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A comparative analysis of $(s,\ Q)$ and $(s,\ S)$ ordering policies in a queuing-inventory system with stock-dependent arrival and queue-dependent service process

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Abstract

This article deals with a Markovian queuing-inventory system (MQIS) under the stochastic modeling technique. The arrival stream of this system is dependent on the present stock level at an instant. Meanwhile, the system focuses on reducing the waiting time of a unit by assuming a queue-dependent service policy (QDSP). The system consists of an infinite waiting hall to receive an arriving unit. The MQIS assumes that no unit of arrival is allowed when the stock level of the system is empty. The discussion of this MQIS runs over the two types of ordering principles named 1) $(s,\ Q)$ 2) $(s,\ S)$. According to both ordering principles, the assumed arrival and service patterns have been considered separately and classified as Model-I (M-I) and Model-II (M-II) respectively. The steady state of the system for both M-I and M-II is analysed and resolved under the Neuts matrix-geometric technique. The system performance measures of the system are also computed. The expected cost function of both M-I and M-II are constructed as well. Further, the necessary numerical illustrations are provided and distinguished for M-I and M-II to explore the proposed model. This paper finds the optimum ordering policy to execute the stock-dependent arrival and queue-dependent service strategies.

Keywords: *stock-dependent arrival, queue-dependent service, infinite waiting hall, ordering principles*

1. Introduction

Successful retail or wholesale businesses almost always have two important aspects: 1) effective management, and 2) inventory control. The observation of an inventory system throughout the whole day provides the knowledge to understand inventory management. In each unit of time, the inventory changes constantly due to sales and service, damage, reordering, and so on. In such a way, effective management

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and inventory control are maintained by limiting these factors. On the other hand, customers play a key role in every inventory business. Inventory management does focus on attracting those customers and making them loyal customers. To do such things, management has to introduce and implement new policies in order to make a predictable profit. For the attraction of customers, the business owners start displaying their products in front of the shop and giving advertisements. In real life, one can see that all the organizations give advertisements through Android phones, television, social media, etc. These advertisements make changes psychologically on the customer's mind to buy the product, and naturally, this will increase the number of arrivals into the inventory system. This idea is applied in the proposed MQIS as it assumes that the arriving jobs occur on the basis of the displayed stock level of the system.

The service facility of the management should be smarter and faster than the other competitors. Because it plays an important role in accommodating the customer in the waiting hall, either to wait for the service or to exit the system. A single server service channel faces customer impatient situation problems in the queuing systems. To eradicate them and generate loyal or happy customers, the management must come up with innovative ideas to develop their service facility. Many organisations try to provide a non-homogeneous service facility in order to avoid customer loss and impatience. When we look into real-life observations, for example, in a single-server fast food restaurant, the server prepares food as fast as the existing queue length. If the queue length decreases, the speed of the server also becomes normal. This phenomenon is to be applied in the proposed MQIS as QDSP. Among these assumptions, the considered MQIS has a detailed discussion based on the two ordering principles: $(s,\, Q)$ or $(s,\, S)$.

1.1. Literature review

The queuing-inventory has received a lot of attention in recent decades for doing research on stochastic modeling. Many varieties of discussions and analyses that exist in the literature are merely related to our proposed model. Since we are all living in a modern technological world, many companies introduce their products for sale with some innovative features. These features are not easily understandable to all customers. They need an explanation about the corresponding products regarding the handling procedures, guarantee, and warranty of the products, etc. Hence, to obtain such an explanation of the product, a customer requires a service facility from the system. According to the queuing-inventory existing literature, Melikov et al. [21] and Sigman et al. [28] introduced the service facility in order to improve customer satisfaction. Subsequently, many authors developed their research with service facilities (see [2, 3, 16, 18, 27]) . The readers can refer to the enlisted papers [1, 4–6, 11, 12, 20, 22, 25] to learn the service facility related interpretations in the stochastic modeling inventory systems.

When we analyse such kinds of MQIS in this modern scenario, every inventory business needs an innovative idea to increase the birth rate of the system. Many businesses try to display their products in the appropriate places in the store in response to the increase in arrivals. Some companies execute new ideas instead of displaying their products. For example, they do advertisements through television, social media, and so on. When adapting this idea to the inventory business, which can be defined as a stock-dependent arrival process, will increase the birth rate of the system. Datta and Pal [7] discussed the inventory model with an inventory-level-dependent demand rate. Karabi et al. [17] analysed the two different arrival rates, which is called the two-component demand rate (TCDR). In this paper, they classified TCDR as two stages of inventory level: one is zero inventory and the other is positive inventory.

In particular, they assumed that the arrival rate is constant when the inventory is empty and a varying arrival rate if the inventory is positive. Moreover, the varying arrival rate was defined as an increasing function and is controlled by the scaling factor, which lies between $[0, 1]$.

Diana Tom Varghese and Dhanya Shajin [31] determined the finite storage inventory system with a variable intensity rate for the arrival process, and it is assumed to be a non-homogeneous Poisson process. Mostly, the traditional inventory systems apply a constant arrival rate to explore their models. However, Sandeepkumar [19] investigated the optimization of an inventory system in which the arriving customer intensity rate is dependent on the current stock level. Recently, Jeganathan et al. [13] presented a comparative study between the (s, S) and (s, Q) ordering policies on the MQIS. This paper explains the stock-dependent arrival rate for those two ordering policies separately. The author makes reference to [25, 29, 32, 33] for the (s, Q) ordering principle as well as [31] for the (s, S) ordering principle. The following listed articles explore the Stock-dependent Arrival Policy (SDAP) [8, 24, 26, 30].

Nowadays, many single-server inventory systems implement a variety of similar practices in order to improve their service facilities. This is because the queue size becomes large in a single server system. When the queue length increases, the waiting time of a customer will also increase, and it will result in the customer's loss. To reduce such losses, some single-server service channels are ready to provide a non-homogeneous service rate, which is assumed to be independently and identically distributed. Jeganathan et al. [10] used two kinds of non-homogeneous service rates in the MQIS, which are determined by the threshold level of queue length. Recently, Jeganathan et al. [15] worked on an MQIS with a retrial facility in which they applied a non-homogeneous service rate based on a queue-dependent service facility. In this, they assumed that after every completion of the service process, the server observes the queue length and then starts the next service at a different rate as per the size of the queue length. This type of service facility can be seen in fast-food restaurants, supermarkets, and so on. Many papers learn more about QDSP, and a few of them will be provided to the readers [9, 14, 15, 34–36].

To the best of authors' knowledge, no paper has been published with SDAP and QDSP that is currently available with an infinite queue size. This idea would be a research gap in the queuing-inventory literature. In order to fill such a research gap, we proposed a stochastic model with the assumption of SDAP and QDSP. In addition, we investigate the two different types of ordering principles known as: 1) (s, Q) 2) (s, S).

In the end, the remaining part of this paper is partitioned as follows: model description in Section 2, analysis of the model under each ordering principle in Section 3, and Section 4. Furthermore, numerical interpretations are given in Section 5, and finally, the concluded results are stated in Section 6.

2. Notations and model description

$\mathbf{0}$	–	zero matrix of an appropriate dimension
$\mathbf{e}$	–	column vector of convenient size having one in each entry
$\mathbf{I}$	–	identity matrix
δ_{ij}	–	$\begin{cases} 1, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$
$H(x)$	–	$\begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$
$\bar{\delta}_{ij}$	–	$1 - \delta_{ij}$

The proposed model describes the MQIS with an infinite queue size that can store up to S items in the inventory. The arrival pattern of a unit holds the SDAP. The intensity of an arriving unit is defined as $\lambda_j(1 \leq j \leq L)$ where $1 \leq L \leq S$ and the arrival process of a unit follows a non-homogeneous Poisson process. Here, L is said to be the threshold limit to terminate the stock-dependent arrival pattern. That is, if the current inventory level exceeds the threshold limit, then the arrival rate of new customers will follow a homogeneous Poisson process. Each arriving unit joins the infinite queue, which is attached to the MQIS. They approach the service channel on the basis of first come, first served (FCFS). The service channel has a single server to provide the best service to the customer. Every customer can purchase only one unit of a product from the MQIS.

The service channel holds the QDSP in order to give their best service facility to an arriving customer. This QDSP is defined as the service facility that is dependent on the number of customers in the queue at an epoch. The intensity of this service process is denoted as $\mu_i(1 \leq i \leq k)$, where k is the threshold limit of the queue length. Due to the practical complications and the assumption of an infinite queue size, the queue-dependent service facility is terminated when the queue size reaches k. At this threshold limit, the service rate of the system is assumed to be non-homogeneous. Once the queue length, i crosses the threshold limit k, the intensity of a service process is defined as μ_k, for all $i \geq k$. At this level, the service rate of the system is assumed to be homogeneous. At the end of service completion, the customer chooses the product with probability p and not with probability q to buy it. The mean service time of the MQIS is assumed to be exponentially distributed. More clearly, the service process of the MQIS does not follow QDSP after the threshold limit point of k.

Moreover, the MQIS does not allow the arriving unit to enter the waiting hall. During the stock-out situation, the already-arrived customer has to wait for the commencement of reordered products. Once the replenishment products are received, the service starts immediately. To perform such replenishment of the products, the proposed MQIS has an analysis of two types of ordering principles along with the above-mentioned assumptions separately.

Definition 1. (s, Q) *ordering principle.* This principle states that when a reorder is triggered, the replenishment quantity of a $Q = S - s$ number of products is always fixed. Such reorder is to be done if the present stock level falls to the reorder limit s.

Definition 2. (s, S) *ordering principle.* This principle states that the replenishment quantity varies in order to fill the maximum capacity of the system when the reorder is triggered. Such reorder is to be done if the present stock level falls to the reorder limit s.

These two ordering principles are to be discussed as Model-I and Model-II, respectively. For each ordering principle, the intensity of the replenishment process is identified as β. The mean reorder time of each principle follows an exponential distribution. Furthermore, the MQIS will consist of defective products. An item in the inventory may become imperfect. So, we use γ to denote a defective rate of an inventory at any time t. The defective rate of a current inventory is defined as $j\gamma$, where $1 \leq j \leq S$. The mean lifetime of a product is assumed to be exponentially distributed.

State space. Let $N(t)$ denote the number of customers in the system at time t and $S(t)$ indicate the present stock level of the system at time t. A stochastic process is formed by the doublet $\{X(t), t \geq 0\}$

$= \{(N(t), S(t)), t \geq 0\}$. It also generates a quasi birth-and-death (QBD) process. The state space of the system is defined by $E = \{(i,\ j) : i = 0, 1, 2, \ldots$ and $j = 0, 1, \ldots, S\}$. Since the state space is discrete, we say that the proposed system comes under the classification of a discrete state and a continuous-time stochastic process. Also, the process $\{X(t), t \geq 0\}$ is said to be a continuous time Markov chain (CTMC).

3. Model-I

This section describes the MQIS with the $(s,\ Q)$ ordering principle.

3.1. Construction of matrices

The CTMC has an infinitesimal generator matrix P as follows

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ k \\ k+1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & \ldots & k & k+1 & \ldots & \end{array} \\ \begin{pmatrix} A_{00} & A_{01} & & & & & \\ A_{10} & A_{11} & A_{01} & & & & \\ & A_{21} & A_{22} & A_{01} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & A_{kk-1} & A_{kk} & A_{01} & \\ & & & & A_{kk-1} & A_{kk} & A_{01} \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix} \end{array}$$

where

$$A_{i,i} = \begin{cases} j\gamma, & i' = i,\ i = 0, 1, 2,\ \ldots, k \\ & j' = j-1, j = 1, 2,\ \ldots, S \\ -\beta & i' = i,\ i = 0, 1, 2,\ \ldots, k \\ & j' = j, j = 0 \\ \beta & i' = i,\ i = 0, 1, 2,\ \ldots, k \\ & j' = Q + j, j = 0, 1, 2,\ \ldots, s \\ -(\bar{\delta}_{i0}(\mu_i + j\gamma) + H(s-j)\beta + \lambda_j) & i' = i,\ i = 0, 1, 2,\ \ldots, k \\ & j' = j, j = 1, 2,\ \ldots, L \\ -(\bar{\delta}_{i0}(\mu_i + j\gamma) + \lambda_L) & i' = i,\ i = 0, 1, 2,\ \ldots, k \\ & j' = j, j = L+1, L+2,\ \ldots, S \\ 0 & \text{otherwise} \end{cases}$$

$$A_{i,i} = \begin{cases} h'_j, & i' = i,\ i = k, k+1, \ldots \\ & j' = j-1, j = 1, 2, \ \ldots, \ S \\ \beta & i' = i,\ i = k, k+1, \ldots \\ & j' = Q+j, j = 0, 1, 2, \ \ldots, \ s \\ f'_j & i' = i,\ i = k, k+1, \ldots \\ & j' = j, j = 0, 1, 2, \ \ldots, \ S \\ 0 & \text{otherwise} \end{cases}$$

where $f'_j = \begin{cases} -(\bar{\delta}_{j0}(\mu_k + j\gamma + \lambda_j) + H(s-j)\beta), & \text{if } j = 0, 1, \ \ldots, L \\ -(\mu_k + j\gamma + \lambda_L + H(s-j)\beta), & \text{if } j = L+1, L+2, \ \ldots, \ S \end{cases}$ and $h'_j = j\gamma$.

$$A_{i,i-1} = \begin{cases} p\mu_i, & i' = i-1, i = 1, 2, \ \ldots, \ k-1 \\ & j' = j-1, j = 1, 2, \ \ldots, \ S \\ q\mu_i, & i' = i-1, i = 1, 2, \ \ldots, \ k-1 \\ & j' = j, j = 1, 2, \ \ldots, \ S \\ 0, & \text{otherwise} \end{cases}$$

$$A_{i,i-1} = \begin{cases} p\mu_k, & i' = i-1, i = k, k+1, \ldots \\ & j' = j-1, j = 1, 2, \ \ldots, \ S \\ q\mu_k, & i' = i-1, i = k, k+1, \ldots \\ & j' = j, j = 1, 2, \ \ldots, \ S \\ 0, & \text{otherwise} \end{cases}$$

$$A_{0,1} = \begin{cases} \lambda_j & i' = i+1, i = 0, 1, \ldots \\ & j' = j, j = 1, 2, \ \ldots, \ L \\ \lambda_L & i' = i+1, i = 0, 1, \ldots \\ & j' = j, j = L+1, L+2, \ \ldots, \ S \\ 0 & \text{otherwise} \end{cases}$$

$$A_K = \begin{cases} h_j, & i' = i-1, i = k \\ & j' = j-1, j = 1, 2, \ \ldots, \ S \\ \beta & i' = i,\ i = k \\ & j' = Q+j, j = 0, 1, 2, \ \ldots, \ s \\ f_j & i' = i,\ i = k \\ & j' = j, j = 0, 1, \ \ldots, \ S \\ 0 & \text{otherwise} \end{cases}$$

where $f_j = -(\bar{\delta}_{j0}(\mu_i + j\gamma) + H(s-j)\beta)$ and $h_j = p\mu_k + j\gamma$.

Lemma 1. The stationary probability vector $\Pi_1 = (\pi_1^{(0)}, \pi_1^{(1)}, \ldots, \pi_1^{(S)})$ to the generator matrix, A_K is determined by

$$\pi_1^{(j)} = \pi_1^{(0)}\Omega_j, \quad j = 0, 1, 2, \ldots, S$$

where

$$\Omega_j = \begin{cases} 1 & j = 0 \\ \dfrac{(-1)^j \prod\limits_{z=0}^{j-1} f_z}{\prod\limits_{z=1}^{j} h_z}, & j = 1, 2, \ldots, Q \\ -\left[\dfrac{\Omega_{j-Q+1}\beta - \Omega_{j-1}f_{j-1}}{h_j}\right], & j = Q+1, Q+2, \ldots, S-1 \\ -\dfrac{\Omega_s \beta}{f_S}, & j = S \end{cases}$$

Proof. Let $A_K = A_{kk-1} + A_{kk} + A_{01}$ and solving $\Pi A_K = \mathbf{0}$, we get

$$\pi_1^{(j)} f_j + \pi_1^{(j+1)} h_{j+1} = 0, \quad j = 0, 1, \ldots, Q-1 \tag{1}$$

$$\pi_1^{(Q-j)}\beta + \pi_1^{(j)} f_j + \pi_1^{(j+1)} h_{j+1} = 0, \quad j = Q, Q+1, \ldots, S-1 \tag{2}$$

$$\pi_1^{(Q-j)}\beta + \pi_1^{(j)} f_j = 0, \quad j = S \tag{3}$$

By solving the above system of equations recursively, we get the stated result. □

3.2. Stability condition

Lemma 2. The stability condition of the system is

$$\mu_k \sum_{j=1}^{S} \Omega_j > \sum_{j=1}^{L} \Omega_j \lambda_j + \lambda_L \sum_{j=L+1}^{S} \Omega_j \tag{4}$$

Proof. By the Neuts result for the stability condition,

$$\Pi_1 A_{kk-1}\mathbf{e} > \Pi_1 A_{01}\mathbf{e} \tag{5}$$

and writing it explicitly, we get L.H.S as $\pi^{(0)} \sum\limits_{j=1}^{S} \Omega_j \mu_k$ and R.H.S as $\pi^{(0)}(\sum\limits_{j=1}^{L} \Omega_j \lambda_j + \lambda_L \sum\limits_{j=L+1}^{S} \Omega_j)$.

Substituting the obtained L.H.S and R.H.S in (5), we obtain

$$\mu_k \sum_{j=1}^{S} \Omega_j > \sum_{j=1}^{L} \Omega_j \lambda_j + \lambda_L \sum_{j=L+1}^{S} \Omega_j$$

□

3.3. Computation of R_1 matrix

Due to the structure of the generator matrix and stationary probability vector, the R_1 matrix can be determined by the matrix equation

$$R_1^2 A_{kk-1} + R_1 A_{kk} + A_{01} = \mathbf{0} \tag{6}$$

where

$$R_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \dots & S \end{array} \\ \begin{pmatrix} 0 & 0 & \dots & 0 \\ r_{10} & r_{11} & \dots & r_{1S} \\ r_{20} & r_{21} & \dots & r_{2S} \\ \vdots & \vdots & \dots & \vdots \\ r_{S0} & r_{S1} & \dots & r_{SS} \end{pmatrix} \end{array}$$

Substituting R_1 in (6) and writing it explicitly, we get the following set of equations. For $j = 1, 2, \dots, S$

$$\begin{aligned} x_{jj'} p\mu_k + r_{jj'} f'_j + r_{jj'+1} h'_{j+1} &= 0, \quad \text{if } j' = 0 \\ x_{jj'} q\mu_k + x_{jj'+1} p\mu_k + r_{jj'} f'_j + r_{jj'+1} h'_{j+1} + \lambda_j \delta_{jj'} &= 0, \quad \text{if } j' = 1, 2, \dots, L \\ x_{jj'} q\mu_k + x_{jj'+1} p\mu_k + r_{jj'} f'_j + r_{jj'+1} h'_{j+1} + \lambda_L \delta_{jj'} &= 0, \quad \text{if } j' = L+1, L+2, \dots, Q-1 \\ x_{jj'} q\mu_k + x_{jj'+1} p\mu_k + r_{jj'-Q}\beta + r_{jj'} f'_j + r_{jj'+1} h'_{j+1} + \lambda_L \delta_{jj'} &= 0, \quad \text{if } j' = Q, Q+1, \dots, S-1 \\ x_{jj'} q\mu_k + r_{jj'-Q}\beta + r_{jj'} f'_j + \lambda_L \delta_{jj'} &= 0, \quad \text{if } j' = S \end{aligned}$$

Solving the above system of non-linear equations by the Gauss–Seidal iterative process, we will obtain the R_1 matrix.

3.4. Partition of the steady state vector

The partition of the steady-state probability vector of the system is defined as follows:

$$\phi_1 = (\phi_1^{(0)}, \phi_1^{(1)}, \phi_1^{(2)}, \dots)$$

$$\phi_1^{(i)} = (\phi_1^{(i,0)}, \phi_1^{(i,1)}, \phi_1^{(i,2)}, \dots, \phi_1^{(i,S)}),\ i = 0, 1, 2, \dots$$

3.5. Computation of steady state probability vector

The entire probability vector of all system states is $\phi_1 = (\phi_1^{(0)}, \phi_1^{(1)}, \phi_1^{(2)}, \dots)$. The system balance equations are given by $\phi_1\ P = \mathbf{0}$ and $\phi_1\ \mathbf{e} = 1$. Then, the steady-state probabilities of the queuing-inventory system are calculated [23] by $\phi_1^{(i)} = \phi_1^{(k)}\ R_1^{(i-k)}$ where $i = k+1, k+2, \dots$ and the initial conditions are represented by the vectors $\phi_1^{(i)} = 0, 1, \dots, k$ are obtained by solving part of the balance equations

$$\phi_1^{(0)} A_{00} + \phi_1^{(1)} A_{10} = \mathbf{0}$$

$$\phi_1^{(0)} A_{01} + \phi_1^{(1)} A_{11} + \phi_1^{(2)} A_{21} = \mathbf{0}$$

$$\phi_1^{(i-1)} A_{01} + \phi_1^{(i)} A_{ii} + \phi_1^{(i+1)} A_{i+1,i} = \mathbf{0}, \quad i = 2, 3, \ldots, k-1$$

$$\phi_1^{(k-1)} A_{01} + \phi_1^{(k)} \left(A_1 + R_1 A_2\right) = \mathbf{0}$$

$$\sum_{n=0}^{k-1} \phi_1^{(n)} \mathbf{e} + \phi_1^{(k)} [I - R_1] \mathbf{e} = 1$$

3.6. System performance measures

The expected system performance of the model under the (s, Q) ordering principle is determined by the following measures:

Expected inventory level. In the MQIS, the expected inventory level of the system is defined as the sum of the product value of the current inventory level and the stationary probability vector

$$\Theta_1 = \sum_{i=0}^{\infty} \sum_{j=1}^{S} j \phi_1^{(i,j)}$$

Expected reorder rate. when the present inventory level reduces to $s+1$, there can either be a service completion happened or an item becomes defective. In the case of either of these two occurrences, the replenishment process is immediately triggered

$$\Theta_2 = \sum_{i=1}^{k} p\mu_i \phi_1^{(i,s+1)} + \sum_{i=k+1}^{\infty} p\mu_k \phi_1^{(i,s+1)} + \sum_{i=0}^{\infty} (s+1)\gamma \phi_1^{(i,s+1)}$$

Expected perishable rate. Since the system may have imperfect items in the storage space, we require an expected perishable rate of the system. This could be done using the sum of the product $j\gamma$ and the stationary probability vector, where j is the current inventory level

$$\Theta_3 = \sum_{i=0}^{\infty} \sum_{j=1}^{S} j\gamma \phi_1^{(i,j)}$$

Expected number of customers in the system. All the customers in the system purchase an item through the first come first serve discipline. The expected number of customers in the system is the sum of the product value of the number of customers present in the system and the stationary probability vector

$$\Theta_4 = \sum_{i=1}^{\infty} \sum_{j=0}^{S} i \phi_1^{(i,j)}$$

Expected arrival rate of a customer in the system. The sum of the product of the average arrival rate of a customer and the stationary probability vector defines the expected arrival rate of a customer

$$\Theta_5 = \sum_{i=0}^{\infty}\sum_{j=1}^{L} \lambda_j \phi_1^{(i,j)} + \sum_{i=0}^{\infty}\sum_{j=L+1}^{S} \lambda_L \phi_1^{(i,j)}$$

Expected waiting time. The expected waiting time of a customer is obtained by Little's formula

$$\Theta_6 = \frac{\Theta_4}{\Theta_5}$$

Expected number of customers lost. The customer loss in the system occurs only at the time of zero inventory level. It is defined by

$$\Theta_7 = \sum_{i=0}^{\infty} \lambda_0 \phi_1^{(i,0)}$$

3.7. Construction of cost function

The expected total cost of the proposed model under the $(s,\, Q)$ ordering principle is constructed by the cost function,

$$Tc = a_1\,\Theta_1 \;+\; a_2\,\Theta_2 \;+\; a_3\,\Theta_3 \;+\; a_4\,\Theta_4 \;+\; a_5\,\Theta_7$$

where a_1 refers to holding cost per item in the system, a_2 refers to set up cost per order, a_3 denotes perishable cost per item, a_4 indicates waiting cost per customer in the system, and a_5 refers to lost cost per customer in the system.

4. Model-II

This section describes the MQIS with the $(s,\, S)$ ordering principle.

4.1. Construction of matrices

The CTMC has an infinitesimal generator matrix P' as follows:

$$P' = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ k \\ k+1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \dots & k & k+1 & \dots \end{array} \\ \begin{pmatrix} A'_{00} & A_{01} & & & & & \\ A_{10} & A'_{11} & A_{01} & & & & \\ & A_{21} & A'_{22} & A_{01} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & A_{kk-1} & A'_{kk} & A_{01} & \\ & & & & A_{kk-1} & A'_{kk} & A_{01} \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix} \end{array}$$

where

$$A'_{i,i} = \begin{cases} j\gamma, & i' = i,\ i = 0,1,2,\ \ldots,\ k \\ & j' = j-1, j = 1,2,\ \ldots,\ S \\ \beta & i' = i,\ i = 0,1,2,\ \ldots,\ k \\ & j' = S, j = 0,1,2,\ \ldots,\ s \\ -\beta & i' = i,\ i = 0,1,2,\ \ldots,\ k \\ & j' = j, j = 0 \\ -(\bar{\delta}_{i0}(\mu_i + j\gamma) + H(s-j)\beta + \lambda_j) & i' = i,\ i = 0,1,2,\ \ldots,\ k \\ & j' = j, j = 1,2,\ \ldots,\ L \\ -(\bar{\delta}_{i0}(\mu_i + j\gamma) + \lambda_L) & i' = i,\ i = 0,1,2,\ \ldots,\ k \\ & j' = j, j = L+1, L+2,\ \ldots,\ S \\ 0 & \text{otherwise} \end{cases}$$

$$A'_{i,i} = \begin{cases} h'_j, & i' = i,\ i = k, k+1,\ \ldots \\ & j' = j-1, j = 1,2,\ \ldots,\ S \\ \beta & i' = i,\ i = k, k+1,\ \ldots \\ & j' = S, j = 0,1,2,\ \ldots,\ s \\ f'_j & i' = i,\ i = k, k+1,\ \ldots \\ & j' = j, j = 0,1,2,\ \ldots,\ S \\ 0 & \text{otherwise} \end{cases}$$

where $f'_j = \begin{cases} -(\bar{\delta}_{j0}(\mu_k + j\gamma + \lambda_j) + H(s-j)\beta), & \text{if } j = 0,1,\ \ldots,L \\ -(\mu_k + j\gamma + \lambda_L + H(s-j)\beta), & \text{if } j = L+1, L+2,\ \ldots,\ S \end{cases}$ and $h'_j = j\gamma$.

$$A_H = \begin{cases} h_j, & i' = i-1, i = k \\ & j' = j-1, j = 1,2,\ \ldots,\ S \\ \beta & i' = i,\ i = k \\ & j' = S, j = 0,1,2,\ \ldots,\ s \\ f_j & i' = i,\ i = k \\ & j' = j, j = 0,1,\ \ldots,\ S \\ 0 & \text{otherwise} \end{cases}$$

where $f_j = -(\bar{\delta}_{j0}(\mu_i + j\gamma) + H(s-j)\beta)$ and $h_j = p\mu_k + j\gamma$.

Lemma 3. The stationary probability vector $\Pi_2 = (\pi_2^{(0)}, \pi_2^{(1)},\ \ldots,\ \pi_2^{(S)})$ to the generator matrix, A_H is determined by

$$\pi_2^{(j)} = \pi_2^{(0)} \Lambda_j, \quad j = 0,1,2,\ \ldots,\ S.$$

where

$$\Lambda_j = \begin{cases} 1 & j = 0 \\ \dfrac{(-1)^j \prod\limits_{z=0}^{j-1} f_z}{\prod\limits_{z=1}^{j} h_z}, & j = 1, 2, \ \ldots, \ S-1 \\ -\dfrac{\sum\limits_{z=0}^{s} \lambda_z \beta}{f_j}, & j = S \end{cases}$$

Proof. Let $A_H = A_{kk-1} + A'_{kk} + A_{01}$ and solving $\Pi A_H = \mathbf{0}$, we get

$$\pi_2^{(j)} f_j + \pi_2^{(j+1)} h_{j+1} = 0, \quad j = 0, 1, \ \ldots, \ S-1, \tag{7}$$

$$\sum_{z=0}^{s} \pi_2^{(z)} \beta + \pi_2^{(j)} f_j = 0, \quad j = S. \tag{8}$$

By solving the above system of equations recursively, we get the stated result.
□

Lemma 4. The stability condition of the system is

$$\mu_k \sum_{j=1}^{S} \Lambda_j > \sum_{j=1}^{L} \Lambda_j \lambda_j + \lambda_L \sum_{j=L+1}^{S} \Lambda_j. \tag{9}$$

Proof. Using the Neuts result for the stability condition on

$$\pi_2 A_{kk-1} \mathbf{e} > \pi_2 A_{01} \mathbf{e} \tag{10}$$

writing it explicitly we get L.H.S as $\pi_2^{(0)} \sum\limits_{j=1}^{S} \Lambda_j \mu_k$ and R.H.S as $\pi_2^{(0)} (\sum\limits_{j=1}^{L} \Lambda_j \lambda_j + \sum\limits_{j=L+1}^{S} \Lambda_j \lambda_L)$.

Substituting the L.H.S and R.H.S on (10), we obtain

$$\mu_k \sum_{j=1}^{S} \Lambda_j > \sum_{j=1}^{L} \Lambda_j \lambda_j + \lambda_L \sum_{j=L+1}^{S} \Lambda_j$$

□

4.2. Computation of R_2 matrix

Due to the structure of the generator matrix and stationary probability vector, R_2 matrix can be determined which satisfies the matrix equation

$$R_2^2 A_{kk-1} + R_2 A'_{kk} + A_{01} = \mathbf{0} \tag{11}$$

where

$$R_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \dots & S \end{array} \\ \begin{pmatrix} 0 & 0 & \dots & 0 \\ y_{10} & y_{11} & \dots & y_{1S} \\ y_{20} & y_{20} & \dots & y_{2S} \\ \vdots & \vdots & \dots & \vdots \\ y_{S0} & y_{S1} & \dots & y_{SS} \end{pmatrix} \end{array}$$

Substituting R_2 in (11), we get the following set of equations. For $j = 1, 2, \dots, S$

$$\begin{aligned} x_{jj'}p\mu_k + y_{jj'}f'_j + y_{jj'+1}h'_{j+1} &= 0, \quad \text{if } j' = 0 \\ x_{jj'}q\mu_k + x_{jj'+1}p\mu_k + y_{jj'}f'_j + y_{jj'+1}h'_{j+1} + \lambda_j\delta_{jj'} &= 0, \quad \text{if } j' = 1, 2, \dots, L \\ x_{jj'}q\mu_k + x_{jj'+1}p\mu_k + y_{jj'}f'_j + y_{jj'+1}h'_{j+1} + \lambda_L\delta_{jj'} &= 0, \quad \text{if } j' = L+1, L+2, \dots, S-1 \\ x_{jj'}q\mu_k + \sum_{z=0}^{s} y_{jz}\beta + y_{jj'}f'_j + \lambda_L\delta_{jj'} &= 0, \quad \text{if } j' = S. \end{aligned}$$

Solving the above system of non-linear equations by the Gauss–Seidal iterative process, we will obtain the R_2 matrix.

4.3. Partition of the steady state vector

The partition of the steady-state probability vector of the system is defined as follows:

$$\phi_2 = (\phi_2^{(0)}, \phi_2^{(1)}, \phi_2^{(2)}, \dots).$$

$$\phi_2^{(i)} = (\phi_2^{(i,0)}, \phi_2^{(i,1)}, \phi_2^{(i,2)}, \dots, \phi_2^{(i,S)}), \quad i = 0, 1, 2, \dots$$

4.4. Computation of steady state probability vector

The entire probability vector of all system states is $\phi_2 = (\phi_2^{(0)}, \phi_2^{(1)}, \phi_2^{(2)}, \dots)$. The system balance equations are given by $\phi_2 P' = \mathbf{0}$ and $\phi_2\mathbf{e} = 1$. Then, the steady-state probabilities of the queuing-inventory system are calculated [23]) by $\phi_2^{(i)} = \phi_2^{(k)}\, R_2^{(i-k)}$ where $i = k+1, k+2, \dots$ and the initial conditions are represented by the vectors $\phi_2^{(i)} = 0, 1, \dots, k$ are obtained by solving part of the balance equations

$$\begin{aligned} \phi_2^{(0)}A'_{00} + \phi_2^{(1)}A_{10} &= \mathbf{0} \\ \phi_2^{(0)}A_{01} + \phi_2^{(1)}A'_{11} + \phi_2^{(2)}A_{21} &= \mathbf{0} \\ \phi_2^{(i-1)}A_{01} + \phi_2^{(i)}A'_{ii} + \phi_2^{(i+1)}A_{i+1,i} &= \mathbf{0} \quad i = 2, 3, \dots, k-1, \\ \phi_2^{(k-1)}A_{01} + \phi_2^{(k)}\,(A'_1 + R_2A_2) &= \mathbf{0} \\ \sum_{n=0}^{k-1}\phi_2^{(n)}\mathbf{e} + \phi_2^{(k)}[I - R_2]\mathbf{e} &= 1. \end{aligned}$$

4.5. System performance measures

The system performance of the model under the $(s,\, S)$ ordering principle is determined as follows:
Expected inventory level. In the MQIS, the expected inventory level of the system is obtained by the sum of the product value of the current inventory level and the stationary probability vector defined by

$$\aleph_1 = \sum_{i=0}^{\infty}\sum_{j=1}^{S} j\phi_2^{(i,\,j)}$$

Expected reorder rate. When the present inventory level reduces to $s+1$, there can either be a service completion happened or an item become defective. In the case of either of these two occurrences, the replenishment process is immediately triggered. It is defined by

$$\aleph_2 = \sum_{i=1}^{k} p\mu_i\phi_2^{(i,\,s+1)} + \sum_{i=k+1}^{\infty} p\mu_k\phi_2^{(i,\,s+1)} + \sum_{i=0}^{\infty}(s+1)\gamma\phi_2^{(i,\,s+1)}$$

Expected perishable rate. Since the system may have imperfect items in the storage space, we require an expected perishable rate of the system. This could be done using the sum of the product $j\gamma$ and the stationary probability vector where j is the current inventory level. It is defined by

$$\aleph_3 = \sum_{i=0}^{\infty}\sum_{j=1}^{S} j\gamma\phi_2^{(i,\,j)}$$

Expected number of customers in the system. All customers in the system purchase an item through the first come first serve discipline. The expected number of customers in the system is the sum of the product value of the number of customers present and the stationary probability vector. It is defined by

$$\aleph_4 = \sum_{i=1}^{\infty}\sum_{j=0}^{S} i\phi_2^{(i,\,j)}$$

The expected arrival rate of a customer enters into the system. The sum of the product of the average arrival rate of a customer and the stationary probability vector is used to define the expected arrival rate of a customer in the system

$$\aleph_5 = \sum_{i=0}^{\infty}\sum_{j=1}^{L} \lambda_j\phi_2^{(i,\,j)} + \sum_{i=0}^{\infty}\sum_{j=L+1}^{S} \lambda_L\phi_2^{(i,\,j)}$$

Expected waiting time. The expected waiting time of a customer is obtained by Little's formula

$$\aleph_6 = \frac{\aleph_4}{\aleph_5}$$

Expected number of customers lost. The customer loss in the system occurs only at the time of zero inventory level. It is defined by

$$\aleph_7 = \sum_{i=0}^{\infty} \lambda_0\phi_2^{(i,\,0)}$$

Construction of cost function. The expected total cost of the proposed model under the (s, S) ordering principle is constructed by the cost function

$$Tc = a_1\aleph_1 + a_2\aleph_2 + a_3\aleph_3 + a_4\aleph_4 + a_5\aleph_7$$

5. Numerical discussion

In this section, we explore the proposed system with the (s, Q) and (s, S) ordering principles by numerical discussion. In such a way, this section explains the total expected cost of the system, the mean number of customers in the waiting hall, the mean number of customers lost, and the mean waiting time of customers in the waiting hall, which is to be discussed for M-I and M-II along with the scaling factors k_1 and k_2. We use the scaling factor k_1 as the controlling factor of a non-homogeneous arrival rate. Similarly, we use k_2 as the controlling factor on the non-homogeneous service rate. On applying those k_1 and k_2, we have given the generalized model. This is because, if we assume $k_1 = k_2 = 0$, this model becomes a purely homogeneous arrival and service rate of the system. Each illustration is explained for homogeneous and non-homogeneous arrival/service rates. The major objective of this section is to find the best ordering principle for the queuing-inventory setup. For this numerical work, we fixed the cost rates as $S = 30, s = 6, a_1 = 0.001, a_2 = 1, a_3 = 0.02, a_4 = 1.1, a_5 = 3$, as well as the rate of parameter $\lambda_0 = 0.4, k_1 = 0.2, k_2 = 0.2, \gamma = 3, \beta = 4.9, \lambda = 1.3, \mu = 12.8, k = 11, L = 7, p = 0.8, q = 0.2, r = k$.

The monotonicity of the parameters is to be assumed for the following examples as follows:

- λ and k_1 increase; the average number of customers entering the system is increased.
- β increases; the average replenishment time is reduced.
- γ increases; the average lifetime of a product is reduced.
- μ and k_2 increase; the average service time per customer is reduced.

5.1. Example I

In this example, we investigate the expected total cost by varying the parameters γ, β, λ, μ and also varying the two distinct scaling factors k_1 and k_2 (Tables 1 and 2)[1]. In Table 1, we mainly present the scaling factor k_1, and how it impacts the expected total cost by varying different parameters. Similarly, Table 2 shows the effect of the scaling factor $k_2$on the expected total cost by varying parameters.

- The expected total cost for M-I and M-II decreases in proportion to the increase in μ for every increment in k_1. This is because the average service time is reduced.
- Both λ and k_1 increase, the expected total cost is increased and also when we compared M-II to M-I (Table 1), M-II gave the minimum ETC. This is because M-II's ordering policy depends on the current stock level which could mean that the mean inventory level is much higher in M-II.
- When the value of γ is increasing, the total cost rate increases, and the total cost rate decrease as β increases. In this case, the ETC of the M-II is lower than that of the M-I. As γ increases, the defective items in the inventory also increase. So that total cost will increase. When β increases, the average time for replenishment decreases. Hence the total cost of the system reduces.

[1]Tables (1–11) are included at the end of the article.

- The total expected cost rate decreases when the scaling vector k_2 increases. And compared to M-I and M-II, M-II has a minimum expected cost value in Table 2. The scaling factor k_2 reduces the service completion time per customer when it increases. Therefore total cost rate is decreased.
- As μ and β increase, the total cost decreases as k_2 increases. When λ and γ increase, the total cost also increases (Table 2). The increase of λ will cause an increase in customers in the system. The waiting cost of each customer in the system reflects the increase in total cost. In this case, M-II has a minimum value of expected total cost compared to M-I.
- From Table 11, we conclude that the expected total cost of the system is high for M-I and low for M-II under the parameter variation S, s, and L. When we increase the inventory level of a system, the expected total cost is increased due to the holding cost. Similarly, the increment in L causes the number of customers in the system. Thus the expected total cost is increased.

This example suggests that the firm has to maintain enough service speed or reorder time to increase profit as well as decrease TC while the number of arriving customers increases and the life of products decreases.

5.2. Example II

In this example, the mean number of customers in the waiting hall is shown by different parameters λ, μ, β, γ with the scaling factor for arrival (k_1) in Table 3 and the scaling factor for the service (k_2) in Table 4. In addition, we compared the M-I and M-II in Tables 3 and 4.

- The mean number of customers in the system is increased when k_1 increases because k_1 causes the increase of arriving customers in the system. The mean number of customers in the waiting hall for M-I is minimum as compared to M-II in Table 3.
- When the service rate μ_2 and the lead time rate β with scaling factor k_1 increases then the mean number of customers in the waiting hall is decreased. And the mean number of customers in the waiting hall increases when the arrival rate λ and the perishable rate γ increases (Table 3). We observe that the service time and lead time will reduce the number of customers in the system when they reduce. In the system, a defective item causes a shortage in the inventory. So customers will face insufficient stock levels in the system. Hence the number of customers in the system will increase.
- Table 4 displayed when the scaling factor for service k_2 increases the mean number of customers in the system decreases. Therefore, the number of customers in the system occurs low for M-II because it helps to reduce the average service time per customer. So that customer in the system quickly reduces when k_2 increases.
- The mean number of customers in the system decreases when μ and β increase. And when λ and γ increase, the mean number of customers in the system increases (Table 4). When we control the defective items in the inventory, we can give service to the customers quickly. If this is to happen in the system, the mean number of customers in the system can be reduced.

From this example, the queuing-inventory-based seller can control the crowd in the queue easily by increasing the service time or reorder time or else increasing the life of the products, so that customers will get service.

5.3. Example III

In this example, we investigate the influence of parameters β and γ with the scaling factors k_1 and k_2 on mean waiting time for M-I and M-II.

- Figure 1 shows that the mean waiting time decreases when β and the mean waiting time increases when k_1 increases for both M-I and M-II. Figure 2 demonstrates that the mean waiting time decreases when both β and k_2 increase for both M-I and M-II.

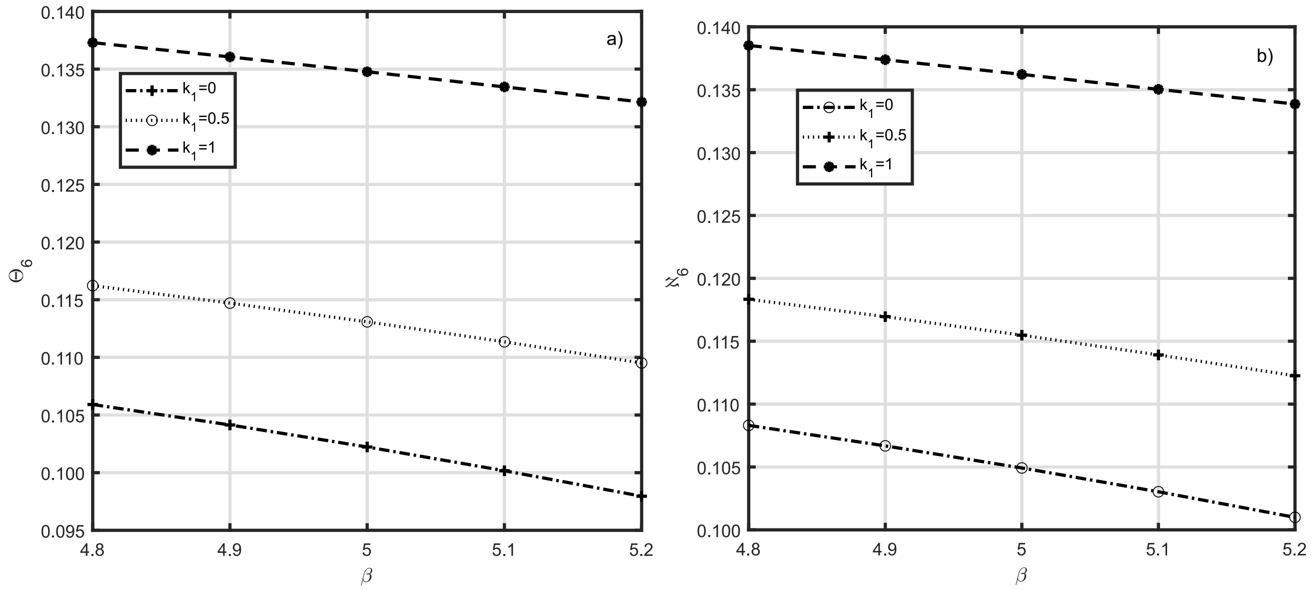

Figure 1. Θ_6 of M-I on k_1 vs. β (a), and $\aleph_6$ of M-II on k_1 vs. β (b)

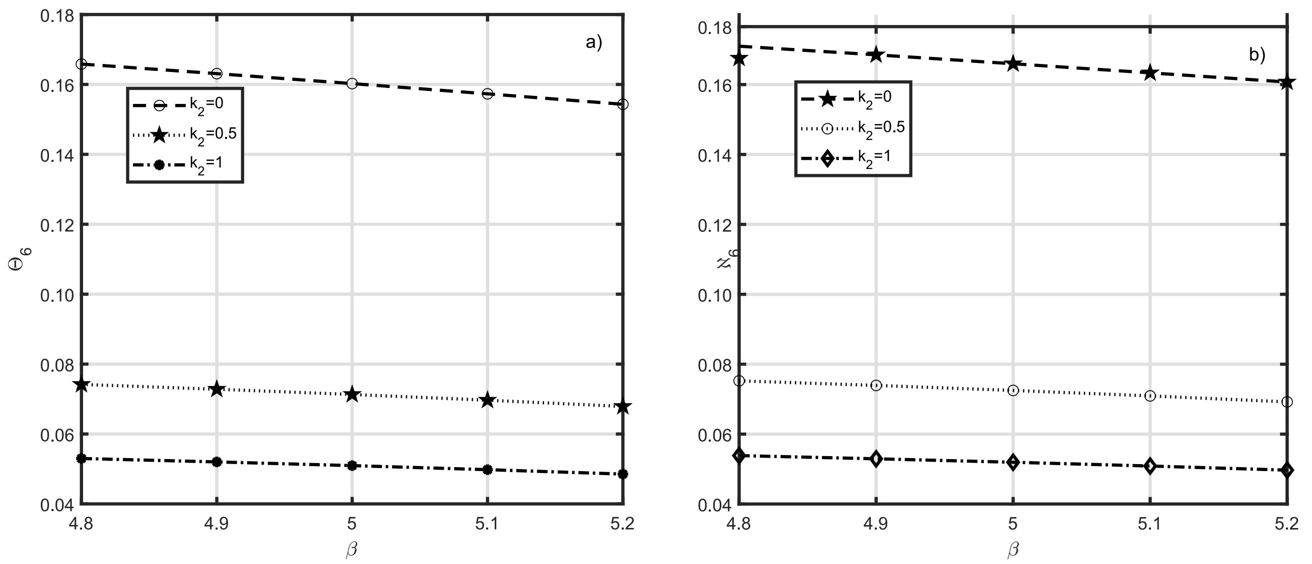

Figure 2. Θ_6 of M-I on k_2 vs. β (a), and $\aleph_6$ of M-II on k_2 vs. β (b)

- Figure 3 shows that the mean waiting time increases when γ and k_1 increase for both M-I and M-II. Furthermore, when the mean waiting time decreases (increases) when k_2 (γ) increases for both M-I and M-II (Figure 4).
- Figure 5 demonstrates that when λ increases, the mean waiting time for both M-I and M-II increases. And the mean waiting time for both M-I and M-II increases when k_1 increases. Next, the mean waiting time for both M-I and M-II decreases when k_2 increases (Figure 6).

- Figure 7 shows that the mean waiting time for both M-I and M-II decreases when μ increases and when k_1 increases the mean waiting time for both M-I and M-II increases. Figure 8 shows that the mean waiting time for both M-I and M-II decreases when k_2 increases.
- Table 11 shows that the mean waiting time per customer increases for M-II rather than M-I under the parameter variation S, s, and L.

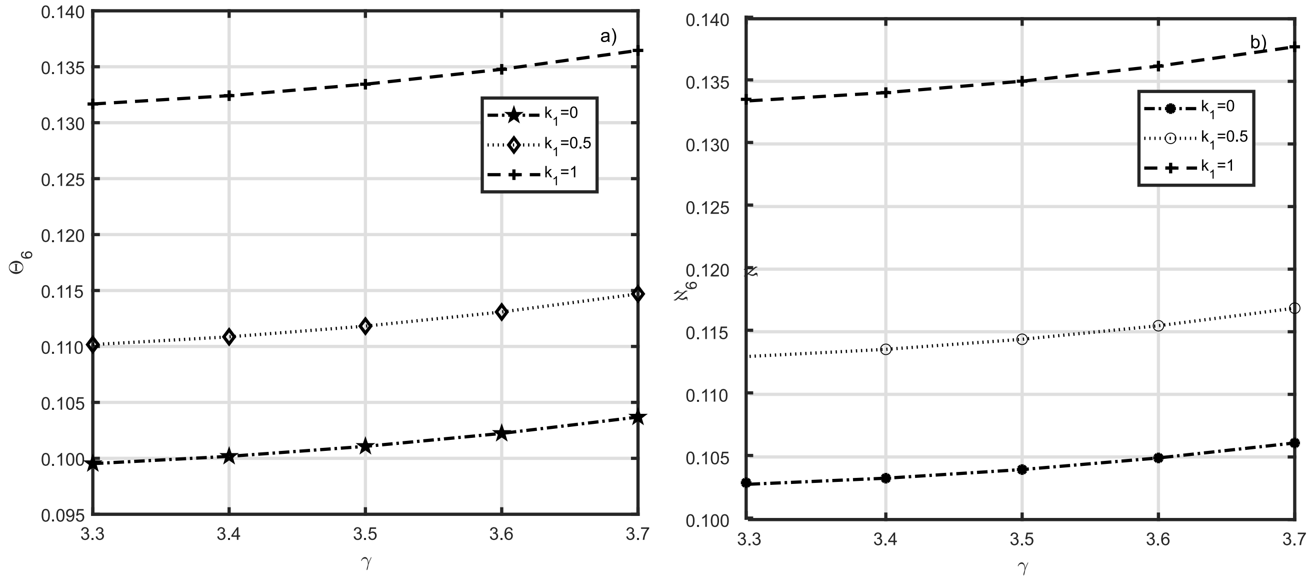

Figure 3. Θ_6 of M-I on k_1 vs. γ (a), and $\aleph$ of M-II on k_1 vs. γ (b)

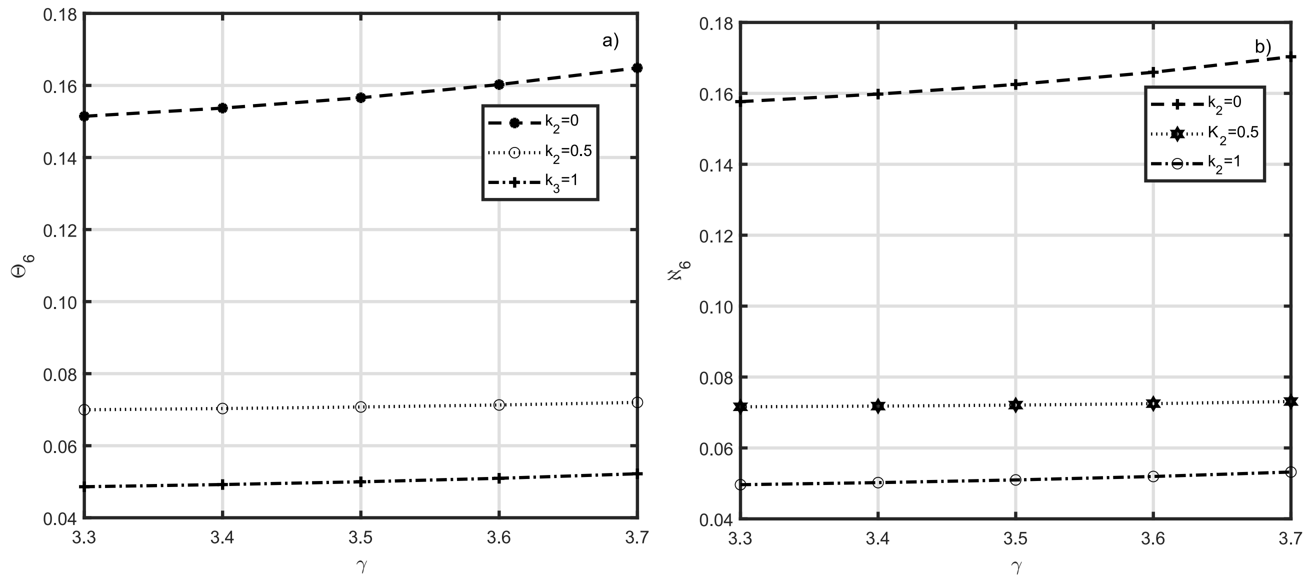

Figure 4. Θ_6 of M-I on k_2 vs. γ (a), and $\aleph_6$ of M-II on k_2 vs. γ (b)

We are witnessing from this example that if a seller provides good service or maintains a good ordering time, the customer's waiting time will decrease, which tends them to visit the shop more times.

5.4. Example IV

This example explores the mean number of customers lost for both M-I and M-II using different parameters λ, μ, β, γ with the scaling factors k_1 and k_2. Additionally, we also compare the M-I and M-II.

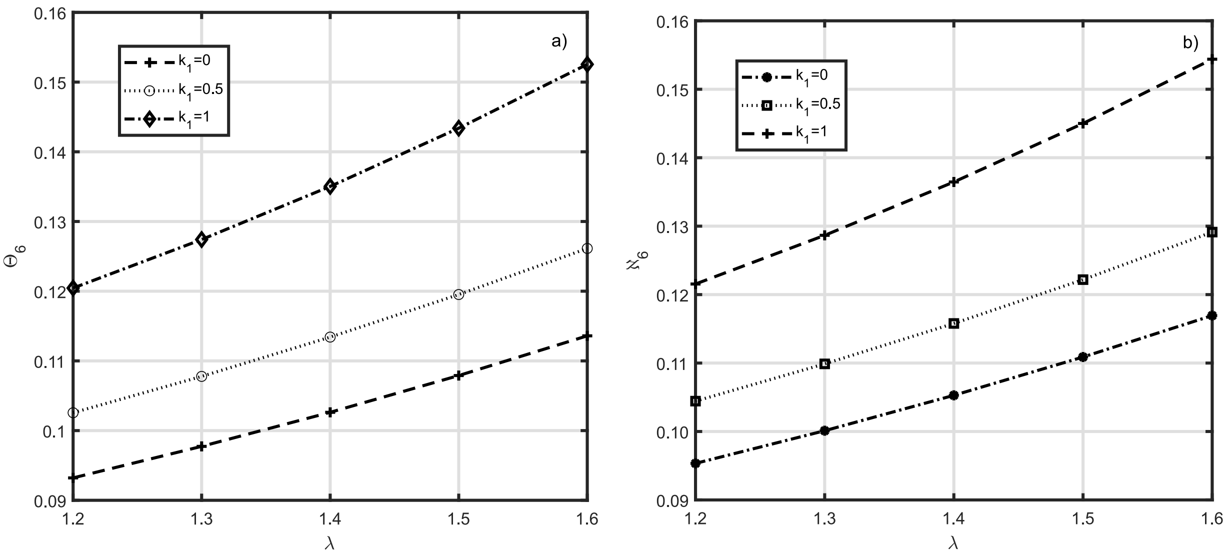

Figure 5. Θ_6 of M-I on k_1 vs. λ (a), and $\aleph_6$ of M-II on k_1 vs. λ (b)

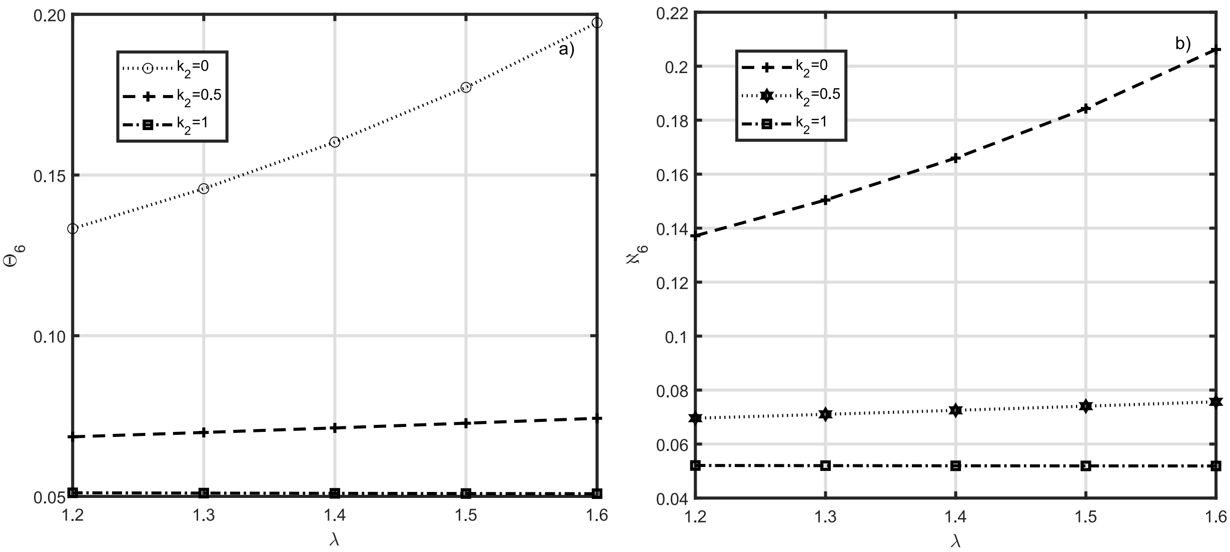

Figure 6. Θ_6 of M-I on k_2 vs. λ (a), and $\aleph_6$ of M-II on k_2 vs. λ (b)

- When k_1 increases, the mean number of lost customers increases. In addition, M-II has a lower mean number of customers lost than M-I in Tables 5 and 6.
- When μ and β increase then the mean number of customers lost decreases and the mean number of customers lost increases when λ and γ increases with k_1 increases (k_2 increases) in Table 5 (6).
- Tables 7–10 show the expected inventory and expected perishable for both M-I and M-II decreases when μ increases with k_1 and k_2 respectively. Table 7 and 8 demonstrate the expected inventory for both M-I and M-II decreases when γ, λ, k_1 and k_2 increases. But the expected inventory for both M-I and M-II increases when β increases in Table 7 and 8.
- The expected perishable for both M-I and M-II increases when β and γ increase, but the expected perishable for both M-I and M-II decreases when λ, k_1 and k_2 increase in Table 9 and 10. Furthermore, M-I is minimum compared to M-II for Tables 7–10.
- In Table 11, we show how the parameters S, s, and L influence the customer loss of the system. The customer loss is higher in M-II and lower in M-I for the increment of S, s, and L.

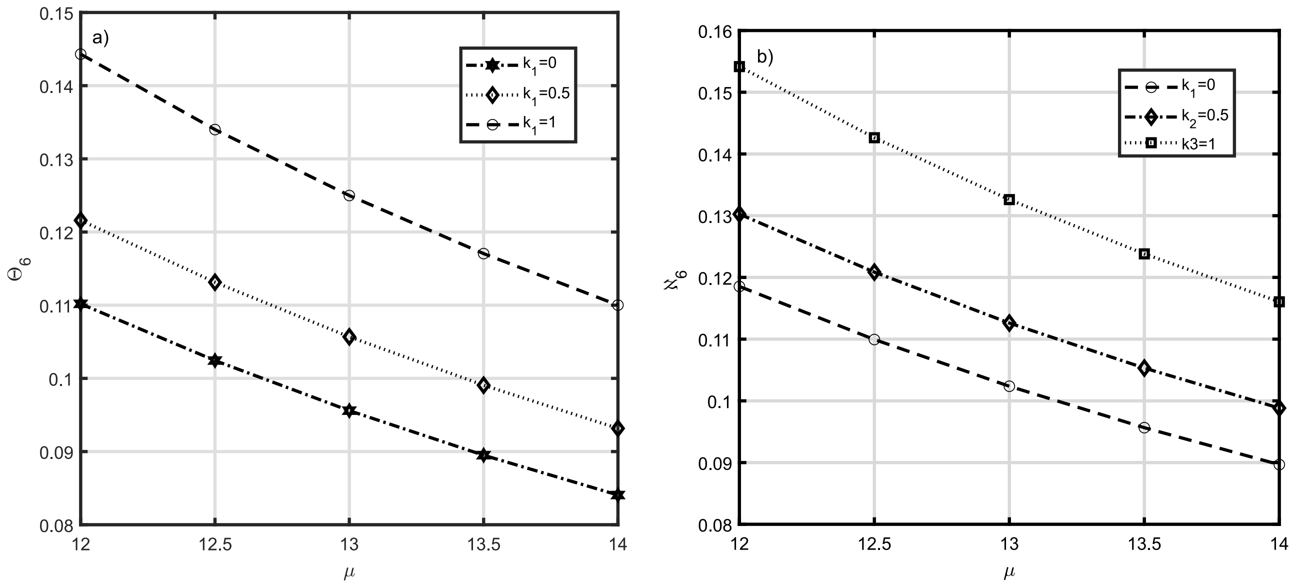

Figure 7. Θ_6 of M-I on k_1 vs. μ (a), and $\aleph_6$ of M-II on k_1 vs. μ (b)

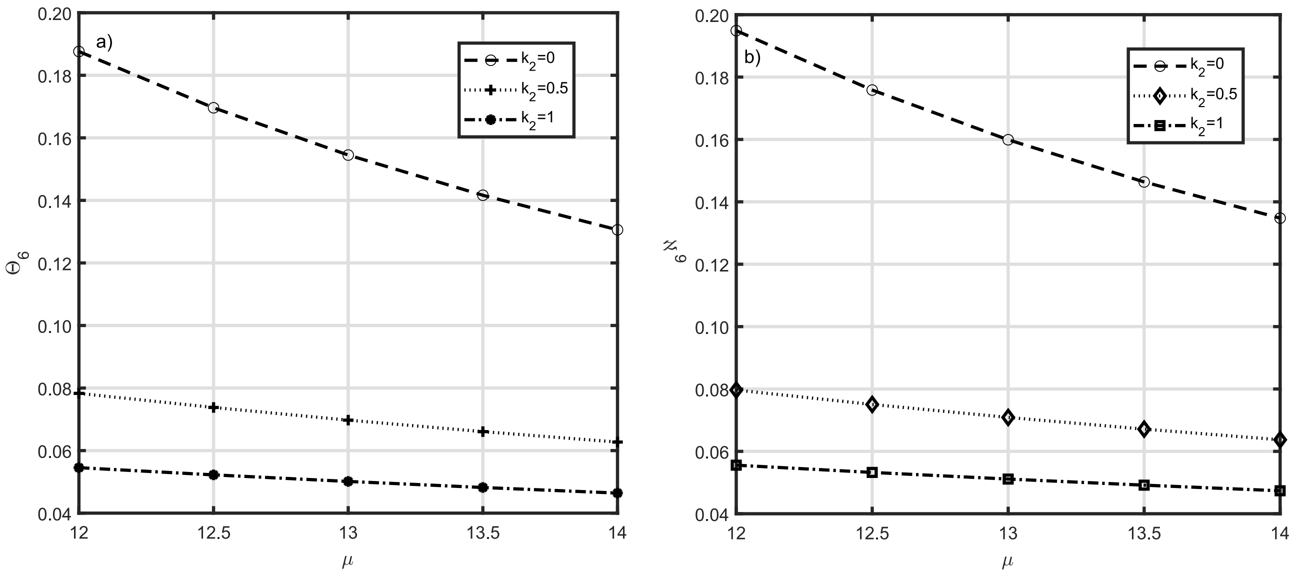

Figure 8. Θ_6 of M-I on k_2 vs. μ (a), and $\aleph_6$ of M-II on k_2 vs. μ (b)

The loss of customers crucially affects the profit of the business. To increase the profit, it is essential to decrease the number of lost customers. It is possible by increasing the reorder time or speed of the service facility or storage capacity or by decreasing the queue size or perishable rate.

6. Conclusion

The single server service channel of the MQIS investigated the SDAP and QDSP with an infinite queue size. The proposed system is a generalised version of the homogeneous and non-homogeneous arrival and service rates, respectively. Mostly, in the existing literature, an inventory system, the discussion with SDAP and QDSP is separate. However, this paper fills such a research gap with an infinite queue in an MQIS. Among these policies, the proposed MQIS deals with two different types of ordering principles. Efficient attention is given to each ordering principle in order to explore and bring managerial inputs to the inventory business. The impact of the parameter variation and the assumption of the ordering principles gave significant results in the numerical section. The output factors will enhance the proposed MQIS economically for every business tycoon. In the future, this model will be extended to a multi-server MQIS. In each of the numerical outputs, we gave the comparison results for both models. In addition, the results are obtained for both homogeneous and non-homogeneous arrival/service rates. According to the limitations of the scaling factor, homogeneous and non-homogeneous classifications are made in each discussion. When we observe the total expected cost of the system, the optimum cost is obtained in M-II. That is, the (s, S) ordering principle set up in a queuing-inventory system produces the minimum total cost rather than the (s, Q) ordering principle. Due to the stock-dependent arrival policy, the mean number of customers in the waiting hall increases in M-II. In the case of the mean number of customers lost in the system, this is mostly reduced for M-II.

In the future, this proposed work will be discussed using the multi-server service facility.

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Table 1. Effect of various parameters with k_1 on the total expected cost

γ	β	λ	k_1	0		0.5		1	
			μ	M-I	M-II	M-I	M-II	M-I	M-II
			12.7	3.918980	3.898881	3.987581	3.966947	4.102751	4.079964
		1.3	12.8	3.911785	3.891095	3.980320	3.959189	4.092319	4.069298
			12.9	3.904758	3.883476	3.973283	3.951659	4.082276	4.059029
			12.7	4.030240	4.013143	4.110393	4.092590	4.263308	4.242081
	4.9	1.4	12.8	4.021557	4.003771	4.101328	4.082924	4.249959	4.228406
			12.9	4.013087	3.994618	4.092546	4.073549	4.237096	4.215227
			12.7	4.136307	4.136307	4.243147	4.229029	4.440848	4.421901
		1.5	12.8	4.125103	4.125103	4.231990	4.217140	4.424054	4.404665
			12.9	4.114178	4.114178	4.221184	4.205616	4.407862	4.388047
			12.7	3.908548	3.887599	3.971571	3.950853	4.066248	4.045050
		1.3	12.8	3.901908	3.880307	3.964950	3.943685	4.056444	4.034988
			12.9	3.895421	3.873169	3.958543	3.936734	4.047019	4.025314
			12.7	4.015275	3.997510	4.089634	4.071963	4.221877	4.202500
3	5	1.4	12.8	4.007216	3.988700	4.081281	4.062955	4.209233	4.189499
			12.9	3.999354	3.980093	4.073197	4.054224	4.197061	4.176981
			12.7	4.129736	4.116028	4.217275	4.203542	4.394020	4.377237
		1.5	12.8	4.120046	4.105460	4.206912	4.192389	4.378025	4.360760
			12.9	4.110605	4.095154	4.196883	4.181584	4.362615	4.344886
			12.7	3.895113	3.872699	3.949930	3.928995	4.015008	3.996266
		1.3	12.8	3.889231	3.866075	3.944195	3.922639	4.006071	3.987037
			12.9	3.883483	3.859585	3.938656	3.916481	3.997501	3.978183
			12.7	3.995669	3.976692	4.061491	4.043905	4.163900	4.147348
	5.1	1.4	12.8	3.988460	3.968639	4.054119	4.035798	4.152224	4.135268
			12.9	3.981428	3.960768	4.046994	4.027947	4.141001	4.123656
			12.7	4.103532	4.088906	4.182161	4.168866	4.328709	4.315188
		1.5	12.8	4.094797	4.079196	4.172885	4.158715	4.313803	4.299744
			12.9	4.086288	4.069724	4.163922	4.148889	4.299459	4.284881
			12.7	4.319918	4.297085	4.352505	4.329099	4.404337	4.380127
		1.3	12.8	4.314752	4.291446	4.347029	4.323202	4.395053	4.370595
			12.9	4.309742	4.285968	4.341765	4.317526	4.386139	4.361445
			12.7	4.420439	4.400533	4.464454	4.444009	4.555732	4.533531
	4.9	1.4	12.8	4.413920	4.393429	4.457302	4.436316	4.543671	4.521120
			12.9	4.407598	4.386530	4.450413	4.428901	4.532069	4.509183
			12.7	4.528199	4.512085	4.585597	4.568996	4.723325	4.703881
		1.5	12.8	4.520125	4.503290	4.576499	4.559214	4.707984	4.688066
			12.9	4.512297	4.494754	4.567729	4.549778	4.693214	4.672841
			12.7	4.306176	4.282566	4.328161	4.304968	4.349321	4.327530
		1.3	12.8	4.301815	4.277675	4.323571	4.299910	4.340846	4.318781
			12.9	4.297593	4.272928	4.319179	4.295059	4.332729	4.310402
			12.7	4.400797	4.380288	4.433911	4.413923	4.494382	4.474949
3.5	5	1.4	12.8	4.395164	4.374008	4.427728	4.407146	4.483224	4.463405
			12.9	4.389709	4.367915	4.421792	4.400631	4.472509	4.452320
			12.7	4.502268	4.485767	4.548429	4.532576	4.655081	4.638815
		1.5	12.8	4.495172	4.477882	4.540396	4.523799	4.640757	4.623970
			12.9	4.488300	4.470236	4.532671	4.515349	4.626984	4.609697
			12.7	4.291955	4.267217	4.301328	4.278333	4.286755	4.267990
		1.3	12.8	4.288490	4.263151	4.297731	4.274211	4.279177	4.260107
			12.9	4.285147	4.259210	4.294316	4.270280	4.271943	4.252582
			12.7	4.380071	4.358621	4.400216	4.380696	4.424859	4.408842
	5.1	1.4	12.8	4.375420	4.353249	4.395113	4.374935	4.414693	4.398250
			12.9	4.370928	4.348043	4.390240	4.369419	4.404954	4.388102
			12.7	4.474634	4.457422	4.507436	4.492373	4.578035	4.565640
		1.5	12.8	4.468617	4.450537	4.500583	4.484705	4.564822	4.551854
			12.9	4.462803	4.443868	4.494019	4.477343	4.552137	4.538620

Table 2. Different parameter effect with k_2 on Total Expected Cost

γ	β	λ	k_2 / μ	0 M-I	0 M-II	0.5 M-I	0.5 M-II	1 M-I	1 M-II
			12.7	4.912393	4.858748	3.628091	3.558163	2.224521	2.183065
		1.3	12.8	4.895226	4.841573	3.615849	3.544975	2.212807	2.171970
			12.9	4.878247	4.824584	3.603630	3.531820	2.201270	2.161052
			12.7	5.240973	5.182143	3.722682	3.653715	2.289261	2.246596
	4.9	1.4	12.8	5.222205	5.163392	3.709963	3.640031	2.277063	2.235023
			12.9	5.203639	5.144839	3.697279	3.626394	2.265049	2.223634
			12.7	5.578656	5.514224	3.818723	3.750797	2.354189	2.310301
		1.5	12.8	5.558335	5.493950	3.805487	3.736577	2.341509	2.298252
			12.9	5.538228	5.473887	3.792302	3.722418	2.329019	2.286393
			12.7	4.752392	4.705203	3.627370	3.557699	2.216088	2.171432
		1.3	12.8	4.736082	4.688832	3.615120	3.544501	2.204363	2.160378
			12.9	4.719947	4.672635	3.602896	3.531337	2.192815	2.149502
			12.7	5.066173	5.014615	3.722339	3.652895	2.280034	2.234122
3	5	1.4	12.8	5.048376	4.996775	3.709606	3.639209	2.267831	2.222597
			12.9	5.030766	4.979120	3.696910	3.625568	2.255812	2.211256
			12.7	5.388237	5.331957	3.818773	3.749608	2.344185	2.297004
		1.5	12.8	5.369004	5.312704	3.805516	3.735392	2.331505	2.285009
			12.9	5.349968	5.293647	3.792312	3.721236	2.319016	2.273203
			12.7	4.570971	4.531734	3.626671	3.557218	2.206929	2.158549
		1.3	12.8	4.555583	4.516226	3.614415	3.544010	2.195189	2.147543
			12.9	4.540357	4.500880	3.602184	3.530834	2.183629	2.136714
			12.7	4.868451	4.825798	3.722014	3.652053	2.270023	2.220336
	5.1	1.4	12.8	4.851694	4.808929	3.709267	3.638363	2.257811	2.208864
			12.9	4.835108	4.792232	3.696559	3.624717	2.245786	2.197577
			12.7	5.173392	5.127038	3.818834	3.748392	2.333342	2.282334
		1.5	12.8	5.155315	5.108866	3.805557	3.734179	2.320661	2.270398
			12.9	5.137419	5.090875	3.792334	3.720025	2.308171	2.258653
			12.7	5.034178	4.990945	4.176268	4.103128	2.625263	2.557871
		1.3	12.8	5.018515	4.975198	4.165956	4.091642	2.611424	2.544704
			12.9	5.003026	4.959624	4.155632	4.080144	2.597775	2.531733
			12.7	5.344160	5.297126	4.264002	4.191803	2.691934	2.623024
	4.9	1.4	12.8	5.327019	5.279917	4.253263	4.179887	2.677622	2.609387
			12.9	5.310066	5.262893	4.242526	4.167973	2.663507	2.595950
			12.7	5.662784	5.611610	4.353095	4.281920	2.758761	2.688338
		1.5	12.8	5.644199	5.592978	4.341892	4.269534	2.743979	2.674231
			12.9	5.625814	5.574544	4.330705	4.257166	2.729399	2.660331
			12.7	4.857390	4.821966	4.172182	4.101419	2.623293	2.553904
		1.3	12.8	4.842562	4.807002	4.161829	4.089936	2.609429	2.540735
			12.9	4.827895	4.792199	4.151467	4.078444	2.595756	2.527762
			12.7	5.152468	5.114093	4.260767	4.190885	2.689658	2.618728
3.5	5	1.4	12.8	5.136268	5.097766	4.249972	4.178957	2.675324	2.605091
			12.9	5.120242	5.081612	4.239182	4.167035	2.661186	2.591655
			12.7	5.455432	5.413825	4.350739	4.281821	2.756187	2.683720
		1.5	12.8	5.437896	5.396177	4.339465	4.269409	2.741385	2.669616
			12.9	5.420545	5.378713	4.328211	4.257017	2.726785	2.655718
			12.7	4.658397	4.632255	4.168323	4.099770	2.621283	2.549780
		1.3	12.8	4.644460	4.618125	4.157929	4.088287	2.607392	2.536610
			12.9	4.630671	4.604145	4.147529	4.076799	2.593694	2.523635
			12.7	4.937159	4.909032	4.257733	4.190005	2.687335	2.614266
	5.1	1.4	12.8	4.921958	4.893640	4.246882	4.178063	2.672977	2.600629
			12.9	4.906917	4.878409	4.236040	4.166131	2.658816	2.587194
			12.7	5.223050	5.192717	4.348558	4.281737	2.753559	2.678927
		1.5	12.8	5.206621	5.176106	4.337213	4.269296	2.738735	2.664826
			12.9	5.190363	5.159665	4.325893	4.256880	2.724113	2.650932

Table 3. Different parameter effect with k_1 on Mean Number of Customers in the Waiting hall

γ	β	λ	k_1	0		0.5		1	
			μ	M-I	M-II	M-I	M-II	M-I	M-II
			12.7	0.761081	0.783707	0.850628	0.871950	1.075663	1.090070
		1.3	12.8	0.751614	0.773803	0.839989	0.860904	1.061463	1.075588
			12.9	0.742360	0.764123	0.829596	0.850115	1.047617	1.061468
			12.7	0.855864	0.883057	0.959172	0.984923	1.222156	1.239813
	4.9	1.4	12.8	0.844869	0.871523	0.946745	0.971990	1.205312	1.222609
			12.9	0.834127	0.860256	0.934614	0.959367	1.188904	1.205853
			12.7	0.958652	0.991178	1.077382	1.108321	1.383884	1.405413
		1.5	12.8	0.945937	0.977801	1.062923	1.093237	1.363970	1.385044
			12.9	0.933523	0.964744	1.048820	1.078526	1.344591	1.365224
			12.7	0.756846	0.779711	0.846534	0.868110	1.071527	1.086189
		1.3	12.8	0.747435	0.769857	0.835952	0.857117	1.057392	1.071768
			12.9	0.738233	0.760225	0.825615	0.846379	1.043611	1.057707
			12.7	0.851048	0.878523	0.954492	0.980546	1.217366	1.235328
3	5	1.4	12.8	0.840120	0.867049	0.942134	0.967676	1.200603	1.218200
			12.9	0.829442	0.855840	0.930071	0.955115	1.184274	1.201517
			12.7	0.953198	0.986053	1.072051	1.103347	1.378350	1.400244
		1.5	12.8	0.940561	0.972749	1.057676	1.088339	1.358536	1.379967
			12.9	0.928223	0.959761	1.043654	1.073703	1.339253	1.360236
			12.7	0.752545	0.775650	0.842371	0.864207	1.067313	1.082236
		1.3	12.8	0.743189	0.765847	0.831848	0.853267	1.053246	1.067877
			12.9	0.734042	0.756265	0.821567	0.842581	1.039529	1.053876
			12.7	0.846160	0.873919	0.949737	0.976098	1.212489	1.230764
	5.1	1.4	12.8	0.835297	0.862505	0.937450	0.963293	1.195809	1.213712
			12.9	0.824684	0.851356	0.925455	0.950795	1.179559	1.197103
			12.7	0.947663	0.980853	1.066638	1.098296	1.372720	1.394987
		1.5	12.8	0.935106	0.967621	1.052347	1.083366	1.353007	1.374803
			12.9	0.922844	0.954703	1.038407	1.068805	1.333821	1.355162
			12.7	0.788648	0.809220	0.879900	0.899361	1.105079	1.117889
		1.3	12.8	0.778634	0.798790	0.868734	0.887809	1.090372	1.102925
			12.9	0.768848	0.788599	0.857829	0.876530	1.076035	1.088338
			12.7	0.888182	0.913143	0.993387	1.017083	1.256471	1.272291
	4.9	1.4	12.8	0.876528	0.900971	0.980320	1.003534	1.239000	1.254489
			12.9	0.865144	0.889086	0.967568	0.990313	1.221984	1.237154
			12.7	0.996410	1.026530	1.117225	1.145911	1.423827	1.443256
		1.5	12.8	0.982903	1.012384	1.101993	1.130079	1.403139	1.422148
			12.9	0.969718	0.998580	1.087140	1.114644	1.383012	1.401613
			12.7	0.784272	0.805079	0.875749	0.895462	1.101009	1.114066
		1.3	12.8	0.774316	0.794701	0.864641	0.883964	1.086367	1.099162
			12.9	0.764584	0.784559	0.853793	0.872736	1.072093	1.084634
			12.7	0.883202	0.908443	0.988636	1.012635	1.251747	1.267866
3.5	5	1.4	12.8	0.871615	0.896333	0.975641	0.999150	1.234356	1.250139
			12.9	0.860298	0.884507	0.962958	0.985993	1.217419	1.232876
			12.7	0.990764	1.021216	1.111807	1.140853	1.418357	1.438146
		1.5	12.8	0.977337	1.007144	1.096661	1.125099	1.397769	1.417130
			12.9	0.964232	0.993412	1.081890	1.109739	1.377737	1.396684
			12.7	0.779821	0.800865	0.871522	0.891493	1.096854	1.110165
		1.3	12.8	0.769922	0.790539	0.860475	0.880049	1.082279	1.095322
			12.9	0.760246	0.780449	0.849684	0.868874	1.068070	1.080854
			12.7	0.878138	0.903663	0.983803	1.008109	1.246929	1.263354
	5.1	1.4	12.8	0.866620	0.891615	0.970879	0.994690	1.229620	1.245703
			12.9	0.855370	0.879850	0.958266	0.981596	1.212762	1.228513
			12.7	0.985026	1.015815	1.106297	1.135709	1.412784	1.432941
		1.5	12.8	0.971681	1.001818	1.091237	1.120034	1.392296	1.412017
			12.9	0.958655	0.988158	1.076551	1.104751	1.372362	1.391661

Table 4. Different parameter effect with k_2 on Mean Number of Customers in the Waiting hall

γ	β	λ	k_2 / μ	0 M-I	0 M-II	0.5 M-I	0.5 M-II	1 M-I	1 M-II
			12.7	1.071866	1.115457	0.595546	0.607018	0.464675	0.473829
		1.3	12.8	1.054598	1.097101	0.589432	0.600735	0.460964	0.470071
			12.9	1.037847	1.079304	0.583433	0.594571	0.457313	0.466374
			12.7	1.259772	1.316459	0.652698	0.665850	0.499489	0.509583
	4.9	1.4	12.8	1.237971	1.293130	0.645896	0.658850	0.495496	0.505538
			12.9	1.216869	1.270565	0.639223	0.651984	0.491568	0.501560
			12.7	1.482057	1.556021	0.711911	0.726911	0.534317	0.545375
		1.5	12.8	1.454420	1.526229	0.704384	0.719154	0.530041	0.541043
			12.9	1.427731	1.497482	0.697002	0.711547	0.525835	0.536783
			12.7	1.065051	1.109041	0.592599	0.604207	0.462650	0.471940
		1.3	12.8	1.047931	1.090827	0.586513	0.597950	0.458954	0.468197
			12.9	1.031324	1.073166	0.580541	0.591812	0.455319	0.464516
			12.7	1.251312	1.308473	0.649471	0.662776	0.497324	0.507567
3	5	1.4	12.8	1.229717	1.285342	0.642699	0.655805	0.493347	0.503539
			12.9	1.208811	1.262965	0.636057	0.648967	0.489436	0.499577
			12.7	1.471483	1.545999	0.708397	0.723570	0.532014	0.543235
		1.5	12.8	1.444131	1.516484	0.700904	0.715845	0.527756	0.538921
			12.9	1.417714	1.488000	0.693555	0.708269	0.523568	0.534678
			12.7	1.058149	1.102544	0.589602	0.601348	0.460588	0.470019
		1.3	12.8	1.041180	1.084472	0.583544	0.595117	0.456909	0.466292
			12.9	1.024717	1.066949	0.577600	0.589006	0.453291	0.462626
			12.7	1.242754	1.300394	0.646189	0.659651	0.495121	0.505517
	5.1	1.4	12.8	1.221366	1.277463	0.639449	0.652709	0.491162	0.501506
			12.9	1.200658	1.255276	0.632837	0.645901	0.487268	0.497561
			12.7	1.460801	1.535874	0.704824	0.720175	0.529672	0.541060
		1.5	12.8	1.433735	1.506637	0.697366	0.712481	0.525433	0.536763
			12.9	1.407591	1.478419	0.690051	0.704938	0.521264	0.532538
			12.7	1.129037	1.170567	0.610000	0.620419	0.477388	0.486414
		1.3	12.8	1.110303	1.150753	0.603692	0.613971	0.473627	0.482609
			12.9	1.092145	1.131557	0.597506	0.607648	0.469928	0.478866
			12.7	1.332702	1.387380	0.668749	0.680742	0.513152	0.523126
	4.9	1.4	12.8	1.308889	1.362033	0.661726	0.673553	0.509108	0.519033
			12.9	1.285859	1.337534	0.654840	0.666504	0.505130	0.515007
			12.7	1.575378	1.647599	0.729654	0.743378	0.548909	0.559851
		1.5	12.8	1.544972	1.615003	0.721878	0.735406	0.544580	0.555469
			12.9	1.515638	1.583579	0.714254	0.727591	0.540323	0.551159
			12.7	1.121941	1.163893	0.606998	0.617556	0.475374	0.484547
		1.3	12.8	1.103366	1.144229	0.600719	0.611135	0.471630	0.480758
			12.9	1.085360	1.125177	0.594561	0.604840	0.467947	0.477031
			12.7	1.323841	1.379031	0.665458	0.677609	0.510998	0.521133
3.5	5	1.4	12.8	1.300249	1.353895	0.658468	0.670450	0.506972	0.517057
			12.9	1.277431	1.329598	0.651613	0.663432	0.503011	0.513048
			12.7	1.564228	1.637058	0.726067	0.739970	0.546617	0.557734
		1.5	12.8	1.534132	1.604761	0.718327	0.732031	0.542307	0.553371
			12.9	1.505092	1.573622	0.710738	0.724250	0.538068	0.549078
			12.7	1.114743	1.157124	0.603940	0.614641	0.473322	0.482646
		1.3	12.8	1.096329	1.137612	0.597691	0.608248	0.469595	0.478873
			12.9	1.078476	1.118705	0.591562	0.601980	0.465929	0.475162
			12.7	1.314865	1.370573	0.662108	0.674420	0.508804	0.519104
	5.1	1.4	12.8	1.291496	1.345650	0.655150	0.667292	0.504796	0.515046
			12.9	1.268890	1.321556	0.648328	0.660304	0.500853	0.511053
			12.7	1.552948	1.626395	0.722417	0.736501	0.544283	0.555580
		1.5	12.8	1.523163	1.594398	0.714713	0.728597	0.539993	0.551235
			12.9	1.494421	1.563545	0.707159	0.720849	0.535773	0.546961

Table 5. Mean number of customers lost for different parameters with k_1

γ	β	λ	k_1	0		0.5		1	
			μ	M-I	M-II	M-I	M-II	M-I	M-II
			12.7	0.003037	0.002970	0.004368	0.004246	0.008618	0.008297
		1.3	12.8	0.002973	0.002907	0.004287	0.004167	0.008505	0.008189
			12.9	0.002911	0.002846	0.004207	0.004089	0.008395	0.008082
			12.7	0.003623	0.003544	0.005157	0.005013	0.009935	0.009558
	4.9	1.4	12.8	0.003548	0.003470	0.005063	0.004921	0.009809	0.009435
			12.9	0.003474	0.003398	0.004971	0.004830	0.009684	0.009315
			12.7	0.004288	0.004196	0.006041	0.005872	0.011367	0.010925
		1.5	12.8	0.004200	0.004110	0.005932	0.005766	0.011226	0.010789
			12.9	0.004114	0.004025	0.005826	0.005662	0.011086	0.010654
			12.7	0.002923	0.002859	0.004198	0.004082	0.008273	0.007967
		1.3	12.8	0.002861	0.002798	0.004120	0.004006	0.008166	0.007864
			12.9	0.002801	0.002739	0.004044	0.003931	0.008060	0.007761
			12.7	0.003488	0.003412	0.004959	0.004821	0.009544	0.009183
3	5	1.4	12.8	0.003415	0.003341	0.004869	0.004733	0.009423	0.009066
			12.9	0.003345	0.003272	0.004780	0.004646	0.009303	0.008950
			12.7	0.004129	0.004041	0.005811	0.005650	0.010926	0.010503
		1.5	12.8	0.004045	0.003958	0.005707	0.005548	0.010790	0.010372
			12.9	0.003963	0.003877	0.005605	0.005448	0.010656	0.010243
			12.7	0.002813	0.002752	0.004037	0.003926	0.007946	0.007654
		1.3	12.8	0.002754	0.002694	0.003962	0.003853	0.007843	0.007555
			12.9	0.002697	0.002637	0.003889	0.003781	0.007741	0.007456
			12.7	0.003359	0.003287	0.004771	0.004639	0.009172	0.008827
	5.1	1.4	12.8	0.003289	0.003218	0.004684	0.004554	0.009055	0.008714
			12.9	0.003221	0.003151	0.004599	0.004471	0.008940	0.008603
			12.7	0.003978	0.003894	0.005593	0.005439	0.010506	0.010102
		1.5	12.8	0.003897	0.003814	0.005493	0.005341	0.010375	0.009976
			12.9	0.003818	0.003736	0.005395	0.005245	0.010246	0.009851
			12.7	0.004307	0.004218	0.006065	0.005902	0.011327	0.010914
		1.3	12.8	0.004223	0.004135	0.005961	0.005800	0.011190	0.010782
			12.9	0.004141	0.004054	0.005859	0.005700	0.011056	0.010653
			12.7	0.005039	0.004936	0.007030	0.006840	0.012864	0.012385
3.5	4.9	1.4	12.8	0.004942	0.004840	0.006911	0.006725	0.012712	0.012239
			12.9	0.004847	0.004747	0.006795	0.006611	0.012563	0.012095
			12.7	0.005854	0.005737	0.008093	0.007875	0.014513	0.013962
		1.5	12.8	0.005744	0.005628	0.007959	0.007744	0.014346	0.013801
			12.9	0.005636	0.005521	0.007828	0.007616	0.014181	0.013642
			12.7	0.004063	0.004148	0.005680	0.005836	0.010496	0.010891
		1.3	12.8	0.003983	0.004068	0.005582	0.005736	0.010369	0.010760
			12.9	0.003905	0.003988	0.005486	0.005638	0.010245	0.010630
			12.7	0.004855	0.004756	0.006768	0.006586	0.012376	0.011918
	5	1.4	12.8	0.004762	0.004665	0.006654	0.006475	0.012230	0.011777
			12.9	0.004671	0.004575	0.006542	0.006366	0.012086	0.011638
			12.7	0.005644	0.005531	0.007795	0.007586	0.013971	0.013443
		1.5	12.8	0.005537	0.005426	0.007666	0.007460	0.013810	0.013287
			12.9	0.005433	0.005323	0.007540	0.007337	0.013650	0.013134
			12.7	0.003998	0.003916	0.005618	0.005469	0.010476	0.010098
		1.3	12.8	0.003920	0.003839	0.005522	0.005375	0.010350	0.009977
			12.9	0.003843	0.003764	0.005427	0.005282	0.010225	0.009856
			12.7	0.004680	0.004586	0.006518	0.006344	0.011912	0.011472
	5.1	1.4	12.8	0.004591	0.004497	0.006408	0.006237	0.011771	0.011337
			12.9	0.004503	0.004411	0.006301	0.006132	0.011632	0.011203
			12.7	0.005443	0.005334	0.007511	0.007311	0.013454	0.012948
		1.5	12.8	0.005340	0.005233	0.007387	0.007190	0.013299	0.012798
			12.9	0.005240	0.005134	0.007265	0.007070	0.013145	0.012650

Table 6. Mean number of customers lost for different parameters with k_2

γ	β	λ	k_2	0		0.5		1	
			μ	M-I	M-II	M-I	M-II	M-I	M-II
			12.7	0.008760	0.008607	0.000440	0.000422	0.000003	0.000003
		1.3	12.8	0.008634	0.008483	0.000427	0.000409	0.000003	0.000003
			12.9	0.008510	0.008360	0.000414	0.000396	0.000003	0.000003
			12.7	0.010204	0.010031	0.000542	0.000519	0.000004	0.000004
	4.9	1.4	12.8	0.010060	0.009889	0.000525	0.000503	0.000004	0.000004
			12.9	0.009919	0.009749	0.000510	0.000488	0.000004	0.000004
			12.7	0.011792	0.011599	0.000661	0.000634	0.000005	0.000005
		1.5	12.8	0.011630	0.011439	0.000641	0.000615	0.000005	0.000005
			12.9	0.011471	0.011281	0.000622	0.000596	0.000005	0.000004
			12.7	0.008416	0.008269	0.000425	0.000407	0.000003	0.000003
		1.3	12.8	0.008295	0.008150	0.000412	0.000395	0.000003	0.000003
			12.9	0.008176	0.008032	0.000399	0.000383	0.000003	0.000003
			12.7	0.009809	0.009643	0.000523	0.000501	0.000004	0.000004
3	5	1.4	12.8	0.009672	0.009507	0.000507	0.000486	0.000004	0.000004
			12.9	0.009536	0.009373	0.000492	0.000471	0.000004	0.000003
			12.7	0.011344	0.011158	0.000638	0.000612	0.000005	0.000005
		1.5	12.8	0.011188	0.011004	0.000619	0.000594	0.000005	0.000005
			12.9	0.011035	0.010853	0.000601	0.000576	0.000005	0.000004
			12.7	0.008089	0.007948	0.000410	0.000393	0.000003	0.000003
		1.3	12.8	0.007973	0.007833	0.000398	0.000381	0.000003	0.000003
			12.9	0.007858	0.007720	0.000386	0.000370	0.000003	0.000003
			12.7	0.009434	0.009274	0.000505	0.000485	0.000004	0.000004
	5.1	1.4	12.8	0.009302	0.009144	0.000490	0.000470	0.000004	0.000003
			12.9	0.009171	0.009015	0.000475	0.000456	0.000004	0.000003
			12.7	0.010917	0.010738	0.000616	0.000592	0.000005	0.000005
		1.5	12.8	0.010767	0.010590	0.000598	0.000574	0.000005	0.000004
			12.9	0.010620	0.010444	0.000580	0.000557	0.000005	0.000004
			12.7	0.011523	0.011319	0.000694	0.000666	0.000006	0.000005
		1.3	12.8	0.011372	0.011170	0.000674	0.000647	0.000005	0.000005
			12.9	0.011223	0.011023	0.000655	0.000628	0.000005	0.000005
			12.7	0.013185	0.012960	0.000836	0.000803	0.000007	0.000007
	4.9	1.4	12.8	0.013016	0.012793	0.000812	0.000780	0.000007	0.000006
			12.9	0.012850	0.012629	0.000789	0.000757	0.000007	0.000006
			12.7	0.014988	0.014741	0.000999	0.000961	0.000009	0.000008
		1.5	12.8	0.014800	0.014556	0.000971	0.000933	0.000009	0.000008
			12.9	0.014615	0.014373	0.000944	0.000907	0.000008	0.000008
			12.7	0.011087	0.010891	0.000670	0.000644	0.000006	0.000005
		1.3	12.8	0.010942	0.010748	0.000651	0.000625	0.000005	0.000005
			12.9	0.010798	0.010607	0.000632	0.000607	0.000005	0.000005
			12.7	0.012695	0.012478	0.000807	0.000776	0.000007	0.000006
3.5	5	1.4	12.8	0.012532	0.012317	0.000784	0.000754	0.000007	0.000006
			12.9	0.012372	0.012159	0.000762	0.000732	0.000006	0.000006
			12.7	0.014439	0.014202	0.000966	0.000928	0.000009	0.000008
		1.5	12.8	0.014258	0.014023	0.000938	0.000902	0.000008	0.000008
			12.9	0.014080	0.013847	0.000912	0.000876	0.000008	0.000007
			12.7	0.010672	0.010483	0.000647	0.000622	0.000005	0.000005
		1.3	12.8	0.010532	0.010346	0.000629	0.000604	0.000005	0.000005
			12.9	0.010394	0.010210	0.000611	0.000587	0.000005	0.000005
			12.7	0.012227	0.012018	0.000780	0.000750	0.000007	0.000006
	5.1	1.4	12.8	0.012070	0.011864	0.000758	0.000729	0.000007	0.000006
			12.9	0.011916	0.011711	0.000737	0.000708	0.000006	0.000006
			12.7	0.013916	0.013687	0.000933	0.000898	0.000009	0.000008
		1.5	12.8	0.013742	0.013515	0.000907	0.000872	0.000008	0.000008
			12.9	0.013570	0.013345	0.000882	0.000847	0.000008	0.000007

Table 7. Effect of different parameters with k_1 on Θ_1 and $\aleph_1$

			μ	12.5		13	
k_1	γ	β	λ	M-I	M-II	M-I	M-II
0	3	4.9	1.3	11.643509	11.978372	11.631575	11.953363
			1.4	11.641260	11.958263	11.632533	11.930972
			1.5	11.635265	11.935409	11.629857	11.905908
		5	1.3	11.677983	12.013947	11.668481	11.988261
			1.4	11.672968	11.993086	11.666719	11.965172
			1.5	11.664200	11.969527	11.661320	11.939454
	3.5	4.9	1.3	11.407928	11.769787	11.404296	11.759722
			1.4	11.397780	11.766476	11.397137	11.753861
			1.5	11.384333	11.760468	11.386783	11.745413
		5	1.3	11.445804	11.784400	11.444268	11.769062
			1.4	11.433539	11.775281	11.435059	11.757684
			1.5	11.417910	11.763716	11.422601	11.743936
0.5	3	4.9	1.3	11.587238	11.958756	11.583968	11.944571
			1.4	11.573839	11.941275	11.563769	11.924666
			1.5	11.556629	11.910899	11.549830	11.901913
		5	1.3	11.634680	11.937981	11.627600	11.883069
			1.4	11.626055	11.921611	11.622047	11.872333
			1.5	11.613700	11.918399	11.612827	11.858802
	3.5	4.9	1.3	11.329030	11.699264	11.324573	11.694649
			1.4	11.311321	11.693515	11.303943	11.691329
			1.5	11.290350	11.685035	11.279981	11.684356
		5	1.3	11.351214	11.740148	11.344664	11.734722
			1.4	11.331290	11.735449	11.321812	11.732386
			1.5	11.308125	11.728070	11.295647	11.727448
1	3	4.9	1.3	11.565393	11.712361	11.544733	11.608410
			1.4	11.547972	11.692887	11.525011	11.586774
			1.5	11.497765	11.570832	11.472479	11.563130
		5	1.3	11.600881	11.901692	11.597011	11.827743
			1.4	11.590723	11.852619	11.578604	11.807160
			1.5	11.551815	11.751607	11.527422	11.744611
	3.5	4.9	1.3	11.495393	11.112961	11.474733	11.108410
			1.4	11.447972	11.092887	11.425011	11.086774
			1.5	11.397765	11.070832	11.372479	11.063130
		5	1.3	11.556811	11.800256	11.531353	11.790498
			1.4	11.550566	11.774972	11.478097	11.763755
			1.5	11.452093	11.747920	11.422409	11.735215

Table 8. Effect of different parameters with k_2 on Θ_1 and $\aleph_1$

			μ	12.5		13	
k_2	γ	β	λ	M-I	M-II	M-I	M-II
0	3	4.9	1.3	12.273132	12.729638	12.223139	12.667039
			1.4	12.265706	12.717481	12.215035	12.654215
			1.5	12.258121	12.705189	12.206772	12.641263
		5	1.3	12.286323	12.737340	12.236624	12.675195
			1.4	12.278776	12.725092	12.228395	12.662276
			1.5	12.271071	12.712707	12.220006	12.649227
	3.5	4.9	1.3	11.963606	12.398156	11.920386	12.341148
			1.4	11.955403	12.385777	11.911545	12.328108
			1.5	11.947072	12.373305	11.902571	12.314976
		5	1.3	11.977397	12.406890	11.934491	12.350366
			1.4	11.969082	12.394423	11.925534	12.337234
			1.5	11.960637	12.381862	11.916442	12.324009
0.5	3	4.9	1.3	11.538341	11.989785	11.518496	11.986791
			1.4	11.514712	11.968154	11.494465	11.964678
			1.5	11.489563	11.945538	11.469089	11.941561
		5	1.3	11.564089	12.053259	11.543576	12.050267
			1.4	11.540045	12.032114	11.519138	12.028639
			1.5	11.514489	12.009973	11.493361	12.005994
	3.5	4.9	1.3	11.370195	11.518305	11.351242	11.515628
			1.4	11.352890	11.496735	11.332821	11.493677
			1.5	11.333995	11.474427	11.312933	11.470974
		5	1.3	11.397107	11.586001	11.377529	11.583320
			1.4	11.379399	11.564838	11.358721	11.561774
			1.5	11.360115	11.542926	11.338456	11.539465
1	3	4.9	1.3	11.212146	11.761392	11.206852	11.741896
			1.4	11.177145	11.728519	11.171214	11.709574
			1.5	11.141009	11.694759	11.134432	11.676530
		5	1.3	11.282098	11.785546	11.276803	11.765213
			1.4	11.247558	11.752286	11.241624	11.732502
			1.5	11.211868	11.718138	11.205283	11.699064
	3.5	4.9	1.3	10.710175	11.589218	10.705675	11.567176
			1.4	10.677162	11.561620	10.672166	11.539326
			1.5	10.643320	11.532969	10.637824	11.510539
		5	1.3	10.783777	11.614952	10.779267	11.592156
			1.4	10.751126	11.587001	10.746118	11.563956
			1.5	10.717631	11.558001	10.712118	11.534822

Table 9. Effect of different parameters with k_1 on Θ_3 and $\aleph_3$

			μ	12.5		13	
k_1	γ	β	λ	M-I	M-II	M-I	M-II
0	3	4.9	1.3	34.930526	35.935117	34.894725	35.860089
			1.4	34.923779	35.874790	34.897599	35.792915
			1.5	34.905795	35.806226	34.889572	35.717724
		5	1.3	35.033950	36.041842	35.005442	35.964784
			1.4	35.018904	35.979257	35.000156	35.895515
			1.5	34.992601	35.908580	34.983960	35.818362
	3.5	4.9	1.3	39.951966	41.194254	39.929131	41.159026
			1.4	39.927749	41.182665	39.915034	41.138514
			1.5	39.892229	41.161638	39.889979	41.108946
		5	1.3	40.054939	41.245400	40.039669	41.191716
			1.4	40.022708	41.213485	40.017686	41.151894
			1.5	39.979103	41.173006	39.984687	41.103775
0.5	3	4.9	1.3	34.090550	35.176269	34.071333	35.133712
			1.4	34.061714	35.153826	34.051904	35.103997
			1.5	34.021517	35.122698	34.021307	35.065738
		5	1.3	34.178164	35.293943	34.166141	35.249206
			1.4	34.141100	35.268832	34.138482	35.217000
			1.5	34.092773	35.235196	34.089768	35.176405
	3.5	4.9	1.3	39.946004	41.097425	39.911606	41.081272
			1.4	39.913801	41.077304	39.900624	41.069653
			1.5	39.829933	41.047622	39.866224	41.048746
		5	1.3	40.039248	41.225041	40.026325	41.161473
			1.4	40.009516	41.194902	39.976341	41.129161
			1.5	39.928438	41.145881	39.884764	41.101157
1	3	4.9	1.3	33.736179	34.238483	33.628200	34.125229
			1.4	33.613917	34.179661	33.475034	34.060322
			1.5	33.433294	34.119496	33.312437	34.009391
		5	1.3	33.940644	34.335075	33.881034	34.273230
			1.4	33.802168	34.201856	33.735811	34.111481
			1.5	33.655445	34.188648	33.582265	34.043834
	3.5	4.9	1.3	33.786179	34.338883	33.724200	34.325229
			1.4	33.643917	34.278661	33.575034	34.260322
			1.5	33.493294	34.212496	33.417437	34.189391
		5	1.3	33.998839	34.380896	33.909735	34.566743
			1.4	33.819815	34.312402	33.623341	34.373144
			1.5	33.732326	34.310718	33.528430	34.273253

Table 10. Effect of different parameters with k_2 on Θ_3 and $\aleph_3$

			μ	12.5		13	
k_2	γ	β	λ	M-I	M-II	M-I	M-II
0	3	4.9	1.3	36.819397	38.188913	36.669418	38.001116
			1.4	36.797117	38.152442	36.645105	37.962646
			1.5	36.774364	38.115567	36.620316	37.923790
		5	1.3	36.858969	38.212021	36.709871	38.025586
			1.4	36.836328	38.175275	36.685184	37.986828
			1.5	36.813212	38.138122	36.660019	37.947681
	3.5	4.9	1.3	41.872622	43.393544	41.721350	43.194018
			1.4	41.843912	43.350219	41.690409	43.148378
			1.5	41.814750	43.306567	41.658997	43.102416
		5	1.3	41.920891	43.424115	41.770719	43.226282
			1.4	41.891787	43.380481	41.739370	43.180320
			1.5	41.862229	43.336516	41.707547	43.134033
0.5	3	4.9	1.3	34.615022	35.969356	34.555489	35.960373
			1.4	34.544135	35.904461	34.483394	35.894035
			1.5	34.468688	35.836613	34.407267	35.824682
		5	1.3	34.692266	36.159776	34.630728	36.150801
			1.4	34.620135	36.096342	34.557414	36.085918
			1.5	34.543466	36.029919	34.480083	36.017983
	3.5	4.9	1.3	39.795683	40.562262	39.729348	40.485116
			1.4	39.735114	40.465670	39.664874	40.387640
			1.5	39.668983	40.365393	39.595265	40.286888
		5	1.3	39.889874	40.652333	39.821350	40.572546
			1.4	39.827897	40.554503	39.755523	40.473845
			1.5	39.760402	40.453005	39.684597	40.371876
1	3	4.9	1.3	33.636437	35.284177	33.620556	35.225687
			1.4	33.531434	35.185557	33.513642	35.128722
			1.5	33.423028	35.084276	33.403295	35.029589
		5	1.3	33.846294	35.356637	33.830410	35.295639
			1.4	33.742675	35.256859	33.724873	35.197505
			1.5	33.635603	35.154413	33.615848	35.097191
	3.5	4.9	1.3	37.485612	40.314068	37.469861	40.304697
			1.4	37.370067	40.238572	37.352582	40.227868
			1.5	37.251619	40.160493	37.232382	40.148410
		5	1.3	37.743220	40.551003	37.727434	40.541620
			1.4	37.628942	40.476932	37.611412	40.466209
			1.5	37.511708	40.400240	37.492413	40.368129

Table 11. Effect of parameters $s,\ S$ and L on Tc, Θ_6, $\aleph_6$, Θ_7, $\aleph_7$ for M-I and M-II

S	s	L	M-I Tc	M-I Θ_6	M-I Θ_7	M-II Tc	M-II $\aleph_6$	M-II $\aleph_7$
25	3	9	3.238036	0.142861	0.015800	3.208242	0.146133	0.015974
		11	3.490347	0.157464	0.016857	3.418891	0.163172	0.016888
		13	3.708020	0.167833	0.017450	3.582784	0.176616	0.017614
	5	9	4.071632	0.151859	0.012341	4.038042	0.157782	0.012591
		11	4.366458	0.167377	0.013698	4.254299	0.178028	0.013765
		13	4.623548	0.177405	0.014476	4.406793	0.194053	0.014683
	7	9	4.787175	0.160186	0.010461	4.768930	0.168921	0.010820
		11	5.164389	0.176973	0.012051	5.016830	0.193346	0.012174
		13	5.492868	0.186689	0.012974	5.167740	0.212878	0.013234
30	3	9	3.321759	0.151609	0.015270	3.294398	0.154038	0.015420
		11	3.691844	0.172875	0.016775	3.628219	0.177302	0.016817
		13	4.069249	0.191854	0.018026	3.953855	0.199015	0.018154
	5	9	4.113452	0.161801	0.011844	4.080144	0.166092	0.012060
		11	4.544993	0.185377	0.013587	4.445382	0.193458	0.013679
		13	4.996804	0.205877	0.015026	4.798062	0.219202	0.015156
	7	9	4.783348	0.171360	0.009929	4.755632	0.177518	0.010230
		11	5.320347	0.197952	0.011865	5.187014	0.210097	0.012026
		13	5.889404	0.220523	0.013498	5.590835	0.241137	0.013619
35	3	9	3.402898	0.158250	0.014815	3.377587	0.160140	0.014946
		11	3.874466	0.185037	0.016650	3.817350	0.188600	0.016699
		13	4.400359	0.211664	0.018436	4.294597	0.217653	0.018529
	5	9	4.166496	0.169140	0.011421	4.134227	0.172420	0.011609
		11	4.713830	0.199246	0.013428	4.624679	0.205639	0.013529
		13	5.344847	0.228925	0.015362	5.163771	0.239888	0.015438
	7	9	4.806328	0.179338	0.009493	4.774310	0.183953	0.009747
		11	5.475793	0.213669	0.011647	5.355552	0.223109	0.011815
		13	6.262317	0.247337	0.013765	5.991815	0.264043	0.013808