and since an exponential term is always positive, we have $f(0) \geq 0$; **A2** is satisfied.

Let us now investigate **A3**. We have

$$f'(x) = abc(1-cx)^{b-1} \exp[a(1-c)^b - a(1-cx)^b]$$

The exponential term is always positive and, since $abc > 0$ and $c \leq 1$, it is clear that $abc(1-cx)^{b-1} \geq 0$. As a result, we have $f'(x) \geq 0$, implying that $f(x)$ is increasing; **A3** holds.

For **A4**, we aim to prove that, under the considered conditions on a, b and c, we have $[x/f(x)]' \geq 0$, implying that $x/f(x)$ is increasing. By using standard differentiation rules and appropriate factorizations, we obtain

$$\left(\frac{x}{f(x)}\right)' = \exp[a(1-cx)^b - a(1-c)^b]\left[1 - abcx(1-cx)^{b-1}\right]$$

The exponential term is always positive. Since $abc > 0$ and $c \leq 1$ imply that $abcx(1-cx)^{b-1} \geq 0$, the property $[x/f(x)]' \geq 0$ is obtained if a, b and c satisfy the following inequality:

$$abc \sup_{x\in[0,1]} x(1-cx)^{b-1} \leq 1 \tag{3}$$

Let us now demonstrate that the considered conditions on a, b and c make this inequality true. By setting $h(x) = x(1-cx)^{b-1}$, we have

$$h'(x) = (1-cx)^{b-2}(1-bcx)$$

Since $c \leq 1$, it is clear that $(1-cx)^{b-2} \geq 0$. Let us now distinguish the cases $bc \leq 1$ and $bc > 1$.

- For $bc \leq 1$, we clearly have $1 - bcx \geq 0$ and $h'(x) \geq 0$. This implies that $h(x)$ is increasing, so $\sup_{x\in[0,1]} h(x) = h(1) = (1-c)^{b-1}$, and the inequality in equation (3) becomes $abc(1-c)^{b-1} \leq 1$.
- For $bc > 1$, the equation $h'(x) = 0$ has the solution $x = 1/(bc) \in [0,1]$. Since $h(x) \geq 0$ with $h(0) = 0$, this is a maximum point, and we have $\sup_{x\in[0,1]} h(x) = h[1/(bc)] = [1/(bc)](1-1/b)^{b-1}$, and the inequality in equation (3) becomes $a(1-1/b)^{b-1} \leq 1$.

As a result, under the additional conditions: $bc \leq 1$ and $abc(1-c)^{b-1} \leq 1$, or $bc > 1$ and $a(1-1/b)^{b-1} \leq 1$, **A4** is satisfied. The desired result is established. □

The proposition below presents an alternative variant; it illustrates a two-parameter polynomial--exponential-like function that satisfies the required assumptions.

Proposition 5. Let $(a,b) \in \mathbb{R}^2$. The following function satisfies the assumptions **A1–A4**:

$$f(x) = 1 - a(1-x)\exp(-bx), \quad x \in [0,1]$$

for $a \in (0,1]$ and $b \geq 0$.

Proof. Since $a \neq 0$, we have $f(1) = 1 - a(1-1)\exp(-b \times 1) = 1$; **A1** is fulfilled. Since $a \in (0,1]$, we have $f(0) = 1 - a(1-0)\exp(-b \times 0) = 1 - a \neq 1$ and $f(0) \geq 0$; **A2** is fulfilled. Furthermore, we have

$$f'(x) = a\exp(-bx)[1 + b(1-x)]$$

The exponential term is always positive. Since $a > 0$ and $b \geq 0$ implying that $1 + b(1-x) \geq 1 \geq 0$, we get $f'(x) \geq 0$. As a result, $f(x)$ is increasing; **A3** holds. On the other hand, we have

$$\left(\frac{x}{f(x)}\right)' = \frac{\exp(bx)\{\exp(bx) - a[1 + bx(1-x)]\}}{[1 - a(1-x)\exp(-bx)]^2}$$

It is clear that $[1-a(1-x)\exp(-bx)]^2 \geq 0$ and $\exp(bx) \geq 0$. Also, since $b \geq 0$, we have $1+bx(1-x) \geq 1 \geq 0$. By using $a \in (0,1]$ and the following well-known exponential inequality: $\exp(u) \geq 1+u$ for $u \in \mathbb{R}$, we get

$$\begin{aligned}\exp(bx) - a[1+bx(1-x)] &\geq \exp(bx) - [1+bx(1-x)] \\ &\geq 1+bx-[1+bx(1-x)] = bx^2 \geq 0\end{aligned}$$

As a result, we have $[x/f(x)]' \geq 0$, so that $x/f(x)$ is increasing; **A4** is fulfilled. This ends the proof. □

We can notice that, for $b=0$, we get $f(x) = 1-a(1-x) = (1-a)+ax$, which is presented in [10, page 388].

Other kinds of parametric functions are considered in the next subsections.

3.3. Trigonometric-like functions

The proposition below illustrates a two-parameter sine-like function that satisfies the required assumptions.

Proposition 6. Let $(a,b) \in \mathbb{R}^2$. The following function satisfies assumptions **A1–A4**:

$$f(x) = \left[\frac{\sin(ax)}{\sin(a)}\right]^b, \quad x \in [0,1]$$

for $a \in (0,\pi/2]$ and $b \in (0,1]$.

Proof. It is clear that $f(1) = [\sin(a)/\sin(a)]^b = 1$; **A1** is satisfied. Since $b \in (0,1]$, we have $f(0) = [\sin(a\times 0)/\sin(a)]^b = 0 \neq 1$ and $f(0) \geq 0$; **A2** are fulfilled. Furthermore, we have

$$f'(x) = \frac{1}{[\sin(a)]^b} ab\cos(ax)[\sin(ax)]^{b-1}$$

Since $a \in (0,\pi/2]$ and $b \in (0,1]$ (in fact, $b > 0$ is enough at this step), it is immediate that $f'(x) \geq 0$. As a result, $f(x)$ is increasing. So **A3** is fulfilled. On the other hand, we have

$$\left(\frac{x}{f(x)}\right)' = [\sin(a)]^b[\sin(ax)]^{-b-1}[\sin(ax) - abx\cos(ax)]$$

Since $b \in (0,1]$ and $ax \in (0,\pi/2]$, by applying the following well-known trigonometric inequality: $\sin(u) \geq u\cos(u)$ for $u \in [0,\pi/2]$, we get

$$\left(\frac{x}{f(x)}\right)' \geq [\sin(a)]^b[\sin(ax)]^{-b-1}[\sin(ax) - ax\cos(ax)] \geq 0$$

Hence $[x/f(x)]' \geq 0$, so that $x/f(x)$ is increasing; **A4** is satisfied. The proof is completed. □

Similarly, the subsequent result presents a two-parameter arctangent-like function that fulfills the necessary assumptions. We recall that $\arctan(x)$ is the inverse function of $\tan(x)$.

Proposition 7. Let $(a, b) \in \mathbb{R}^2$. The following function satisfies assumptions **A1–A4**:

$$f(x) = \left[\frac{\arctan(ax)}{\arctan(a)}\right]^b, \quad x \in [0, 1]$$

for $a > 0$ and $b \in (0, 1]$.

Proof. Since $a \neq 0$, we have $f(1) = [\arctan(a)/\arctan(a)]^b = 1$; **A1** is satisfied. Also, since $b \in (0, 1]$, we have $f(0) = [\arctan(a \times 0)/\arctan(a)]^b = 0 \neq 1$ and $f(0) \geq 0$; **A2** is fulfilled. Furthermore, we have

$$f'(x) = \frac{ab}{(1 + a^2x^2)\arctan(ax)} \left[\frac{\arctan(ax)}{\arctan(a)}\right]^b$$

Since $a > 0$ and $b \in (0, 1]$, all the main terms are positive, and it is clear that $f'(x) \geq 0$. As a result, $f(x)$ is increasing; **A3** holds. On the other hand, we have

$$\left(\frac{x}{f(x)}\right)' = \frac{1}{(1 + a^2x^2)\arctan(ax)} \left[\frac{\arctan(ax)}{\arctan(a)}\right]^{-b} [(1 + a^2x^2)\arctan(ax) - abx]$$

Only the last term in the square bracket needs a sign study, the other main terms being positive. By the following well-known arctangent inequality: $\arctan(u) \geq u/(1 + u^2)$ for $u \geq 0$, we establish that

$$\left(\frac{x}{f(x)}\right)' \geq \frac{1}{(1 + a^2x^2)\arctan(ax)} \left[\frac{\arctan(ax)}{\arctan(a)}\right]^{-b} \left[(1 + a^2x^2)\frac{ax}{1 + a^2x^2} - abx\right]$$

$$= \frac{1}{(1 + a^2x^2)\arctan(ax)} ax \left[\frac{\arctan(ax)}{\arctan(a)}\right]^{-b} (1 - b)$$

Thus, for $b \in (0, 1]$, we have $[x/f(x)]' \geq 0$, so that $x/f(x)$ is increasing; **A4** is fulfilled. This ends the proof. □

3.4. Logarithmic-like functions

The subsequent proposition elucidates a logarithmic-like function with two parameters, designed to meet the required assumptions.

Proposition 8. Let $(a, b) \in \mathbb{R}^2$. The following function satisfies assumptions **A1–A4**:

$$f(x) = \left[\frac{\log(1 + ax)}{\log(1 + a)}\right]^b, \quad x \in [0, 1]$$

for $b \in (0, 1]$, $a \geq b - 1$ and $a \neq 0$.

Proof. Since $b \in (0, 1]$, $a \geq b - 1 > -1$, $a \neq 0$, let us mention that $f(x)$ is well-defined because $\log(1+a)$ and $\log(1+ax)$ are of the same sign. Since $a \neq 0$, we have $f(1) = [\log(1+a)/\log(1+a)]^b = 1$;

A1 is satisfied. Moreover, since $b \in (0,1]$, we have $f(0) = [\log(1+a\times 0)/\log(1+a)]^b = 0 \neq 1$ and $f(0) \geq 0$; **A2** is fulfilled. Furthermore, we have

$$f'(x) = b\frac{a}{(1+ax)\log(1+ax)}\left[\frac{\log(1+ax)}{\log(1+a)}\right]^b$$

Since $a > -1$, a and $\log(1+ax)$ of the same sign implying that $a/\log(1+ax) \geq 0$, $1+ax \geq 1-x \geq 0$ and $b \in (0,1]$, we have $f'(x) \geq 0$. As a result, $f(x)$ is increasing; **A3** holds. On the other hand, we have

$$\left(\frac{x}{f(x)}\right)' = \frac{1}{(1+ax)\log(1+ax)}\left[\frac{\log(1+ax)}{\log(1+a)}\right]^{-b}[(1+ax)\log(1+ax) - abx]$$

Let us distinguish the cases $a \in (-1,0)$ and $a > 0$.

For $a \in (-1,0)$, we have $\log(1+ax) \leq 0$, $a/\log(1+ax) \geq 0$, $1+ax \geq 0$ and, by the following well-known logarithmic inequality: $\log(1+u) \leq u$ for $u > -1$, we have

$$\left(\frac{x}{f(x)}\right)' \geq \frac{1}{(1+ax)\log(1+ax)}\left[\frac{\log(1+ax)}{\log(1+a)}\right]^{-b}[(1+ax)ax - abx]$$

$$= \frac{a}{(1+ax)\log(1+ax)}x\left[\frac{\log(1+ax)}{\log(1+a)}\right]^{-b}(ax+1-b)$$

$$\geq \frac{a}{(1+ax)\log(1+ax)}x\left[\frac{\log(1+ax)}{\log(1+a)}\right]^{-b}(a+1-b)$$

Thus, for $a \geq b-1$ and $b \in (0,1]$, we have $[x/f(x)]' \geq 0$, so that $x/f(x)$ is increasing; **A4** is satisfied.

For $a > 0$, we have $\log(1+ax) \geq 0$, $a/\log(1+ax) \geq 0$, $1+ax \geq 0$ and, by the following well-known logarithmic inequality: $\log(1+u) \geq u/(1+u)$ for $u > -1$, we have

$$\left(\frac{x}{f(x)}\right)' \geq \frac{1}{(1+ax)\log(1+ax)}\left[\frac{\log(1+ax)}{\log(1+a)}\right]^{-b}\left[(1+ax)\frac{ax}{1+ax} - abx\right]$$

$$= \frac{a}{(1+ax)\log(1+ax)}x\left[\frac{\log(1+ax)}{\log(1+a)}\right]^{-b}(1-b)$$

Thus, for $b \in (0,1]$, we have $[x/f(x)]' \geq 0$, so that $x/f(x)$ is increasing; **A4** is fulfilled. The list of assumptions is proven. □

3.5. Hyperbolic-like functions

The result below presents a two-parameter hyperbolic cosine-like function that fulfills the necessary assumptions. We recall that $\cosh(x) = [\exp(x) + \exp(-x)]/2$.

Proposition 9. Let $(a, b) \in \mathbb{R}^2$. The following function satisfies assumptions **A1–A4**:

$$f(x) = \left[\frac{\cosh(ax)}{\cosh(a)}\right]^b, \quad x \in [0, 1]$$

for $a > 0$, $b > 0$ and $a^2 b \in (0, 1]$.

Proof. Since $a \neq 0$, we have $f(1) = [\cosh(a)/\cosh(a)]^b = 1$; **A1** is satisfied. We have $f(0) = [\cosh(a \times 0)/\cosh(a)]^b = 1/[\cosh(a)]^b \neq 1$ and $f(0) \geq 0$; **A2** is fulfilled. Furthermore, we have

$$f'(x) = ab \tanh(ax)[\operatorname{sech}(a)\cosh(ax)]^b$$

where $\tanh(x) = \sinh(x)/\cosh(x)$ and $\operatorname{sech}(x) = 1/\cosh(x)$. Since $a > 0$ and $b > 0$, all the main terms are positive, and it is clear that $f'(x) \geq 0$. As a result, $f(x)$ is increasing; **A3** holds. On the other hand, we have

$$\left(\frac{x}{f(x)}\right)' = [1 - abx\tanh(ax)][\operatorname{sech}(a)\cosh(ax)]^{-b}$$

By the following well-known hyperbolic tangent inequality: $\tanh(u) \leq u$ for $u \geq 0$, we get

$$\left(\frac{x}{f(x)}\right)' \geq (1 - a^2bx^2)[\operatorname{sech}(a)\cosh(ax)]^{-b} \geq (1 - a^2b)[\operatorname{sech}(a)\cosh(ax)]^{-b}$$

Thus, for $a^2b \in (0, 1]$, we have $[x/f(x)]' \geq 0$, so that $x/f(x)$ is increasing; **A4** is satisfied. This ends the proof. □

A variant is proposed in the proposition below; it illustrates a two-parameter hyperbolic sine-like function that fulfills the necessary assumptions. We recall that $\sinh(x) = [\exp(x) - \exp(-x)]/2$.

Proposition 10. Let $(a, b) \in \mathbb{R}^2$. The following function satisfies assumptions **A1–A4**:

$$f(x) = \left[\frac{\sinh(ax)}{\sinh(a)}\right]^b, \quad x \in [0, 1]$$

for $a > 0$, $b > 0$ and $(a + 1)b \in (0, 1]$.

Proof. Since $a \neq 0$, we have $f(1) = [\sinh(a)/\sinh(a)]^b = 1$; **A1** is fulfilled. We have $f(0) = [\sinh(a \times 0)/\sinh(a)]^b = 0 \neq 1$ and $f(0) \geq 0$; **A2** is satisfied. Furthermore, we have

$$f'(x) = ab \operatorname{cotanh}(ax)[\operatorname{csch}(a)\sinh(ax)]^b$$

where $\operatorname{cotanh}(x) = \cosh(x)/\sinh(x)$ and $\operatorname{csch}(x) = 1/\sinh(x)$. Since $a > 0$ and $b > 0$, all the main terms are positive, and it is clear that $f'(x) \geq 0$. As a result, $f(x)$ is increasing; **A3** holds. On the other hand, we have

$$\left(\frac{x}{f(x)}\right)' = [1 - abx\operatorname{cotanh}(ax)][\operatorname{csch}(a)\sinh(ax)]^{-b}$$

Owing to the following well-known hyperbolic cotangent inequality: $\operatorname{cotanh}(u) \leq 1 + 1/[u(1+u)]$ for $u \geq 0$, since $b/(1+ax) \leq b$, we get

$$\left(\frac{x}{f(x)}\right)' \geq \left(1 - abx - \frac{b}{1+ax}\right)[\operatorname{csch}(a)\sinh(ax)]^{-b} \geq [1-(a+1)b][\operatorname{csch}(a)\sinh(ax)]^{-b}$$

Thus, for $(a+1)b \in (0,1]$, we have $[x/f(x)]' \geq 0$, so that $x/f(x)$ is increasing; **A4** is fulfilled. This finishes the proof. □

This completes the main list of proposed functions satisfying the assumptions of Durante's method. However, other types of functions, including special functions (integral, series, etc.), can be considered. A complementary study supporting this claim is given below.

3.6. Complement. A special function

To begin, let us recall that the error function is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)dt, \quad x \geq 0$$

The result below exhibits a two-parameter error-like function that satisfies the assumptions of Durante's method.

Proposition 11. Let $(a,b) \in \mathbb{R}^2$. The following function satisfies assumptions **A1–A4**:

$$f(x) = \left[\frac{\operatorname{erf}(ax)}{\operatorname{erf}(a)}\right]^b, \quad x \in [0,1]$$

for $a > 0$ and $b \in (0,1]$.

Proof. It is immediate that $f(1) = [\operatorname{erf}(a)/\operatorname{erf}(a)]^b = 1$; **A1** is satisfied. Since $b \in (0,1]$ and $\operatorname{erf}(0) = 0$, we have $f(0) = [\operatorname{erf}(a \times 0)/\operatorname{erf}(a)]^b = 0 \neq 1$ and $f(0) \geq 0$; **A2** fulfilled. On the other hand, we have

$$f'(x) = \frac{2}{\sqrt{\pi}\operatorname{erf}(ax)} ab \exp(-a^2x^2)\left[\frac{\operatorname{erf}(ax)}{\operatorname{erf}(a)}\right]^b$$

Since $a > 0$ and $b \in (0,1]$, it is immediate that $f'(x) \geq 0$ as the product of positive functions. Therefore, $f(x)$ is increasing. So **A3** is fulfilled. On the other hand, we have

$$\left(\frac{x}{f(x)}\right)' = \left[1 - \frac{2}{\sqrt{\pi}\operatorname{erf}(ax)} abx \exp(-a^2x^2)\right]\left[\frac{\operatorname{erf}(ax)}{\operatorname{erf}(a)}\right]^{-b}$$

Since $a > 0$ and $\exp(-t^2)$ is a decreasing positive function, we have

$$\frac{\sqrt{\pi}\operatorname{erf}(ax)}{2} = \int_0^{ax} \exp(-t^2)dt \geq \exp(-a^2x^2)\int_0^{ax} dt = ax\exp(-a^2x^2)$$

Using this inequality and $b \in (0,1]$, we obtain

$$\left(\frac{x}{f(x)}\right)' \geq \left[1 - \frac{2}{\sqrt{\pi}\,\mathrm{erf}(ax)} ax \exp(-a^2x^2)\right] \left[\frac{\mathrm{erf}(ax)}{\mathrm{erf}(a)}\right]^{-b} \geq 0$$

Hence $[x/f(x)]' \geq 0$, so that $x/f(x)$ is increasing; **A4** is satisfied. The proof is completed. □

The previous result is encouraging for further research in the direction of original functions to use in a copula creation scheme.

4. Examples of copulas

This section shows how Durante's method and the findings of the previous section can be combined to create original multi-parameter copulas with versatile-shaped density.

4.1. Discussion on the baseline copulas

In addition to the choice of $f(x)$ and $g(x)$ in Durante's method, there is a need to select two baseline copulas, $A(x, y)$ and $B(x, y)$, as described in Proposition 1. However, in view of Durante's method, the baseline copulas $A(x, y)$ and $B(x, y)$ must not be too complex to obtain a tractable copula. With this in mind, we will focus on the list of copulas presented in Table 1.

4.2. First series of new copulas

Owing to Durante's method presented in Proposition 1 and the function described in Proposition 2, for any copula $A(x, y)$ and $B(x, y)$, we define a new copula by

$$\begin{aligned} C(x, y) =& A\left\{\frac{x}{[a_1 + (1-a_1)x^{b_1}]^{1/b_1}}, \frac{y}{[a_2 + (1-a_2)y^{b_2}]^{1/b_2}}\right\} \\ & \times B\left\{[a_1 + (1-a_1)x^{b_1}]^{1/b_1}, [a_2 + (1-a_2)y^{b_2}]^{1/b_2}\right\}, \quad (x, y) \in [0,1]^2 \end{aligned}$$

with $(a_1, a_2) \in (0,1)^2$ and $(b_1, b_2) \in \mathbb{R}^2$. In particular, by taking $A(x, y)$ as the independence copula to simplify the situation, i.e., $A(x, y) = xy$, we obtain the following expression:

$$\begin{aligned} C(x, y) =& \frac{xy}{[a_1 + (1-a_1)x^{b_1}]^{1/b_1} [a_2 + (1-a_2)y^{b_2}]^{1/b_2}} \\ & \times B\left\{[a_1 + (1-a_1)x^{b_1}]^{1/b_1}, [a_2 + (1-a_2)y^{b_2}]^{1/b_2}\right\}, \quad (x, y) \in [0,1]^2 \end{aligned}$$

Then we can choose any of the copulas in Table 1 for $B(x, y)$, except the independence copula, which leads to itself for $C(x, y)$. In this way, the obtained copulas are described in Table 2. The following name changes are given: FGM new 1 (FGM-N1) copula, AMH new 1 (AMH-N1) copula, etc.

Table 2. New copulas based on Durante's method (see Propositions 1, 2, and Table 1)

Name	Copula $C(x, y)$	Parameter(s)
FGM-N1	$xy\left[1+\alpha\left\{1-\left[a_1+(1-a_1)x^{b_1}\right]^{1/b_1}\right\}\left\{1-\left[a_2+(1-a_2)y^{b_2}\right]^{1/b_2}\right\}\right]$	$\alpha \in [-1,1]$, $(a_1,a_2)\in(0,1)^2$, $(b_1,b_2)\in\mathbb{R}^2$
AMH-N1	$\dfrac{xy}{1-\alpha\left\{1-\left[a_1+(1-a_1)x^{b_1}\right]^{1/b_1}\right\}\left\{1-\left[a_2+(1-a_2)y^{b_2}\right]^{1/b_2}\right\}}$	$\alpha \in [-1,1]$, $(a_1,a_2)\in(0,1)^2$, $(b_1,b_2)\in\mathbb{R}^2$
C-N1	$\dfrac{xy}{\left[a_1+(1-a_1)x^{b_1}\right]^{1/b_1}\left[a_2+(1-a_2)y^{b_2}\right]^{1/b_2}}\times \left\{\left[a_1+(1-a_1)x^{b_1}\right]^{-\alpha/b_1}+\left[a_2+(1-a_2)y^{b_2}\right]^{-\alpha/b_2}-1\right\}^{-1/\alpha}$	$\alpha>0$, $(a_1,a_2)\in(0,1)^2$, $(b_1,b_2)\in\mathbb{R}^2$
F-N1	$-\dfrac{xy}{\alpha\left[a_1+(1-a_1)x^{b_1}\right]^{1/b_1}\left[a_2+(1-a_2)y^{b_2}\right]^{1/b_2}}\times \log\left\{1+\dfrac{\left[\exp\left\{-\alpha\left[a_1+(1-a_1)x^{b_1}\right]^{1/b_1}\right\}-1\right]\left[\exp\left\{-\alpha\left[a_2+(1-a_2)y^{b_2}\right]^{1/b_2}\right\}-1\right]}{\exp(-\alpha)-1}\right\}$	$\alpha\in\mathbb{R}/\{0\}$, $(a_1,a_2)\in(0,1)^2$, $(b_1,b_2)\in\mathbb{R}^2$
GH-N1	$\dfrac{xy}{\left[a_1+(1-a_1)x^{b_1}\right]^{1/b_1}\left[a_2+(1-a_2)y^{b_2}\right]^{1/b_2}}\times \exp\left\{-\left[\left\{-\dfrac{1}{b_1}\log\left[a_1+(1-a_1)x^{b_1}\right]\right\}^{\alpha}+\left\{-\dfrac{1}{b_2}\log\left[a_2+(1-a_2)y^{b_2}\right]\right\}^{\alpha}\right]^{1/\alpha}\right\}$	$\alpha\geq 1$, $(a_1,a_2)\in(0,1)^2$, $(b_1,b_2)\in\mathbb{R}^2$
GB-N1	$xy\exp\left\{-\dfrac{\alpha}{b_1b_2}\log\left[a_1+(1-a_1)x^{b_1}\right]\log\left[a_2+(1-a_2)y^{b_2}\right]\right\}$	$\alpha\in[0,1]$, $(a_1,a_2)\in(0,1)^2$, $(b_1,b_2)\in\mathbb{R}^2$
CC-N1	$xy\exp\left[-\alpha\left\{1-\left[a_1+(1-a_1)x^{b_1}\right]^{1/b_1}\right\}\left\{1-\left[a_2+(1-a_2)y^{b_2}\right]^{1/b_2}\right\}\right]$	$\alpha \in [-1,1]$, $(a_1,a_2)\in(0,1)^2$, $(b_1,b_2)\in\mathbb{R}^2$
Ch-N1	$xy\exp\left[-\alpha\left\{1-\left[a_1+(1-a_1)x^{b_1}\right]^{-1/b_1}\right\}\left\{1-\left[a_2+(1-a_2)y^{b_2}\right]^{-1/b_2}\right\}\right]$	$\alpha\in[0,1]$, $(a_1,a_2)\in(0,1)^2$, $(b_1,b_2)\in\mathbb{R}^2$

To the best of our knowledge, all the presented copulas are new. As one of the most intriguing examples, let us focus on the FGM-N1 copula, which is defined by

$$C(x,\, y) = xy \left[1 + \alpha \left\{1 - \left[a_1 + (1 - a_1)x^{b_1}\right]^{1/b_1}\right\} \left\{1 - \left[a_2 + (1 - a_2)y^{b_2}\right]^{1/b_2}\right\}\right], \;\; (x,\, y) \in [0, 1]^2$$

where $\alpha \in [-1, 1]$, $(a_1, a_2) \in (0, 1)^2$ and $(b_1, b_2) \in \mathbb{R}^2$.

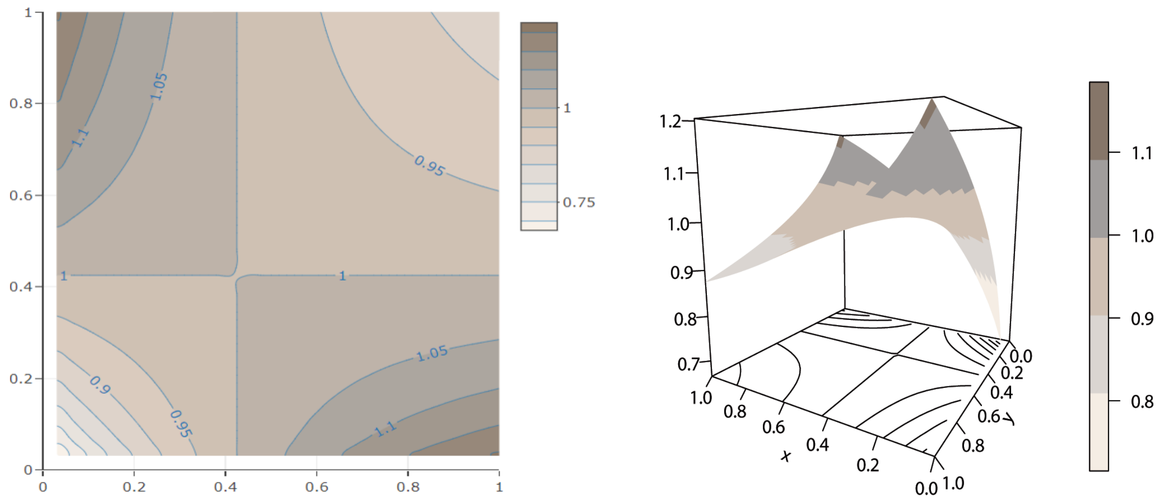
The FGM-N1 copula density is obtained as the mixed derivative of $C(x,\, y)$, i.e., $c(x,\, y) = \partial^2 C(x,\, y)/(\partial x \partial y)$. After some mathematical manipulations, we get

$$\begin{aligned} c(x,\, y) =& 1 + \alpha(1 - a_1)(1 - a_2)x^{b_1}y^{b_2}\left[a_1 + (1 - a_1)x^{b_1}\right]^{1/b_1 - 1}\left[a_2 + (1 - a_2)y^{b_2}\right]^{1/b_2 - 1} \\ &- \alpha(1 - a_1)x^{b_1}\left[a_1 + (1 - a_1)x^{b_1}\right]^{1/b_1 - 1}\left\{1 - \left[a_2 + (1 - a_2)y^{b_2}\right]^{1/b_2}\right\} \\ &- \alpha(1 - a_2)y^{b_2}\left\{1 - \left[a_1 + (1 - a_1)x^{b_1}\right]^{1/b_1}\right\}\left[a_2 + (1 - a_2)y^{b_2}\right]^{1/b_2 - 1} \\ &+ \alpha\left\{1 - \left[a_1 + (1 - a_1)x^{b_1}\right]^{1/b_1}\right\}\left\{1 - \left[a_2 + (1 - a_2)y^{b_2}\right]^{1/b_2}\right\}, \quad (x,\, y) \in [0, 1]^2 \end{aligned}$$

The more the shapes of such a copula density are varied, the more the subjacent dependence's structure is versatile and can adapt to various dependence scenarios.

To understand the possibilities of the FGM-N1 copula density, let us plot it for several values of the parameters α, a_1, a_2, b_1 and b_2. Figures 1–4 represent it for the following parameter configurations:

Conf1: $\alpha = -0.5$, $a_1 = 0.5$, $a_2 = 0.5$, $b_1 = -0.5$ and $b_2 = -0.5$,
Conf2: $\alpha = 0.5$, $a_1 = 0.1$, $a_2 = 0.7$, $b_1 = -1$ and $b_2 = -0.1$,
Conf3: $\alpha = 0.3$, $a_1 = 0.7$, $a_2 = 0.2$, $b_1 = 10$ and $b_2 = -7$,
Conf4.$\alpha = 0.7$, $a_1 = 0.1$, $a_2 = 0.8$, $b_1 = 1.5$ and $b_2 = -2$.



a) b)

Figure 1. Plots of the FGM-N1 copula density under Conf1: a) contours, and b) shapes

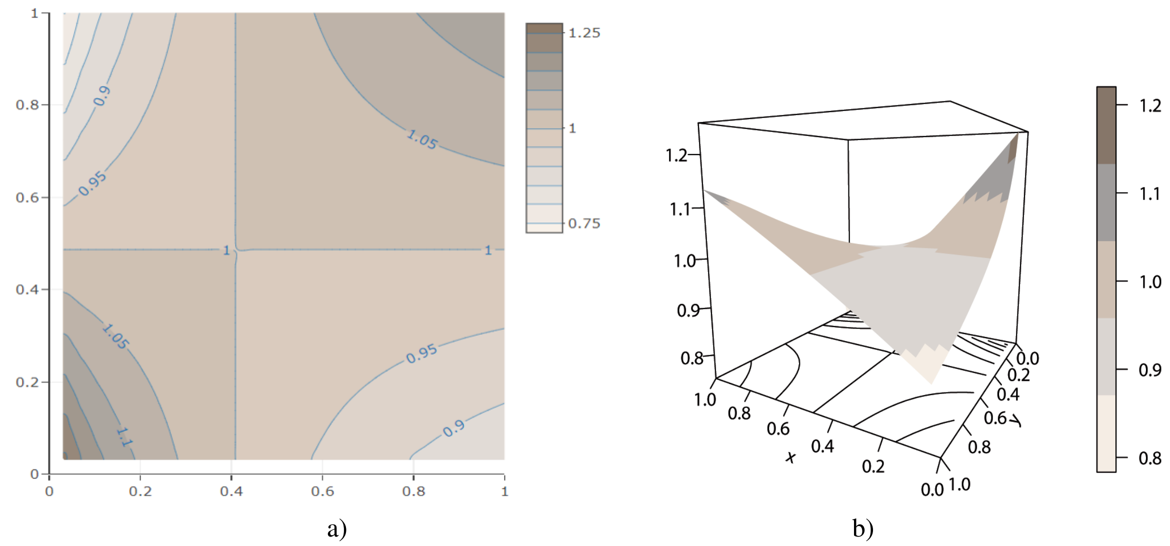


Figure 2. Plots of the FGM-N1 copula density under Conf2: a) contours, and b) shapes

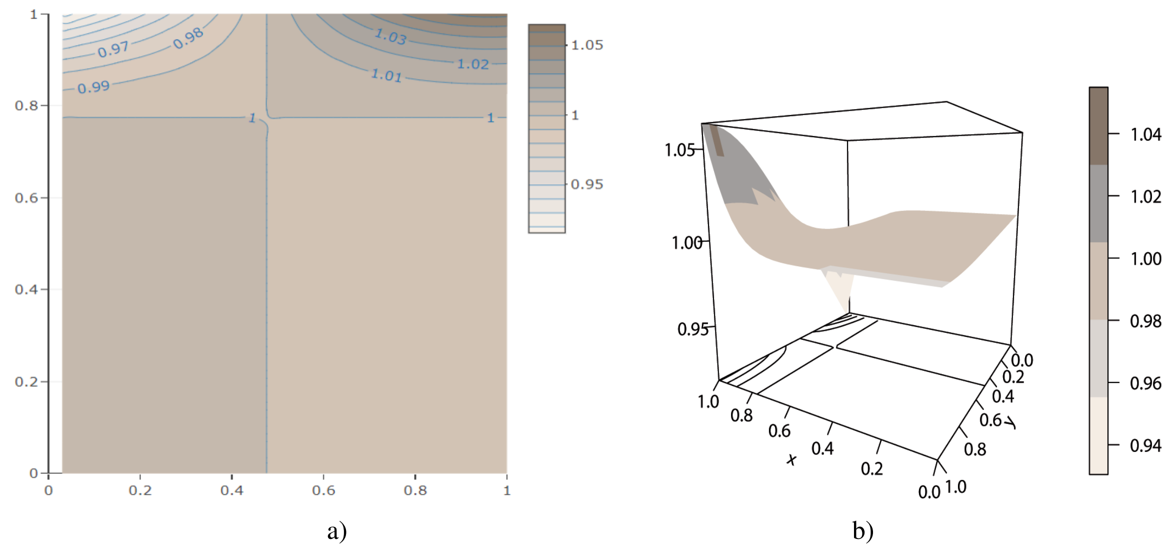


Figure 3. Plots of the FGM-N1 copula density under Conf3: a) contours, and b) shapes

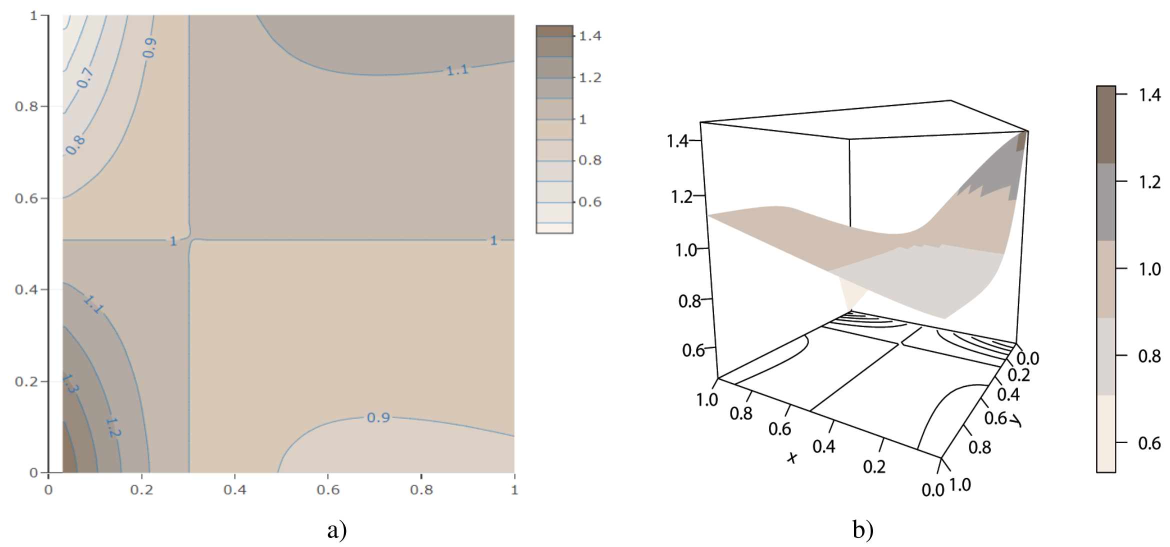


Figure 4. Plots of the FGM-N1 copula density under Conf4: a) contours, and b) shapes

These figures reveal completely different shapes depending on the parameter values: we observe bump-like, spike-like, plate-like, and corner-like shapes. This demonstrates the versatile dependence modeling aspect of the FGM-N1 copula density.

In addition, the following comments on the FGM-N1 copula can be made: it is non-exchangeable for $a_1 \neq a_2$ or $b_1 \neq b_2$, it is not Archimedean, the medial correlation is obtained as

$$M = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1 = \left[1 + \alpha\left\{1 - \left[a_1 + (1 - a_1)2^{-b_1}\right]^{1/b_1}\right\}\right.$$
$$\left.\times \left\{1 - \left[a_2 + (1 - a_2)2^{-b_2}\right]^{1/b_2}\right\}\right] - 1$$

Kendall's τ and Spearman's ρ have the ranges $[-0.22, 0.22]$ and $[-0.33, 0.33]$, respectively, and a limit analysis shows that it has no lower and upper tail dependences, i.e., $\lambda_L = \lambda_U = 0$, where λ_L and λ_U denote the lower and upper tail dependence coefficients, respectively, which are similar characteristics to those of the FGM copula. The main interest of the FGM-N1 copula remains in its high diversity in its shape density structure, which can make a significant difference in capturing the maximum amount of dependence information behind random variables.

4.3. Second series of new copulas

Based on Durante's method and the function described in Proposition 5, for any copula $A(x, y)$ and $B(x, y)$, we define a new copula by

$$C(x, y) = A\left[\frac{x}{1 - a_1(1 - x)\exp(-b_1 x)}, \frac{y}{1 - a_2(1 - y)\exp(-b_2 y)}\right]$$
$$\times B\left[1 - a_1(1 - x)\exp(-b_1 x), 1 - a_2(1 - y)\exp(-b_2 y)\right], \quad (x, y) \in [0, 1]^2$$

with $(a_1, a_2) \in (0, 1]^2$ and $(b_1, b_2) \in [0, \infty)^2$. In particular, by taking $A(x, y)$ as the independence copula to simplify the situation, the main copula is reduced to

$$C(x, y) = \frac{xy}{[1 - a_1(1 - x)\exp(-b_1 x)][1 - a_2(1 - y)\exp(-b_2 y)]}$$
$$\times B\left[1 - a_1(1 - x)\exp(-b_1 x), 1 - a_2(1 - y)\exp(-b_2 y)\right], \quad (x, y) \in [0, 1]^2$$

Then we can choose any copula in Table 1 for $B(x, y)$. In particular, the CC-N2 copula is defined by

$$C(x, y) = xy\exp\left[-\alpha a_1 a_2 (1 - x)(1 - y)\exp(-b_1 x - b_2 y)\right], \quad (x, y) \in [0, 1]^2$$

where $\alpha \in [-1, 1]$, $(a_1, a_2) \in (0, 1]^2$ and $(b_1, b_2) \in [0, \infty)^2$.

Let us notice that it can be rewritten as

$$C(x, y) = xy\exp\left[-\theta(1 - x)(1 - y)\exp(-b_1 x - b_2 y)\right], \quad (x, y) \in [0, 1]^2$$

where $\theta = \alpha a_1 a_2 \in [-1, 1]$. The parameter θ can be viewed as a single product parameter independent of b_1 and b_2, making the CC-N2 copula a three-parameter one.

Table 3. New copulas based on Durante's method; Proposition 5 and Table 1

Name	Copula $C(x,\, y)$	Parameter(s)
FGM-N2	$xy\left[1+\alpha a_1a_2(1-x)(1-y)\exp(-b_1x-b_2y)\right]$	$\alpha\in[-1,1]$, $(a_1,a_2)\in(0,1]^2$, $(b_1,b_2)\in[0,\infty)^2$
AMH-N2	$\dfrac{xy}{1-\alpha a_1a_2(1-x)(1-y)\exp(-b_1x-b_2y)}$	$\alpha\in[-1,1]$, $(a_1,a_2)\in(0,1]^2$, $(b_1,b_2)\in[0,\infty)^2$
C-N2	$\dfrac{xy}{[1-a_1(1-x)\exp(-b_1x)][1-a_2(1-y)\exp(-b_2y)]}\times$ $\left\{[1-a_1(1-x)\exp(-b_1x)]^{-\alpha}+[1-a_2(1-y)\exp(-b_2y)]^{-\alpha}-1\right\}^{-1/\alpha}$	$\alpha>0$, $(a_1,a_2)\in(0,1]^2$, $(b_1,b_2)\in[0,\infty)^2$
F-N2	$-\dfrac{xy}{\alpha[1-a_1(1-x)\exp(-b_1x)][1-a_2(1-y)\exp(-b_2y)]}\times$ $\log\left\{1+\dfrac{[\exp\{-\alpha[1-a_1(1-x)\exp(-b_1x)]\}-1]\,[\exp\{-\alpha[1-a_2(1-y)\exp(-b_2y)]\}-1]}{\exp(-\alpha)-1}\right\}$	$\alpha\in\mathbb{R}/\{0\}$, $(a_1,a_2)\in(0,1]^2$, $(b_1,b_2)\in[0,\infty)^2$
GH-N2	$\dfrac{xy}{[1-a_1(1-x)\exp(-b_1x)][1-a_2(1-y)\exp(-b_2y)]}\times$ $\exp\left\{-\left[\{-\log[1-a_1(1-x)\exp(-b_1x)]\}^{\alpha}+\{-\log[1-a_2(1-y)\exp(-b_2y)]\}^{\alpha}\right]^{1/\alpha}\right\}$	$\alpha\geq1$, $(a_1,a_2)\in(0,1]^2$, $(b_1,b_2)\in[0,\infty)^2$
GB-N2	$xy\exp\left\{-\alpha\log[1-a_1(1-x)\exp(-b_1x)]\log[1-a_2(1-y)\exp(-b_2y)]\right\}$	$\alpha\in[0,1]$, $(a_1,a_2)\in(0,1]^2$, $(b_1,b_2)\in[0,\infty)^2$
CC-N2	$xy\exp\left[-\alpha a_1a_2(1-x)(1-y)\exp(-b_1x-b_2y)\right]$	$\alpha\in[-1,1]$, $(a_1,a_2)\in(0,1]^2$, $(b_1,b_2)\in[0,\infty)^2$
Ch-N2	$xy\exp\left\{-\alpha\left[1-\dfrac{1}{1-a_1(1-x)\exp(-b_1x)}\right]\left[1-\dfrac{1}{1-a_2(1-y)\exp(-b_2y)}\right]\right\}$	$\alpha\in[0,1]$, $(a_1,a_2)\in(0,1]^2$, $(b_1,b_2)\in[0,\infty)^2$

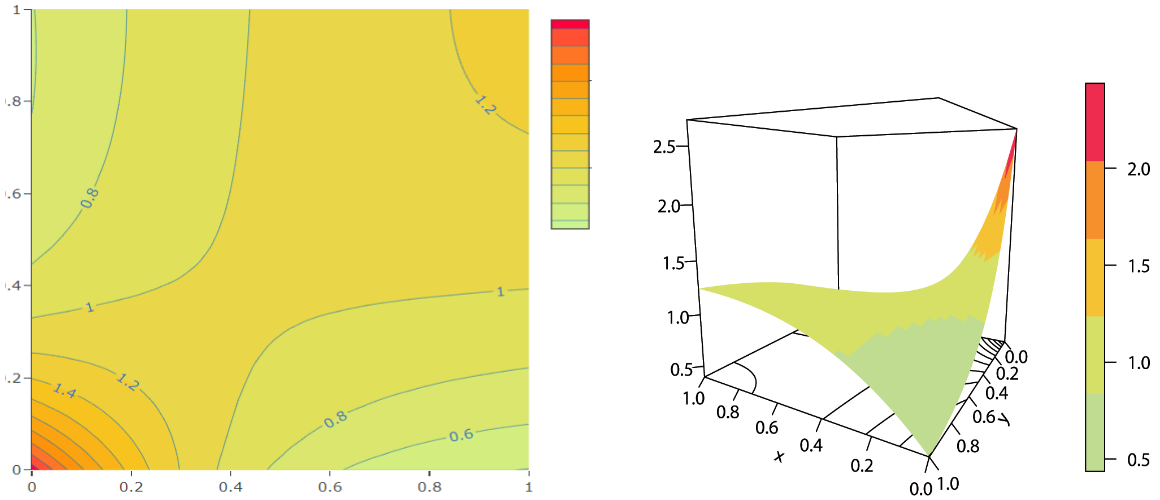
Based on this simplified expression, the CC-N2 copula density is obtained as

$$
\begin{aligned}
c(x, y) = {} & x \exp\left[-\theta(1-x)(1-y)\exp(-b_1 x - b_2 y)\right] \\
& \times \left[\theta(1-y)\exp(-b_1 x - b_2 y) + \theta b_1 (1-x)(1-y)\exp(-b_1 x - b_2 y)\right] \\
& + xy \exp\left[-\theta(1-x)(1-y)\exp(-b_1 x - b_2 y)\right]\left[\theta(1-x)\exp(-b_1 x - b_2 y)\right. \\
& \left. + \theta b_2 (1-x)(1-y)\exp(-b_1 x - b_2 y)\right] \\
& \times \left[\theta(1-y)\exp(-b_1 x - b_2 y) + \theta b_1 (1-x)(1-y)\exp(-b_1 x - b_2 y)\right] \\
& + \exp\left[-\theta(1-x)(1-y)\exp(-b_1 x - b_2 y)\right] \\
& + y \exp\left[-\theta(1-x)(1-y)\exp(-b_1 x - b_2 y)\right] \\
& \times \left[\theta(1-x)\exp(-b_1 x - b_2 y) + \theta b_2 (1-x)(1-y)\exp(-b_1 x - b_2 y)\right] \\
& + xy \exp\left[-\theta(1-x)(1-y)\exp(-b_1 x - b_2 y)\right] \\
& \times \left[-\theta \exp(-b_1 x - b_2 y) - \theta b_1 (1-x)\exp(-b_1 x - b_2 y)\right. \\
& \left. - \theta b_2 (1-y)\exp(-b_1 x - b_2 y) - \theta b_1 b_2 (1-x)(1-y)\exp(-b_1 x - b_2 y)\right], \quad (x, y) \in [0,1]^2
\end{aligned}
$$

Let us plot this copula density for various values of the parameters θ, b_1, and b_2 in order to comprehend its possible shapes. Figures 5–8 represent it for the following parameter configurations:

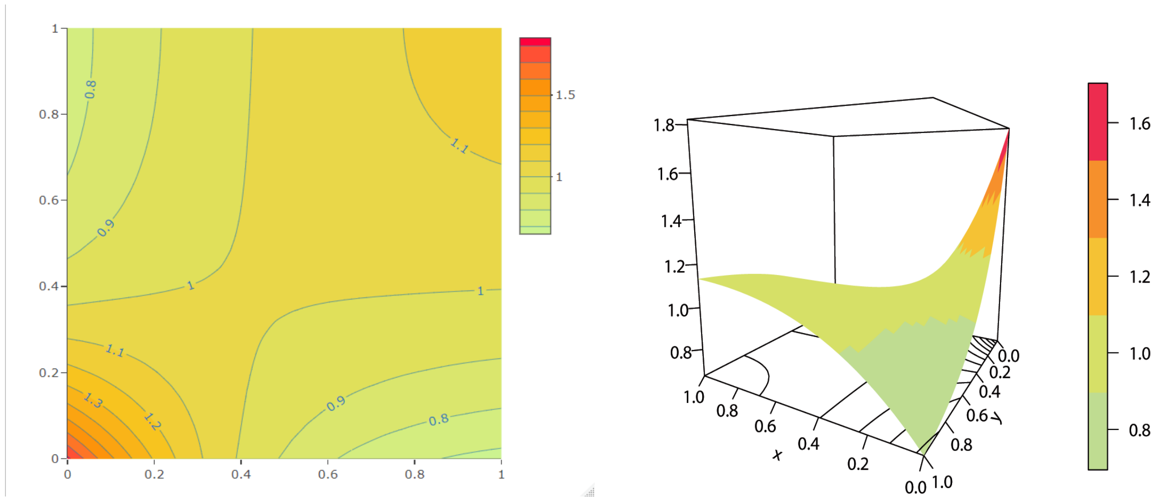
Conf1: $\theta = -1$, $b_1 = 0.9$ and $b_2 = 0.5$, Conf2: $\theta = -0.6$, $b_1 = 0.9$ and $b_2 = 0.6$,
Conf3: $\theta = 0.3$, $b_1 = 0.4$ and $b_2 = 0.1$, Conf4: $\theta = 0.9$, $b_1 = 0.7$ and $b_2 = 2$.

These figures illustrate the versatile dependence modeling aspect of the CC-N2 copula density, revealing entirely different shapes depending on the parameter values.



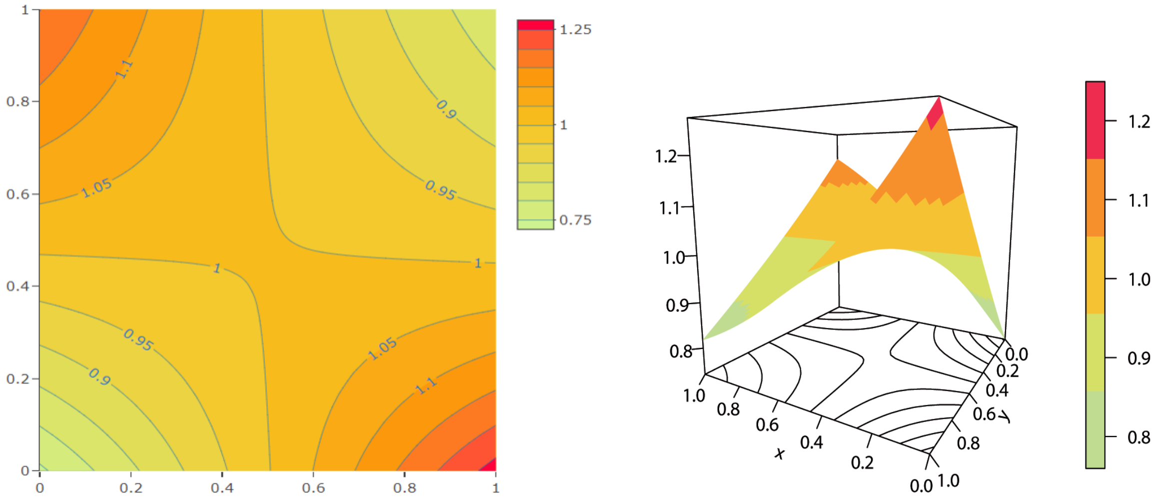
a) b)

Figure 5. Plots of the CC-N2 copula density under Conf1: a) contours, and b) shapes



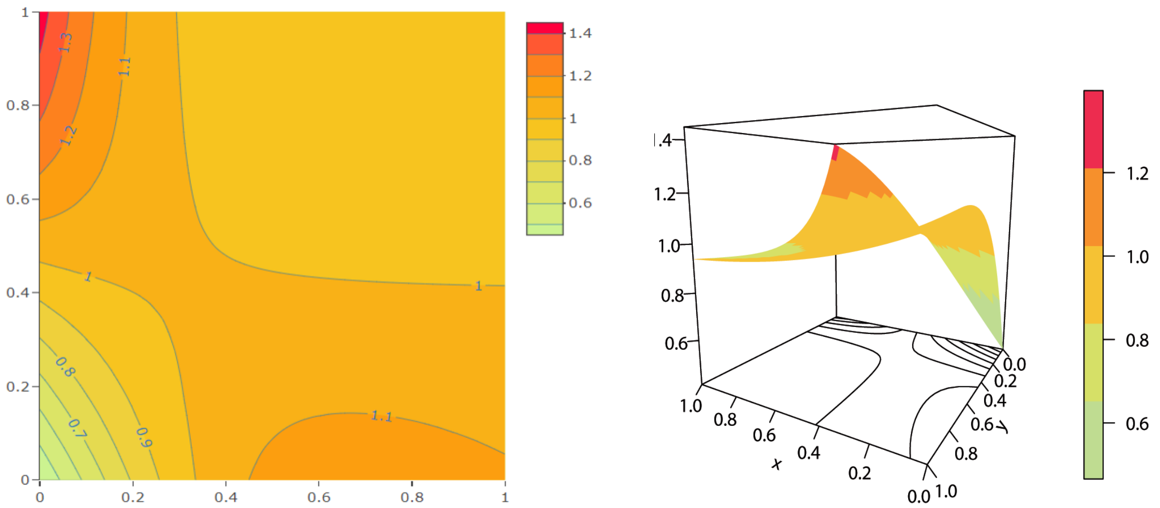
a) b)

Figure 6. Plots of the CC-N2 copula density under Conf2: a) contours, and b) shapes



a) b)

Figure 7. Plots of the CC-N2 copula density under Conf3: a) contours, and b) shapes



a) b)

Figure 8. Plots of the CC-N2 copula density under Conf4: a) contours, and b) shapes

On the other hand, the following comments on the CC-N2 copula can be made: it is non-exchangeable for $b_1 \neq b_2$, it is not Archimedean, the medial correlation is obtained as

$$M = \exp\left\{-\frac{\theta}{4}\exp\left[-\frac{1}{2}(b_1+b_2)\right]\right\} - 1$$

Kendall's τ and Spearman's ρ have the ranges $[-0.20, 0.26]$ and $[-0.29, 0.38]$, respectively, and it has no lower and upper tail dependences. These features are close to those of the CC copula. Thus, the main interest of the CC-N2 copula remains in its high diversity in its shape density structure.

4.4. Third series of new copulas

Owing to Durante's method and the function described in Proposition 6, for any copula $A(x, y)$ and $B(x, y)$, we define a new copula by

$$C(x, y) = A\left\{x\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1}, y\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2}\right\} B\left\{\left[\frac{\sin(a_1x)}{\sin(a_1)}\right]^{b_1}, \left[\frac{\sin(a_2y)}{\sin(a_2)}\right]^{b_2}\right\}, \quad (x, y) \in [0,1]^2$$

with $(a_1, a_2) \in (0, \pi/2]^2$ and $(b_1, b_2) \in (0, 1]^2$. In particular, by taking $A(x, y)$ as the independence copula to simplify the situation, the main copula is reduced to

$$C(x, y) = xy\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1}\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2} B\left\{\left[\frac{\sin(a_1x)}{\sin(a_1)}\right]^{b_1}, \left[\frac{\sin(a_2y)}{\sin(a_2)}\right]^{b_2}\right\}, \quad (x, y) \in [0,1]^2$$

Then we can choose any copula in Table 1 for $B(x, y)$. In particular, the Ch-N3 copula is defined by

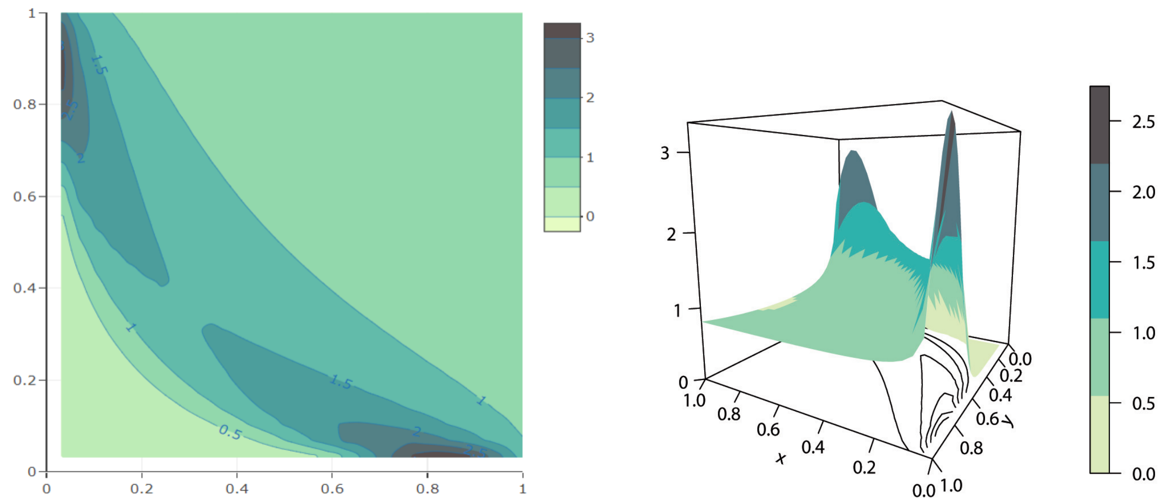
$$C(x, y) = xy\exp\left[-\alpha\left\{1-\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1}\right\}\left\{1-\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2}\right\}\right], \quad (x, y) \in [0,1]^2$$

where $\alpha \in [0, 1]$, $(a_1, a_2) \in (0, \pi/2]^2$ and $(b_1, b_2) \in (0, 1]^2$. The Ch-N3 copula density is obtained as

$$\begin{aligned} c(x, y) = {} & \exp\left[-\alpha\left\{1-\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1}\right\}\left\{1-\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2}\right\}\right] \\ & \times\left[1+\alpha^2 a_1a_2b_1b_2xy\sin(a_1)\sin(a_2)\left\{\frac{\cos(a_1x)}{[\sin(a_1x)]^2}\right\}\left\{\frac{\cos(a_2y)}{[\sin(a_2y)]^2}\right\} es\right. \\ & \times\left\{1-\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1}\right\}\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1-1}\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2-1}\left\{1-\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2}\right\} \\ & -\alpha a_1a_2b_1b_2xy\sin(a_1)\sin(a_2)\left\{\frac{\cos(a_1x)}{[\sin(a_1x)]^2}\right\}\left\{\frac{\cos(a_2y)}{[\sin(a_2y)]^2}\right\}\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1-1}\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2-1} \\ & -\alpha a_1b_1x\sin(a_1)\left\{\frac{\cos(a_1x)}{[\sin(a_1x)]^2}\right\}\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1-1}\left\{1-\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2}\right\} \\ & \left.-\alpha a_2b_2y\sin(a_2)\left\{\frac{\cos(a_2y)}{[\sin(a_2y)]^2}\right\}\left\{1-\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1}\right\}\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2-1}\right], \quad (x, y) \in [0,1]^2 \end{aligned}$$

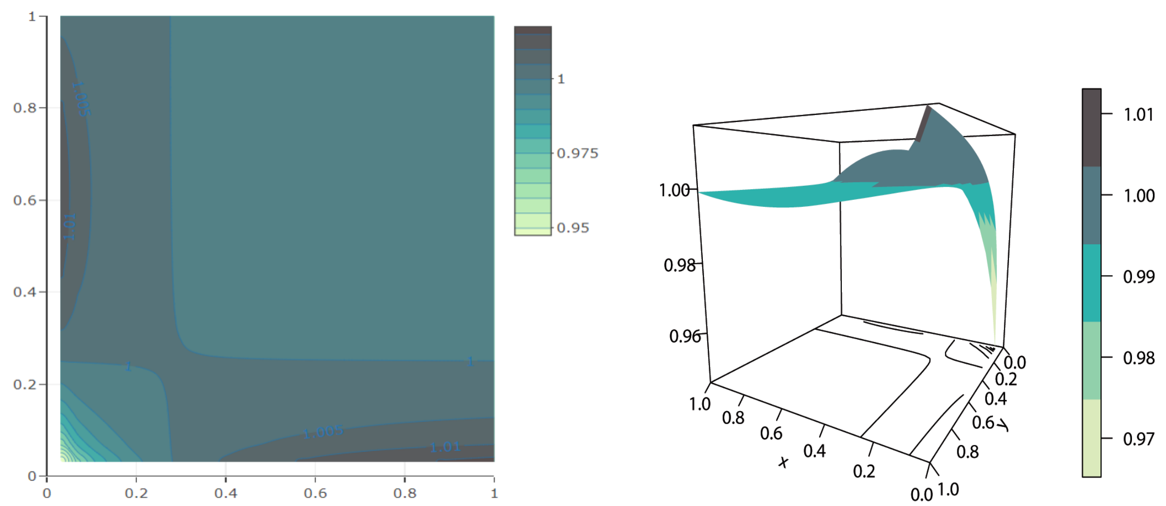
Table 4. New copulas based on Durante's method (see Proposition 6 and Table 1)

Name	Copula $C(x,\, y)$	Parameter(s)
FGM-N3	$xy\left[1+\alpha\left\{1-\left[\frac{\sin(a_1x)}{\sin(a_1)}\right]^{b_1}\right\}\left\{1-\left[\frac{\sin(a_2y)}{\sin(a_2)}\right]^{b_2}\right\}\right]$	$\alpha\in[-1,1]$, $(a_1,a_2)\in(0,\pi/2]^2$, $(b_1,b_2)\in(0,1]^2$
AMH-N3	$\frac{xy}{1-\alpha\left\{1-\left[\frac{\sin(a_1x)}{\sin(a_1)}\right]^{b_1}\right\}\left\{1-\left[\frac{\sin(a_2y)}{\sin(a_2)}\right]^{b_2}\right\}}$	$\alpha\in[-1,1]$, $(a_1,a_2)\in(0,\pi/2]^2$, $(b_1,b_2)\in(0,1]^2$
C-N3	$xy\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1}\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2}\left\{\left[\frac{\sin(a_1x)}{\sin(a_1)}\right]^{-\alpha b_1}+\left[\frac{\sin(a_2y)}{\sin(a_2)}\right]^{-\alpha b_2}-1\right\}^{-1/\alpha}$	$\alpha>0$, $(a_1,a_2)\in(0,\pi/2]^2$, $(b_1,b_2)\in(0,1]^2$
F-N3	$-xy\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1}\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2}\frac{1}{\alpha}\times$ $\log\left\{1+\frac{\left[\exp\left\{-\alpha\left[\frac{\sin(a_1x)}{\sin(a_1)}\right]^{b_1}\right\}-1\right]\left[\exp\left\{-\alpha\left[\frac{\sin(a_2y)}{\sin(a_2)}\right]^{b_2}\right\}-1\right]}{\exp(-\alpha)-1}\right\}$	$\alpha\in\mathbb{R}/\{0\}$, $(a_1,a_2)\in(0,\pi/2]^2$, $(b_1,b_2)\in(0,1]^2$
GH-N3	$xy\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1}\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2}\times$ $\exp\left\{-\left[\left\{-b_1\log\left[\frac{\sin(a_1x)}{\sin(a_1)}\right]\right\}^{\alpha}+\left\{-b_2\log\left[\frac{\sin(a_2y)}{\sin(a_2)}\right]\right\}^{\alpha}\right]^{1/\alpha}\right\}$	$\alpha\geq 1$, $(a_1,a_2)\in(0,\pi/2]^2$, $(b_1,b_2)\in(0,1]^2$
GB-N3	$xy\exp\left\{-\alpha b_1b_2\log\left[\frac{\sin(a_1x)}{\sin(a_1)}\right]\log\left[\frac{\sin(a_2y)}{\sin(a_2)}\right]\right\}$	$\alpha\in[0,1]$, $a_1\in(0,\pi/2]$, $a_2\in(0,\pi/2]$, $(b_1,b_2)\in(0,1]^2$
CC-N3	$xy\exp\left[-\alpha\left\{1-\left[\frac{\sin(a_1x)}{\sin(a_1)}\right]^{b_1}\right\}\left\{1-\left[\frac{\sin(a_2y)}{\sin(a_2)}\right]^{b_2}\right\}\right]$	$\alpha\in[-1,1]$, $(a_1,a_2)\in(0,\pi/2]^2$, $(b_1,b_2)\in(0,1]^2$
Ch-N3	$xy\exp\left[-\alpha\left\{1-\left[\frac{\sin(a_1)}{\sin(a_1x)}\right]^{b_1}\right\}\left\{1-\left[\frac{\sin(a_2)}{\sin(a_2y)}\right]^{b_2}\right\}\right]$	$\alpha\in[0,1]$, $(a_1,a_2)\in(0,\pi/2]^2$, $(b_1,b_2)\in(0,1]^2$



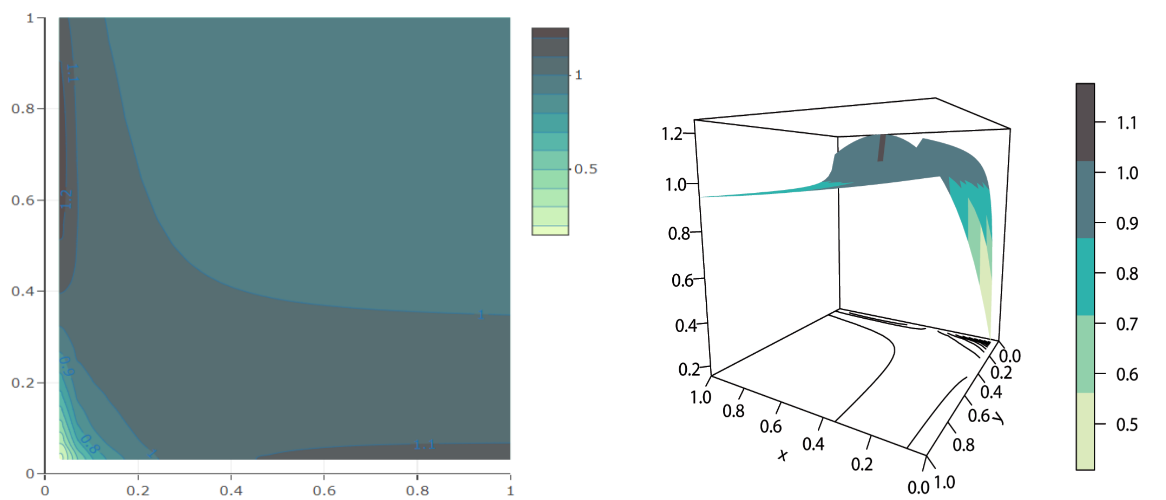
a) b)

Figure 9. Plots of the Ch-N3 copula density under Conf1: a) contours, and b) shapes



a) b)

Figure 10. Plots of the Ch-N3 copula density under Conf2: a) contours, and b) shapes



a) b)

Figure 11. Plots of the Ch-N3 copula density under Conf3: a) contours, and b) shapes

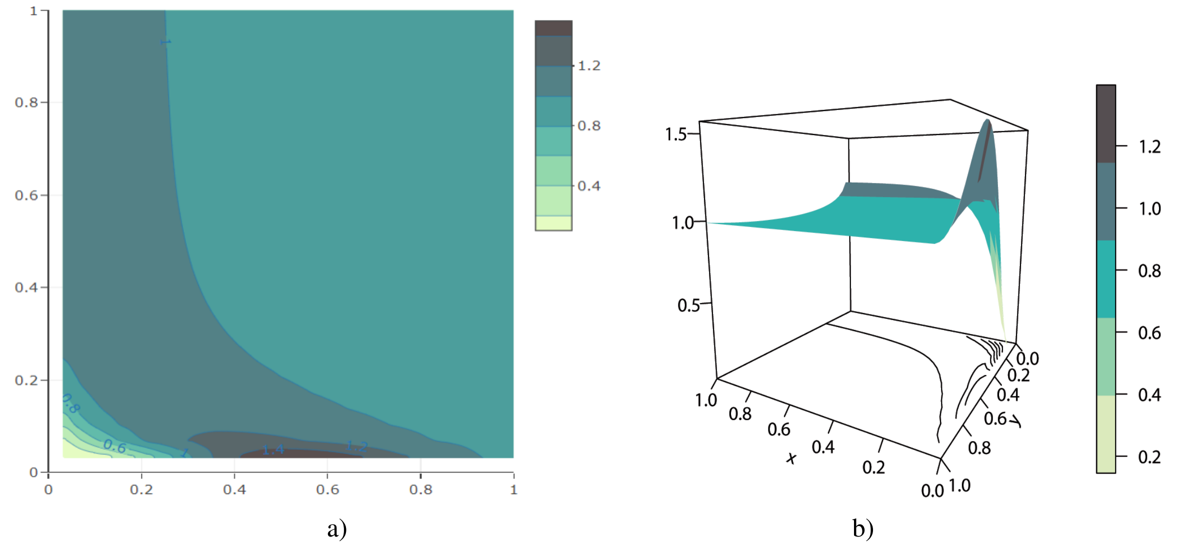


Figure 12. Plots of the Ch-N3 copula density under Conf4: a) contours, and b) shapes

To understand its shape possibilities, let us plot this copula density for several values of the parameters α, a_1, a_2, b_1 and b_2.

Figures 9, 10, 11 and 12 represent it for the following parameter configurations:

Conf1: $\alpha = 0.9$, $a_1 = 0.5$, $a_2 = 1.3$, $b_1 = 0.9$ and $b_2 = 0.7$,

Conf2: $\alpha = 0.2$, $a_1 = 1.5$, $a_2 = 0.3$, $b_1 = 0.1$ and $b_2 = 0.4$,

Conf3: $\alpha = 0.8$, $a_1 = 0.1$, $a_2 = 0.6$, $b_1 = 0.1$ and $b_2 = 0.8$,

Conf4: $\alpha = 1$, $a_1 = 0.5$, $a_2 = 1.5$, $b_1 = 1$ and $b_2 = 0.1$.

These figures show how the Ch-N3 copula density can be used for versatile dependence modeling, with completely different shapes revealed depending on the parameter values.

In addition, the following comments on the Ch-N3 copula can be made: it is non-exchangeable for $a_1 \neq a_2$ or $b_1 \neq b_2$, it is not Archimedean, the medial correlation is obtained as

$$M = \exp\left[-\alpha\left\{1-\left[\frac{\sin(a_1)}{\sin(a_1/2)}\right]^{b_1}\right\}\left\{1-\left[\frac{\sin(a_2)}{\sin(a_2/2)}\right]^{b_2}\right\}\right]-1$$

Kendall's τ and Spearman's ρ have the ranges $[-0.65, 0]$ and $[-0.83, 0]$, respectively, and a limit analysis reveals that it is upper and lower tail dependent, which are characteristics also shared by the Ch copula. Therefore, the gain is mainly in the flexibility of the Ch-N3 copula density shapes, which demonstrate a higher capacity to adapt to adapt to various functional dependence details.

5. Conclusion

In conclusion, this article explores the often overlooked potential of Durante's methodology introduced in 2009, demonstrating its remarkable ability to generate a diverse range of copulas through the use of various parametric functions. The rigorous exploration, including power, exponential, trigonometric, logarithmic, and hyperbolic functions, supported by extensive proofs and mathematical techniques, establishes the robustness of the results. A summary of the considered functions is given in Table 5.

Table 5. List of functions $f(x)$ (or $g(x)$) satisfying Durante's method assumptions considered in this article

Name	Function $f(x)$	Parameter(s)
Func1	$[a+(1-a)x^b]^{1/b}$	$a \in (0,1)$, $b \in \mathbb{R}$
Func2	$1+a(1-c)^b - a(1-cx)^b$	$c \leq 1$, $abc > 0$, $a[1-(1-c)^b] \leq 1$, and either $a(1-b)b < 0$, or $a(1-b)b \geq 0$ and $abc(1-c)^{b-1} \leq 1$
Func3	$\exp[a(1-c)^b - a(1-cx)^b]$	$c \leq 1$, $abc > 0$, $bc \leq 1$ and either $abc(1-c)^{b-1} \leq 1$, or $bc > 1$ and $a(1-1/b)^{b-1} \leq 1$
Func4	$1-a(1-x)\exp(-bx)$	$a \in (0,1]$, $b \in [0,\infty)$
Func5	$\left[\dfrac{\sin(ax)}{\sin(a)}\right]^b$	$a \in (0,\pi/2]$, $b \in (0,1]$
Func6	$\left[\dfrac{\arctan(ax)}{\arctan(a)}\right]^b$	$a > 0$, $b \in (0,1]$
Func7	$\left[\dfrac{\log(1+ax)}{\log(1+a)}\right]^b$	$b \in (0,1]$, $a \geq b-1$, $a \neq 0$
Func8	$\left[\dfrac{\cosh(ax)}{\cosh(a)}\right]^b$	$a > 0$, $b > 0$, $a^2 b \in (0,1]$
Func9	$\left[\dfrac{\sinh(ax)}{\sinh(a)}\right]^b$	$a > 0$, $b > 0$, $(a+1)b \in (0,1]$
FuncComp	$\left[\dfrac{\text{erf}(ax)}{\text{erf}(a)}\right]^b$	$a > 0$, $b \in (0,1]$

Based on these functions, the determination of three distinct sets of eight new copulas further contributes to the theory. A graphical analysis is made to support the shape versatility of the associated copula densities. Beyond theoretical advances, this study introduces a new perspective on the construction of copulas in various contexts, opening new avenues for both creation and analysis. The "more than two-dimensional case is also an interesting direction which needs more investigation.

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References

[1] Ali, M. M., Mikhail, N. N., and Haq, M. S. A class of bivariate distributions including the bivariate logistic. *Journal of Multivariate Analysis 8*, 3 (1978), 405–412.

[2] Barnett, V. Some bivariate uniform distributions. *Communications in Statistics - Theory and Methods 9*, 4 (1980), 453–461.

[3] Celebioğlu, S. A way of generating comprehensive copulas. *Gazi University Journal of Institute of Science and Technology 10*, 1 (1997), 57–61.

[4] Chesneau, C. On new types of multivariate trigonometric copulas. *AppliedMath 1*, 1 (2021), 3–17.

[5] Chesneau, C. On the Gumbel–Barnett extended Celebioglu–Cuadras copula. *Japanese Journal of Statistics and Data Science 6*, 2 (2023), 759–781.

[6] Chesneau, C. A revisit of the modified Celebioglu-Cuadras copula. *Electronic Journal of Mathematical Analysis and Applications 11*, 2 (2023), 10.

[7] Clayton, D. G. A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika 65*, 1 (1978), 141–151.

[8] Cuadras, C. M. Constructing copula functions with weighted geometric means. *Journal of Statistical Planning and Inference 139*, 11 (2009), 3766–3772.

[9] Dolati, A., Mohseni, S., and Úbeda Flores, M. Some results on a transformation of copulas and quasi-copulas. *Information Sciences 257* (2014), 176–182.

[10] Durante, F. Construction of non-exchangeable bivariate distribution functions. *Statistical Papers 50*, 2 (2009), 383–391.

[11] Durante, F., and Sempi, C. *Principles of Copula Theory*. Taylor & Francis, New York, 2015.

[12] El Ktaibi, F., Bentoumi, R., Sottocornola, N., and Mesfioui, M. Bivariate copulas based on counter-monotonic shock method. *Risks 10*, 11 (2022), 202.

[13] Eyraud, H. The principles of correlation measurement. *Annales de l'Université de Lyon. Sciences. Section A, Sciences Mathématiques et Astronomie 1*, 30–47 (1936), 111 (in French).

[14] Frank, M. J. On the simultaneous associativity of F(x, y) and x + y - F(x, y). *Aequationes mathematicae 19* (1979), 194–226.

[15] Gumbel, E. J. Bivariate exponential distributions. *Journal of the American Statistical Association 55*, 292 (1960), 698–707.

[16] Gumbel, E. J. Bivariate logistic distributions. *Journal of the American Statistical Association 56*, 294 (1961), 335–349.

[17] Hougaard, P. A class of multivariate failure time distributions. *Biometrika 73*, 3 (1986), 671–678.

[18] Joe, H. *Multivariate Models and Multivariate Dependence Concepts*. Vol. 73 of *Monographs on Statistics & Applied Probability*. Chapman & Hall/CRC, London, 1997.

[19] Michimae, H., and Emura, T. Likelihood inference for copula models based on left-truncated and competing risks data from field studies. *Mathematics 10*, 13 (2022), 2163.

[20] Mukherjee, S., Lee, Y., Kim, J.-M., Jang, J., and Park, J.-S. Construction of bivariate asymmetric copulas. *Communications for Statistical Applications and Methods 25*, 2 (2018), 217–234.

[21] Nelsen, R. B. *An Introduction to Copulas*. Springer Series in Statistics. Springer, New York, 2007.

[22] Poonia, P. K., and Singh, V. V. Stochastic analysis of delayed reporting of faults in a computer network using copula distribution. *International Journal of Quality Engineering and Technology 9*, 2 (2023), 161–182.

[23] Shih, J.-H., Konno, Y., Chang, Y.-T., and Emura, T. Copula-based estimation methods for a common mean vector for bivariate meta-analyses. *Symmetry 14*, 2 (2022), 186.

[24] Strauch, O., and Baláž, V. Copulas. *Uniform distribution theory 18*, 1 (2023), 47–200.

[25] Susam, S. O. Parameter estimation of some Archimedean copulas based on minimum Cramér-von-Mises distance. *Journal of the Iranian Statistical Society 19*, 1 (2019), 163–183.

[26] Susam, S. O. A new family of Archimedean copula via trigonometric generator function. *Gazi University Journal of Science 33*, 3 (2020), 806–813.

[27] Vaughan, H. E. The expression 0^0. *The Mathematics Teacher 63* (1970), 111–112.

[28] Yeh, C.-T., Liao, G.-Y., and Emura, T. Sensitivity analysis for survival prognostic prediction with gene selection: A copula method for dependent censoring. *Biomedicines 11*, 3 (2023), 797.