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Reliability analysis of N-policy vacation-based FTC system subject to standby switching failures

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Abstract

The paper is aimed to investigate the reliability metrics of a multi-unit fault-tolerant control (FTC) system wherein the units are subject to failure and those are repairable by two heterogeneous servers. Server 1 remains permanently available for essential service of failed units, whereas server 2 goes on vacation and renders service based on the N-policy threshold, which may also provide optional and essential services. Server 1 may break down at a steady rate during its servicing period but immediately gets repaired and resume servicing the failed units. When the working unit fails, the available warm standby unit holds responsibility for the smooth operation of the system. The transition of standby units to operational mode may be unsuccessful with switching failure probability. We develop a Markovian model to obtain the steady-state probabilities. We explore computational and sensitivity analysis of different performance measures for various variability of the parameters.

Keywords: *reliability, standby switching failure, N-policy, server breakdown, optional service*

1. Introduction

Many researchers developed new design approaches and fault control techniques for better system reliability and availability from time to time. Fault-tolerant control (FTC) systems are very crucial in many life-basic applications such as power plants, flight control frameworks, telecommunication, manufacturing, space applications, and so on. The modern FTC complex system has been developed with tolerance capabilities and fault accommodation. The presence of faults can disturb the functionality of the system or sometimes destroy the whole system. The fault-tolerant computing system has the ability that preserves the continuity of operation without interruption in the presence of one or more faults [6, 18]. Hence the design of fault-tolerant control (FTC) systems is necessarily validated from the reliability perspective. The reliability of FTC systems with various parameters has been investigated by many researchers. In a repairable system, warm standby provisioning is taken as a significant part of improving the quality and

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accessibility. In most of the FTC systems with warm standby provisioning, the exchange of standbys in a working state is prevalent. The process of switching from standby to operating mode may be unsuccessful after a period of time. Such imperfect switching of standby units needs to be analyzed in the FTC system to make it adaptable [15, 16].

FTC systems with server vacation should be effectively quantified for finding/maintaining the reliability and other performance indices of dependability. The term "server vacation" refers to a period of time when a server is not available for service for a random period of time [19]. The idea of threshold N-policy vacation is utilization in the reliability and machine repair problems for maximum utilization of the server properly. In N-policy, when N or more than N failed units are gathered in the system, the subsequent server turns on, and when there is no failed unit in the system, the server turns off and it may go for vacation [2]. It is expected that the server is consistently accessible in the system on a regular basis to provide the service to failed units and the system never fails. In any FTC system, service interruption due to server failure may happen which can show a direct impact on desired system performance as well as in reaching the intended output. Many researchers looked into the repairable service system, in which the server may break down. Choudhary and Tadj [4] discussed a repairable system with two phases of the server when it is subjected to server breakdown. The servers are set up to provide two types of services: essential and optional. After the completion of the first essential service, an optional service is provided to the units. Optional services reflect a significant impact in increasing the system's performance. Jain and Chauhan [11] discussed the repairable Markovian model with optional service assistance whereas the bulk arrival queuing system with optional services has been suggested by Singh et al. [20]. In this context, the reliability estimation for the machining system under the server vacation threshold rule with essential and optional service has been discussed by Gupta and Agrawal [7].

We present some notable contributions in the direction of real-time systems from a reliability viewpoint with concepts of imperfect switching, N-policy, and server breakdown. Choudhary and Tadj [4] discussed a repairable system with two phases of the server when it is subjected to server breakdown. Wang and Chen [21] examined the reliability metrics of three frameworks with general service time, reboot delay, and switching failure. A reliability perspective quality investigation of FTC framework on automated airborne vehicles has been done by Hu and Seiler [9]. Yang and Wu [24] discussed a working vacation repairable system with server breakdown for minimizing the cost. Jain et al. [14] investigated N-policy-based time-shared repairable system with mixed spares, whereas Jain and Jain [10] studied the multiple server repairable system with inconsistent server and two kinds of spares. He et al. [8] discussed a multi-component system with N-policy and vacation. Some new investigations have also been discussed on the reliability forecast with N-policy by Jain and Gupta [12]. Jain and Meena [13] developed a Markov mathematical model for a repairable system with two heterogeneous servers where recovery of units follows some threshold rule. Chen et al. [3] assessed aeronautics system reliability measures with imperfect fault coverage. Wu and Yang [23] and Fang et al. [5] discussed the reliability of the repairable system with switching failure subject to an unreliable repairman.

From the literature survey, it is evident that very few research articles have been published on the performance analysis of production and manufacturing systems with server breakdown, multiple types of spare provisioning, and essential and optional services under the N-policy threshold for server vacationing. From the best knowledge of the authors, it is related that a research gap in the area of reliability modeling of fault-tolerant control systems with switching failure, server interruption, and the option of essential/optional services. Performance modeling of the FTC system under the assumptions of switching failure rate, vacation rate, operational and standby units failure and service rates, breakdown and service rate of servers have a wide range of applications. We are motivated to investigate various reliability indices of the FTC system. A clear comparison of notable features used in recent-past studies with the proposed study is given in Table 1.

Table 1. Special features used in the proposed model different from the other relevant works

Authors	Threshold policy	Standbys /spares	Vacation /sorking vacation	Breakdown	Optional service
Jain et al. [14]	✓	✓	×	✓	×
Jain and Meena [13]	✓	✓	✓	×	×
He et al. [8]	✓	×	✓	×	×
Singh et al. [20]	×	×	×	✓	✓
Yen et al. [26]	✓	×	✓	✓	✓
Yang and Wu [25]	×	✓	✓	✓	×
Kumar et al. [17]	✓	✓	✓	×	×
Wang et al. [22]	×	✓	×	✓	×
Proposed model	✓	✓	✓	✓	✓

In the present investigation, we are concerned with Markov analysis of fault-tolerant control system (FTC) by developing a machine repair model with two heterogeneous servers and provision of k-types warm standby units. To explore the performance metrics of the FTC system with k-types warm standby support by incorporating realistic assumptions, a Markov model in the general setup can be framed. We investigate the reliability, mean time to system failure, and sensitivity check of an FTC system.

We refer to a fault-tolerant process control system that is responsible for controlling, monitoring, and documenting production/manufacturing processes (cf. [1]). The systematic diagram for a fault-tolerant process control system is given in Figure 1. The system comprises operating hardware units (Racks, CPUs, synchronization cables and modules, communication processors) along with their redundant copies that simultaneously participate in the control tasks. These hardware operating units are highly fault tolerant. However, service unavailability of any unit is a possible case for a period of time due to processor speed, network bandwidth, and improper switching from failed to standby component, server vibration, improper connections. Hence the system components must be repaired on a regular basis. The fault-tolerant control system enables one to configure two maintenance supervisors (server 1 and server 2) redundantly for fault-tolerant operations which ensures process monitoring, controlling, and fault repairing at all times (essential tasks/service). Server 1 is subject to the process error at any point during its functioning (server breakdown). In order to ensure high availability of service, broken down server 1 is immediately sent to the repair facility to check and repair it instantly. In the real-time fault-tolerant process control system, when some components are detected as failed during the vacation period of the supervisor (server 2), it may join as soon as possible to service the failed component in the system. The supervisor (server 2) is also responsible to monitor and control the environmental noise like a magnetic field in actuators and sensors (optional services).

The remainder of this paper is organized as follows. In Section 2, we discuss the complete model description with the help of a state transition diagram. The system state transition equations are presented in Section 3. In Section 4, we present the analysis of system governing equations by employing a matrix approach. In Section 5, we discuss some performance estimates. Mathematical outcomes under numerical simulation are given in Section 6. Finally, Section 7 contains the conclusion of the complete work.

2. Model description

In this section, we develop a multi-unit Markovian model for the fault-tolerant control system in which the units are subject to failures and two servers are helpful to repair the failed units. The units may demand two types of services, namely essential and optional. We assume total $(M+S)$ units of FTC system which is composed of M identical operating units and S warm standby units with two heterogeneous servers – 1 and 2. Following are the notations and assumptions which we have used for modeling purposes.

2.1. Notations and assumptions

- The S warm standbys are supposed to be of k types such that $S = \sum_{j=1}^{k} S_j$, which are different as per different failure rates. The lifetimes of M operating units and $j\mathrm{th}$ $(j = 1, 2, \ldots, k)$ type standby units are exponentially distributed. The failure rates of units are state-dependent and units may also fail due to common cause failure.
- The failure rate of $j\mathrm{th}$ type standby unit is higher than that of the $(j+1)\mathrm{th}$ type standby unit, i.e., $\alpha_j \geq \alpha_{j+1}$. When an operating unit fails, it is immediately replaced by a warm standby unit which has a higher failure rate.
- For a normal functioning of the system, M operational units are required and units operate simultaneously in parallel. When all the warm standby units are exhausted and when there are less than M but more than m operating units, the system still works in a degraded mode. The system fails when L or more than L units fail in the system, where $L = M + S - m + 1$ $(1 < m < M)$.
- When the failed units are gathered in the system, server 1 becomes available to service them in normal busy periods, but there may be the possibility of server interruption in servicing process. Hence, server 1 may be broken down.
- The lifetime and service time of server 1 during his breakdown period are exponentially distributed with rates α and β, respectively.
- In a normal busy period, initially server 2 is unavailable for servicing the failed units as he can go on vacation for a random period of time. But server 2 returns from vacation to a normal busy state only when there are N or more failed units accumulated in the system as per the N-policy rule, and then both servers 1 and 2 become available to repair the failed units as an essential service. The vacation time of server 2 is assumed to follow an exponential distribution.
- Only server 2 is responsible for performing optional service on demand for servicing failed units with probability $\overline{p} = 1 - p$. If there is no demand/requirement for units, units will be serviced by probability p.
- The switch-over times of the units from standby to operational state is instantaneous.
- We assume automatic switching of standby units in operating state is imperfect.
- The service times of both servers 1 and 2 are exponentially distributed.

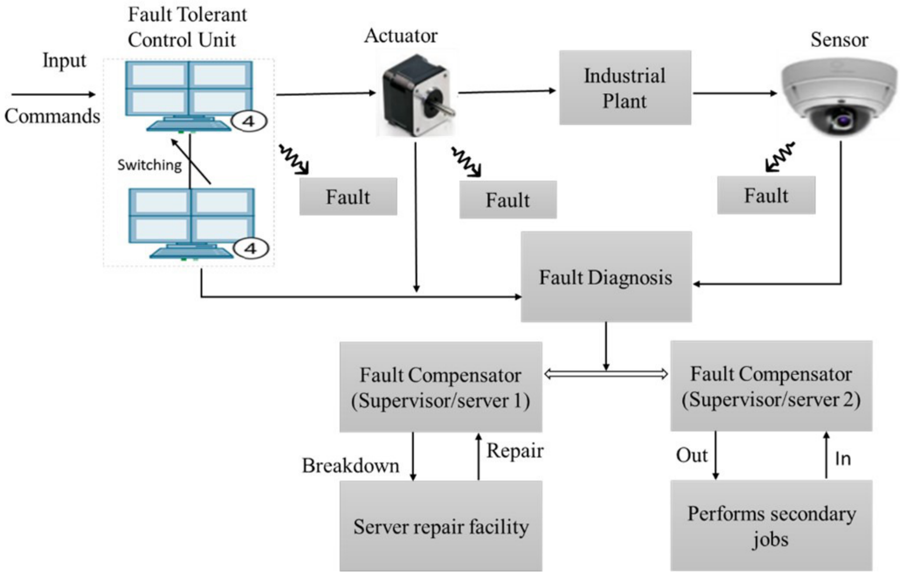

Figure 1. Schematic diagram of fault-tolerant process control system

M	–	identical operating units
S	–	warm standby units
S_j	–	$j\text{th}$ type warm standby/spares unit, $j = 1, 2 \ldots, k$
λ	–	failure rate of operating unit
α_j	–	failure rate of $j\text{th}$ type standby unit
λ_c	–	common cause failure rate
λ_d	–	degraded failure rate
α	–	server 1 breakdown rate
β	–	server 1 recovery (repair) rate
N	–	threshold parameter of server 2 when it returns from vacation
θ	–	server 2 vacation return rate
q	–	standby switching failure probability
μ_1	–	service rate of server 1
μ_2	–	service rate of server 2 for essential service
μ_2'	–	service rate of server 2 for optional service

$P_{n,0}(t)$	–	probability when there are n $(0 \le n \le L)$ failed units in the system at time t and server 1, is available for essential service while server 2 is on vacation
$P_{n,1}(t)$	–	probability when there are n $(0 \le n \le L)$ failed units in the system at time t and server 1 is in a broken down state and server 2 is not available
$P_{n,2}(t)$	–	probability when there are n $(N \le n \le L)$ failed units in the system at time t and essential service is being rendered by server 1 and server 2
$P_{n,3}(t)$	–	probability when there is n $(N \le n \le L)$ failed units in the system at time t and only server 2 is available for optional service to the failed units

We consider bivariate Markov process $\eta(t) = \{(\chi(t), \xi(t), \, t \geq 0\}$ to develop the Markov model. Here $\chi(t)$ denotes the number of failed units in the system at time t and $\xi(t)$ denotes the state of the server at time t, respectively,

where

$$\xi(t) = \begin{cases} 0, & \text{server 1 is available for essential service and server 2 is on vacation} \\ 1, & \text{only server 1 is in breakdown state, server 2 is not available} \\ 2, & \text{both servers 1 and 2 are available for essential service} \\ 3, & \text{only server 2 is available for optional service} \end{cases}$$

The transition failure rate in the system can be given as:

$$\lambda_n = \begin{cases} M\lambda + (S^{(1)} - n)\alpha_1 + \sum\limits_{i=2}^{k} S_i\alpha_i + \lambda_c, \quad 0 \leq n \leq S^{(1)} \\ M\lambda + (S^{(l)} - n)\alpha_l + \sum\limits_{i=l+1}^{k} S_i\alpha_i + \lambda_c, \quad S^{(l-1)} \leq n \leq S^{(l)}, \, l = 2, \, 3, \, \ldots, \, k \\ (M + S - n)\lambda_d, \quad S^{(k)} \leq n \leq M + S^{(k)} - m = L - 1 \\ 0, \quad \text{otherwise} \end{cases}$$

where $S^{(l)} = \sum\limits_{j=1}^{l} S_j$.

3. System state transition equations

In this section, we build the transient difference differential equations for the Markov model of the fault-tolerant control system by utilizing the birth-death process (see Figure 2), which are mentioned below for different levels.

Case 1. Server 1 is available for essential service and server 2 is on vacation, $\xi(t) = 0, \; 0 \leq n \leq L$

$$P'_{0,0}(t) = -(\lambda_0 + M\lambda q + \alpha)P_{0,0}(t) + \mu_1 P_{1,0}(t) + \beta P_{0,1}(t) \tag{1}$$

$$P'_{1,0}(t) = -(\lambda_1 + \mu_1 + M\lambda q + \alpha)P_{1,0}(t) + \lambda_0 P_{0,0}(t) + \mu_1 P_{2,0}(t) + \beta \tag{2}$$

$$\begin{aligned} P'_{n,0}(t) = &-(\lambda_n + \mu_1 + M\lambda q + \alpha)P_{n,0}(t) + \lambda_{n-1}P_{n-1,0}(t) + \mu_1 P_{n+1,0}(t) \\ &+ \sum_{r=0}^{n-2} M\lambda q^{n-r-1}(1-q)P_{r,0}(t) + \beta P_{n,1}(t), \;\; 2 \leq n \leq N-2 \end{aligned} \tag{3}$$

$$\begin{aligned} P'_{N-1,0}(t) = &-(\lambda_{N-1} + \mu_1 + M\lambda q + \alpha)P_{N-1,0}(t) + \lambda_{N-2}P_{N-2,0}(t) + \mu_1 P_{n,0}(t) \\ &+ \sum_{r=0}^{N-3} M\lambda q^{N-r-2}(1-q)P_{r,0}(t) + (\mu_1 + p\mu_2)P_{n,2}(t) + \mu'_2 P_{n,3}(t) + \beta P_{N-1,1}(t) \end{aligned} \tag{4}$$

$$P'_{n,0}(t) = -(\lambda_n + \mu_1 + \theta + M\lambda q + \alpha)P_{n,0}(t) + \lambda_{n-1}P_{n-1,0}(t) + \mu_1 P_{n+1,0}(t) \\ + \sum_{r=0}^{n-2} M\lambda q^{n-r-1}(1-q)P_{r,0}(t) + \beta P_{n,1}(t), \;\; N \le n \le S^{(k)} - 1 \tag{5}$$

$$P'_{S^{(k)},0}(t) = -\left(\lambda_{S^{(k)}} + \mu_1 + \theta + \alpha\right)P_{S^{(k)},0}(t) + \lambda_{S^{(k)}-1}P_{S^{(k)}-1,0}(t) + \mu_1 P_{S^{(k)}+1,0}(t) \\ + \sum_{r=0}^{S^{(k)}-2} M\lambda q^{S^{(k)}-r-1}(1-q)P_{r,0}(t) + \beta P_{S^{(k)},1}(t) \tag{6}$$

$$P'_{S^{(k)}+1,0}(t) = -(\lambda_{S^{(k)}+1} + \mu_1 + \theta + \alpha)P_{S^{(k)}+1,0}(t) + \lambda_{S^{(k)}}P_{S^{(k)},0}(t) + \mu_1 P_{S^{(k)}+2,0}(t) \\ + \sum_{r=0}^{S^{(k)}-1} M\lambda q^{S^{(k)}-r}P_{r,0}(t) + \beta P_{S^{(k)}+1,1}(t) \tag{7}$$

$$P'_{n,0}(t) = -(\lambda_n + \mu_1 + \theta + \alpha)P_{n,0}(t) + \lambda_{n-1}P_{n-1,0}(t) \\ + \mu_1 P_{n+1,0}(t) + \beta P_{n,1}(t), S^{(k)}+2 \le n \le L-2 \tag{8}$$

$$P'_{L-1,0}(t) = -(\lambda_{L-1} + \mu_1 + \theta + \alpha)P_{L-1,0}(t) + \lambda_{L-2}P_{L-2,0}(t) + \beta P_{L-1,1}(t) \tag{9}$$

$$P'_{L,0}(t) = \lambda_{L-1}P_{L-1,0}(t) \tag{10}$$

Case 2. Only server 1 is in a breakdown state, server 2 is not available, $\xi(t) = 1$, $0 \le n \le L$

$$P'_{0,1}(t) = -(\lambda_0 + M\lambda q + \beta)P_{0,1}(t) + \alpha P_{0,0}(t) \tag{11}$$

$$P'_{1,1}(t) = -(\lambda_1 + M\lambda q + \beta)P_{1,1}(t) + \lambda_0 P_{0,1}(t) + \alpha P_{1,0}(t) \tag{12}$$

$$P'_{n,1}(t) = -(\lambda_n + M\lambda q + \beta)P_{n,1}(t) + \lambda_{n-1}P_{n-1,1}(t) \\ + \sum_{r=0}^{n-2} M\lambda q^{n-r-1}(1-q)P_{r,1}(t) + \alpha P_{n,0}(t), \;\; 2 \le N \le S^{(k)} - 1 \tag{13}$$

$$P'_{S^{(k)},1}(t) = -(\lambda_{S^{(k)}} + \beta)P_{S^{(k)},1}(t) + \lambda_{S^{(k)}-1}P_{S^{(k)}-1,1}(t) \\ + \sum_{r=0}^{S^{(k)}-2} M\lambda q^{S^{(k)}-r-1}(1-q)P_{r,1}(t) + \alpha P_{S^{(k)},0}(t) \tag{14}$$

$$P'_{S^{(k)}+1,1}(t) = -(\lambda_{S^{(k)}+1} + \beta)P_{S^{(k)}+1,1}(t) + \lambda_{S^{(k)}}P_{S^{(k)},1}(t) \\ + \sum_{r=0}^{S^{(k)}-1} M\lambda q^{S^{(k)}-r}P_{r,1}(t) + \alpha P_{S^{(k)}+1,0}(t) \tag{15}$$

$$P'_{n,1}(t) = -(\lambda_n + \beta)P_{n,1}(t) + \lambda_{n-1}P_{n-1,1}(t) + \alpha P_{n,0}(t), \quad S^{(k)}+2 \le n \le L-1 \tag{16}$$

$$P'_{L,1}(t) = \lambda_{L-1}P_{L-1,1}(t) \tag{17}$$

Case 3. Both server 1 and server 2 are available for essential service, $\xi(t) = 2$, $N \le n \le L$

$$P'_{n,2}(t) = -(\lambda_N + \mu_1 + \mu_2 + M\lambda q)P_{n,2}(t) + \theta P_{n,0}(t) + \mu'_2 P_{N+1,3}(t) + (\mu_1 + p\mu_2)P_{N+1,2}(t), \tag{18}$$

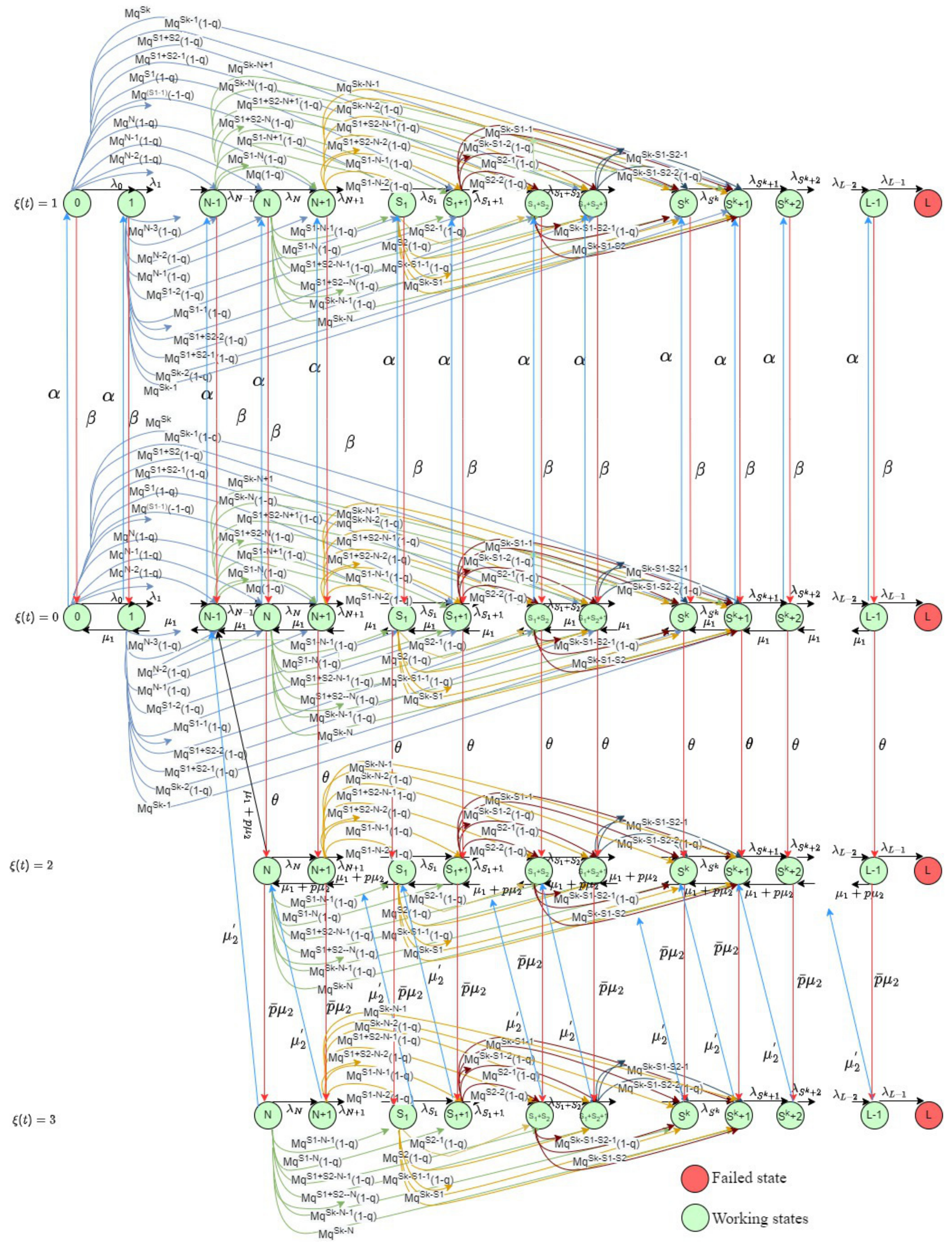

Figure 2. State transition diagram

$$P'_{N+1,2}(t) = -(\lambda_{N+1} + \mu_1 + \mu_2 + M\lambda q)P_{N+1,2}(t) + \theta P_{N+1,0}(t) + \mu'_2 P_{N+2,3}(t) \\ +(\mu_1{+}p\mu_2)P_{N+2,2}(t) + \lambda_N P_{n,2}(t) \tag{19}$$

$$P'_{n,2}(t) = -(\lambda_n + \mu_1 + \mu_2 + M\lambda q)P_{n,2}(t) + \theta P_{n,0}(t) + \mu'_2 P_{n+1,3}(t) + (\mu_1{+}p\mu_2)P_{n+1,2}(t) \\ +\lambda_{n-1}P_{n-1,2}(t) + \sum_{r=N}^{n-2} M\lambda q^{n-r-1}(1-q)P_{r,2}(t), \quad N+2 \le n \le S^{(k)}-1 \tag{20}$$

$$P'_{S^{(k)},2}(t) = -(\lambda_{S^{(k)}} + \mu_1 + \mu_2)P_{S^{(k)},2}(t) + \theta P_{S^{(k)},0}(t) + \mu'_2 P_{S^{(k)}+1,3}(t) \\ +(\mu_1{+}p\mu_2)P_{S^{(k)}+1,2}(t) + \lambda_{S^{(k)}-1}P_{S^{(k)}-1,2}(t) + \sum_{r=N}^{S^{(k)}-2} M\lambda q^{S^{(k)}-r-1}(1-q)P_{r,2}(t) \tag{21}$$

$$P'_{S^{(k)}+1,2}(t) = -(\lambda_{S^{(k)}+1} + \mu_1 + \mu_2)P_{S^{(k)}+1,2}(t) + \theta P_{S^{(k)}+1,0}(t) + \mu'_2 P_{S^{(k)}+2,3}(t) \\ +(\mu_1{+}p\mu_2)P_{S^{(k)}+2,2}(t) + \lambda_{S^{(k)}}P_{S^{(k)},2}(t) + \sum_{r=N}^{S^{(k)}-1} M\lambda q^{S^{(k)}-r}P_{r,2}(t) \tag{22}$$

$$P'_{n,2}(t) = -(\lambda_n + \mu_1 + \mu_2)P_{n,2}(t) + \theta P_{n,0}(t) + \mu'_2 P_{n+1,3}(t) + (\mu_1{+}p\mu_2)P_{n+1,2}(t) \\ +\lambda_{n-1}P_{n-1,2}(t), \quad S^{(k)}{+}2 \le n \le L-2 \tag{23}$$

$$P'_{L-1,2}(t) = -(\lambda_{L-1} + \mu_1 + \mu_2)P_{L-1,2}(t) + \theta P_{L-1,0}(t) + \lambda_{L-2}P_{L-2,2}(t) \tag{24}$$

$$P'_{L,2}(t) = \lambda_{L-1}P_{L-1,2}(t) \tag{25}$$

Case 4. Only server 2 is available for optional service, $\xi(t) = 3$, $N \le n \le L$

$$P'_{n,3}(t) = -\left(\lambda_N + \mu'_2 + M\lambda q\right)P_{n,3}(t) + \overline{p}\mu_2 P_{n,2}(t) \tag{26}$$

$$P'_{N+1,3}(t) = -\left(\lambda_{N+1} + \mu'_2 + M\lambda q\right)P_{N+1,3}(t) + \lambda_N P_{(n,3)}(t) + \overline{p}\mu_2 P_{N+1,2}(t) \tag{27}$$

$$P'_{n,3}(t) = -(\lambda_n + \mu'_2 + M\lambda q)P_{n,3}(t) + \lambda_{n-1}P_{n-1,3}(t) + \overline{p}\mu_2 P_{n,2}(t) \\ +\sum_{r=N}^{n-2} M\lambda q^{n-r-1}(1-q)P_{r,3}(t), \quad N+2 \le n \le S^{(K)}-1 \tag{28}$$

$$P'_{S^{(k)},3}(t) = -\left(\lambda_{S^{(k)}} + \mu'_2\right)P_{S^{(k)},3}(t) + \lambda_{S^{(k)}-1}P_{S^{(k)}-1,3}(t) + \overline{p}\mu_2 P_{S^{(k)},2}(t) \\ +\sum_{r=N}^{S^{(k)}-2} M\lambda q^{S^{(k)}-r-1}(1-q)P_{r,3}(t) \tag{29}$$

$$P'_{S^{(k)}+1,3}(t) = -\left(\lambda_{S^{(k)}+1} + \mu'_2\right)P_{S^{(k)}+1,3}(t) + \lambda_{S^{(k)}}P_{S^{(k)},3}(t) + \overline{p}\mu_2 P_{S^{(k)}+1,2}(t) \\ +\sum_{r=N}^{S^{(k)}-1} M\lambda q^{S^{(k)}-r}P_{r,3}(t) \tag{30}$$

$$P'_{n,3}(t) = -\left(\lambda_n + \mu'_2\right)P_{n,3}(t) + \lambda_{n-1}P_{n-1,3}(t) + \overline{p}\mu_2 P_{n,2}(t), S^{(k)}{+}2 \le n \le L-1 \tag{31}$$

$$P'_{L,3}(t) = \lambda_{L-1}P_{L-1,3}(t). \tag{32}$$

4. The analysis

The concept of matrix analytic approach can be used to evaluate equations (1)–(32). For the solution purpose, we introduce the Laplace transform of $P_{i,\xi(t)}(t)$ as

$$P^*_{i,\xi(t)}(s) = \int_0^\infty e^{-st} P_{i,\xi(t)}(t)dt, \quad s \geq 0 \tag{33}$$

We assume that all units are in a good state initially so that $P_{0,0}(0) = 1$, $P_{n,0}(0) = 0$, when $1 \leq n \leq L$ and $P_{n,1}(0) = P_{n,2}(0) = P_{n,3}(0) = 0$ when $0 \leq n \leq L$.

Expressing equations (1)–(32) in matrix notations as

$$D(s)P^*(s) = P(0) \tag{34}$$

where $D(s)$ denotes the transition rate matrix of order $(4(L+1) - 2N) \times (4(L+1) - 2N)$ it may be given as

$$D(s) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{35}$$

and its submatrices M_{11}, M_{12}, M_{21}, M_{22} one can find in appendix A,

$$P^*(s) = \begin{Bmatrix} P^*_{0,0}(s), P^*_{1,0}(s), \ldots, P^*_{n,0}(s), P^*_{N+1,0}(s), \ldots, P^*_{S^k,0}(s), P^*_{S^{(k)}+1,0}(s), \ldots, P^*_{L,0}(s), \\ P^*_{0,1}(s), P^*_{1,1}(s), \ldots, P^*_{n,1}(s), P^*_{N+1,1}(s), \ldots, P^*_{S^{(k)},1}(s), P^*_{S^{(k)}+1,1}(s), \ldots, P^*_{L,1}(s), \\ P^*_{n,2}(s), P^*_{N+1,2}(s), \ldots, P^*_{S^{(k)},2}(s), P^*_{S^{(k)}+1,2}(s), \ldots, P^*_{L,2}(s), \\ P^*_{n,3}(s), P^*_{N+1,3}(s), \ldots, P^*_{S^{(k)},3}(s), P^*_{S^{(k)}+1,3}(s), \ldots, P^*_{L,3}(s) \end{Bmatrix}^T$$

and

$$P(0) = \{1, 0, 0, \ldots, 0\}^T$$

Now we present Cramer's rule to solve equation (34) and obtain an explicit expression for the last element of column vector $P^*_{i,\xi(t)}(s)$ as

$$P^*_{i,\xi(t)}(s) = \frac{\det[D_{i+1}(s)]}{\det[D(s)]}, \quad \xi(t) = 0, 1, 2, 3 \tag{36}$$

Here, $\det[D(s)]$ is the determinant of matrix $D(s)$, and the determinant $\det[D_{i+1}(s)]$ is obtained by replacing $(L + i + 1 - \overline{N-1})$th column in matrix $D(s)$ by initial vector $P(0) = \{1, 0, 0, \ldots, 0\}^T$.

Now, we compute the characteristic roots of matrix $D(s)$, $s = 0$ is clearly a root of $\det[D(s)] = 0$.

When we substitute $s = -r$, we obtain

$$D(-r) = D - rI$$

Here, identity matrix I and $D = D(0)$ are matrix of $(4(L+1) - 2N) \times (4(L+1) - 2N)$ order.

Thus equation (34) becomes

$$D(-r)P^{*}(s)=(D-rI)P^{*}(s)=P(0) \tag{37}$$

To find the distinct eigenvalues $r_f(f\neq 0)$ which may be real or complex. We set the determinant of matrix $(D-rI)=0$ and $f=1,\,2,\,\dots,\,(4(L+1)-2N)$.

We assume that there are g distinct real eigenvalues (with zero), say $r_1,r_2,\,\dots,\,r_g$ and h pairs of distinct complex conjugate eigenvalues, say $(r_{g+1},\overline{r}_{g+1}),(r_{g+2},\overline{r}_{g+2}),\,\dots,\,(r_{g+h},\overline{r}_{g+h})$ where g and h satisfy $g+2h=4(L+1)-2N$. It is noticed that $h=0$ represents all eigenvalues that are real and $g=0$ denotes all eigenvalues (exclude zero) are complex.

Then, we evaluate $\det[D_{L-i+1}(s)]$. Then we substitute the values of $\det[D_{L-i+1}(s)]$ and $\det[D(s)]$ in equation (36)

$$\begin{aligned} P^{*}_{i,\xi(t)}(s)=&\frac{b_{i,1}}{s+r_1}+\frac{b_{i,g}}{s+r_g}+\frac{(c_{i,1})\,s+d_{i,1}}{s^2+(r_{g+1}+\overline{r}_{g+1})\,s+(r_{g+1}.\overline{r}_{g+1})}+\dots\\ &+\frac{(c_{i,h})\,s+d_{i,h}}{s^2+(r_{g+h}+\overline{r}_{g+h})\,s+(r_{g+h}.\overline{r}_{g+h})},\quad i=0,1,\dots,L \end{aligned} \tag{38}$$

where $b_{i,1},\,\dots,\,b_{i,g},c_{i,1},\,\dots,\,c_{i,h},d_{i,1},\,\dots,\,d_{i,h}$ are unknown real numbers.

Let us consider u_f and v_f are real and imaginary parts of h pairs of distinct complex conjugate eigenvalues. By taking inverse Laplace transform of equation (38), we get explicit expressions of $P_{i,\xi(t)}(t)$ given by

$$\begin{aligned} P_{i,\xi(t)}(t)=&\sum_{f=1}^{g}b_{i,f}\mathrm{e}^{-u_f t}\\ &+\sum_{f=1}^{h}\Big(c_{i,f}\mathrm{e}^{-u_f t}\cos(v_f t)+\frac{d_{i,f}-c_{i,f}u_f}{v_f}\mathrm{e}^{-u_f t}\sin(v_f t)\Big),\quad i=0,1,\,\dots,\,L \end{aligned} \tag{39}$$

5. Performance measures

In this section, we establish various performance measures to check the operational efficiency of the fault-tolerant control system and to know which parameter will be helpful to enhance it even in the presence of server breakdown, and standby switching failures wherein the server remains on vacation for a random time period.

- Expected number of failed units in the system

$$E_f(t)=\sum_{i=1}^{L}iP_{i,0}(t)+\sum_{i=1}^{L}iP_{i,1}(t)+\sum_{i=N}^{L}iP_{i,2}(t)+\sum_{i=N}^{L}iP_{i,3}(t)$$

- Expected number of standby units in the system

$$E_s(t) = \sum_{i=0}^{S-1} (S-i)\, P_{i,0}(t) + \sum_{i=0}^{S-1} (S-i)\, P_{i,1}(t) + \sum_{i=N}^{S-1} (S-i)\, P_{i,2}(t) + \sum_{i=N}^{S-1} (S-i)\, P_{i,3}(t)$$

- Probability that server 1 is busy

$$P_b^1(t) = \sum_{i=1}^{L} P_{i,0}(t) + \sum_{i=N}^{L} P_{i,2}(t)$$

- Probability that server 2 is busy

$$P_b^2(t) = \sum_{i=N}^{L} P_{i,2}(t) + \sum_{i=N}^{L} P_{i,3}(t)$$

- Probability that server 1 is in a breakdown state

$$P_{bd}(t) = \sum_{i=0}^{L} P_{i,1}(t)$$

- The system reliability

$$R(t) = 1 - \sum_{j=0}^{3} P_{L,j}(t), \;\; t \geq 0$$

- Mean time to failure

$$MTTF = \lim_{s \to 0} R^*(s) = \lim_{s \to 0} \left(\frac{1}{s} - \sum_{j=0}^{3} P_{L,j}(t) \right)$$

- The system availability

$$A(t) = 1 - \frac{E_f(t)}{L}$$

- Failure frequency of the system

$$Wf = \lambda_{L-1}\, (P_{L,0}(t) + P_{L,1}(t) + P_{L,2}(t) + P_{L,3}(t))$$

6. Numerical results and sensitivity analysis

We explore computational and sensitivity analysis of different performance measures for various variability of the parameters with the help of MATLAB software.

6.1. Numerical simulation of system reliability

The numerical illustration is taken to check the sensitivity of parameters for assuming ten operating units $(M = 10)$ for fault-tolerant control system along three types of warm standbys $(S = S_1 + S_2 + S_3)$

taking as $S_1 = 4$, $S_2 = 3$ and $S_3 = 2$. Also, we assume that server 2 becomes active to render the service only after returning from vacation when $N = 3$. Following are the default parameters that we have taken for computational results of the system reliability indices as $\lambda = 0.1$, $\lambda_d = 0.01$, $\lambda_c = 1.5$, $\alpha_1 = 0.01$, $\alpha_2 = 0.1$, $\alpha_3 = 0.5$, $q = 0.7$, $p = 0.1$, $\overline{p} = 0.9$, $\mu_1 = 0.02$, $\mu_2 = 0.1$, $\mu_2' = 1$, $\theta = 0.8$, $\alpha = 1.5$, $\beta = 0.5$. The numerical findings are summarised in Figures 3–7 and Tables 2–7 which allow us to investigate the effect of various parameters on performance measures as time passes.

- **Time dependencies of the system reliability under varying failure parameters of operating and standby units.** In Figure 3, we show the effect of the system's reliability with respect to time. Initially, as time increases, reliability $R(t)$ decreases sharply. After that, it decreases gradually and lastly becomes nearly constant as time t further increases. It is also visible that the reliability $R(t)$ of the system decreases with increasing parameters λ, λ_d, λ_c and α_3.
- **Time dependencies of the system reliability under the varying server 1 breakdown and repair parameters.** As previously, we get a decreasing pattern of the system reliability upon time and then the constant value (Figure 4). Further, we yield in Figure 4a that as we increase the breakdown rate of server 1, the reliability decreases. On the contrary, in Figure 4b the reliability $R(t)$ increases as its recovery rate increases.
- **Time dependencies of the system reliability under varying switching failure and vacation parameters.** Upon time increasing, we observe a gradual decrease in the system's reliability $R(t)$which then becomes asymptotically constant for different values of switching failure probability q and vacation rates θ (Figure 5). We can see that as q increases, $R(t)$ decreases whereas, when θ increases, $R(t)$ also increases.
- **Time dependencies of the system reliability with a service rate of server 1 by the varying server 1 breakdown and repair parameters.** We show the sensitivity of system reliability for the service rate (μ_1) of server 1 in Figure 6. Initially, a sharp jump is found in $R(t)$ for lower values of μ_1 then it starts to increase gradually. Further, upon increasing the breakdown rate (α) of server 1, the reliability decreases which matches with a realistic scenario (Figure 6a). Also, $R(t)$ increases as the repair rate of server 1 in the breakdown period (β) increases (Figure 6b).
- **Time dependencies of the system reliability with a service rate of server 2 by varying switching failure and vacation parameters.** Figure 7 depicts the effect of system reliability with respect to service rate (μ_2) of server 2 by varying switching failure probability (q) and vacation rate (θ). We get an increasing pattern upon μ_2 increasing up to a constant value. As we increase the switching failure probability, the reliability decreases (Figure 7a). On the contrary, the reliability $R(t)$ increases as the vacation rate increases (Figure 7b).
- **Effect of various operational characteristics by varying parameters.** The numeric outcomes of operational characteristics such as the expected number of failed and standby units, machine availability, and probabilities for busy servers with respect to different parameters are given in Tables 2–7. It is quite clear to see the effect of parameters λ, q, β, μ_1 on various performance measures by varying the time in Tables 2–5. In Tables 2–3, the indices $E_f, P_b^1, P_b^2, P_{bd}, Wf$ increase while A and E_s decrease with increase in λ and q. With the increase in the indices β and μ_1, $E_f, P_b^1, P_b^2, P_{bd}, Wf$ decrease while $A(t)$ and E_s increase (see Tables 4–5).

Tables 6 and 7 depict the effect of θ and α on system measures by varying time t. In Table 6,

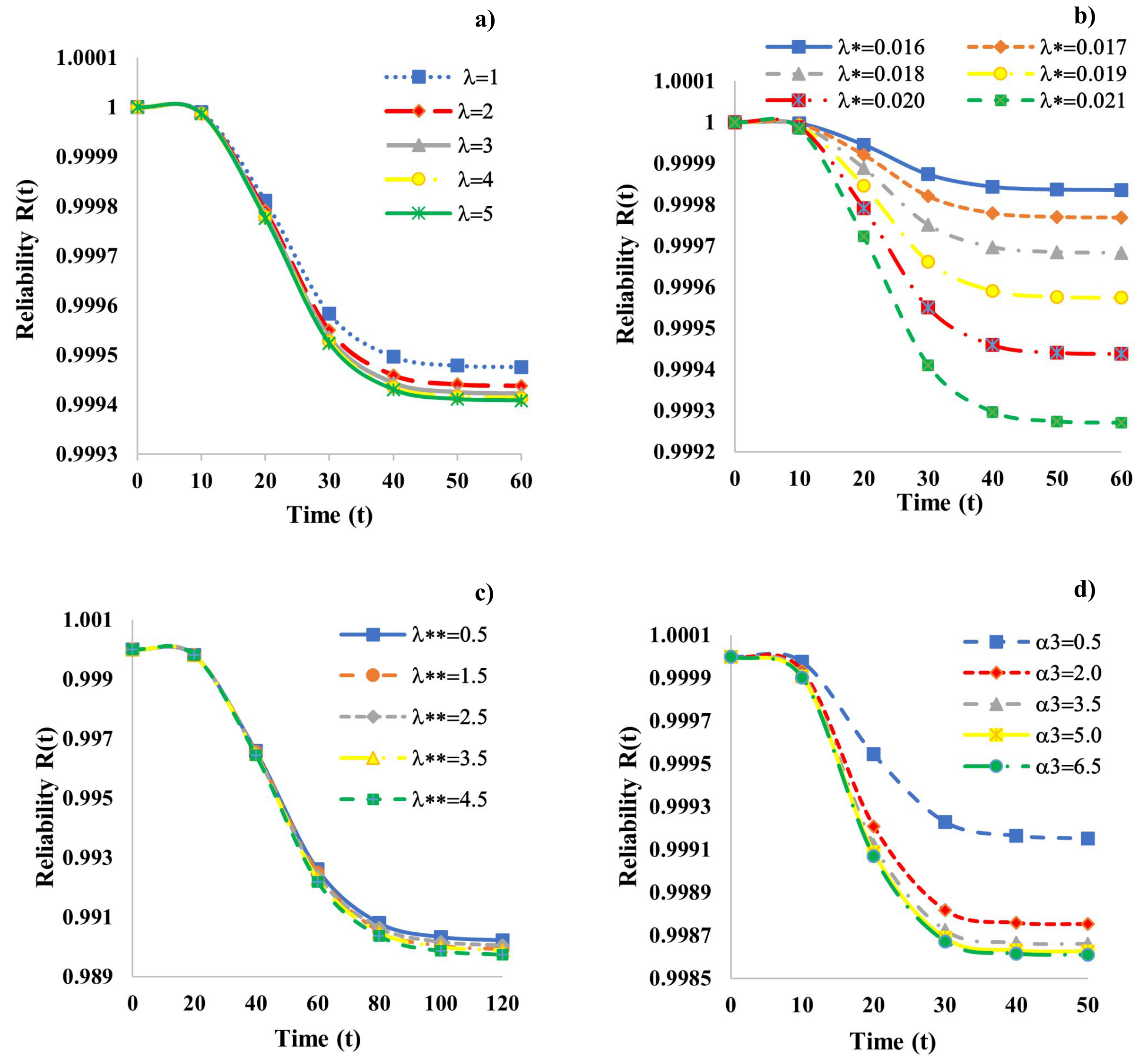

Figure 3. Reliability vs. time by varying: a) λ, b) λ_d, c) λ_c, d) α_3, $\lambda_d = \lambda^*$ in Figure 3b and $\lambda_c = \lambda^{**}$ in 3c)

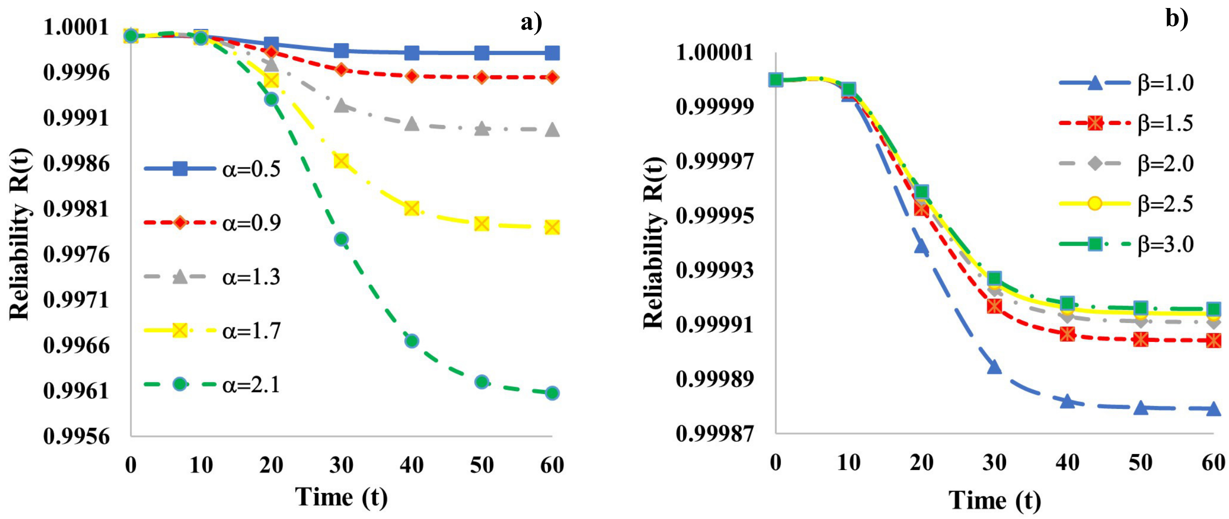

Figure 4. Reliability vs. time by varying a) α, and b) β

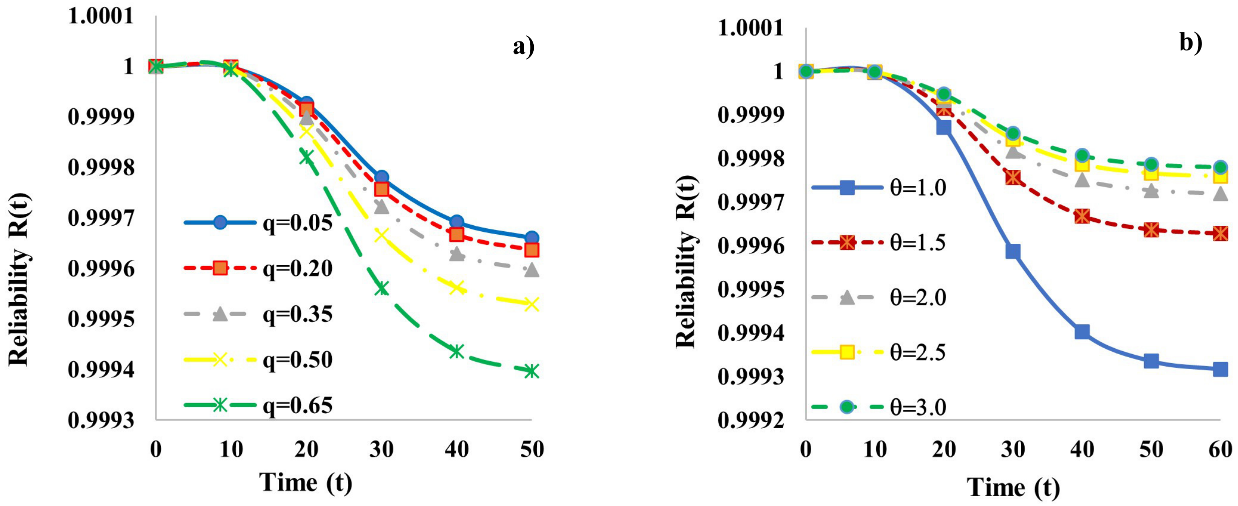

Figure 5. Reliability vs. time by varying a) q, and b) θ

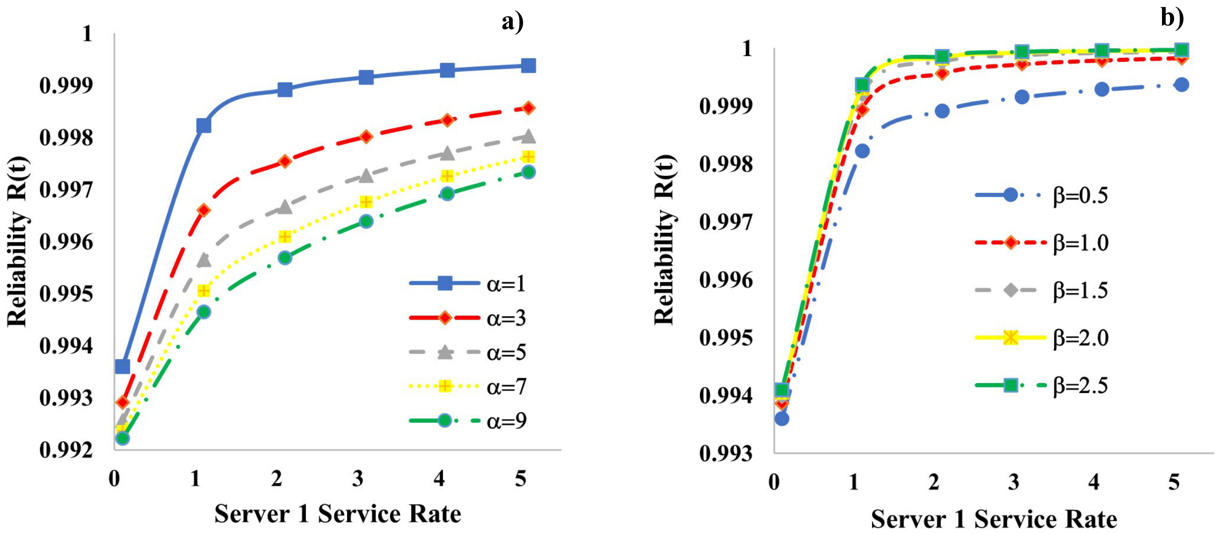

Figure 6. Reliability vs. server 1 service rate (μ_1) by varying a) α, and b) β

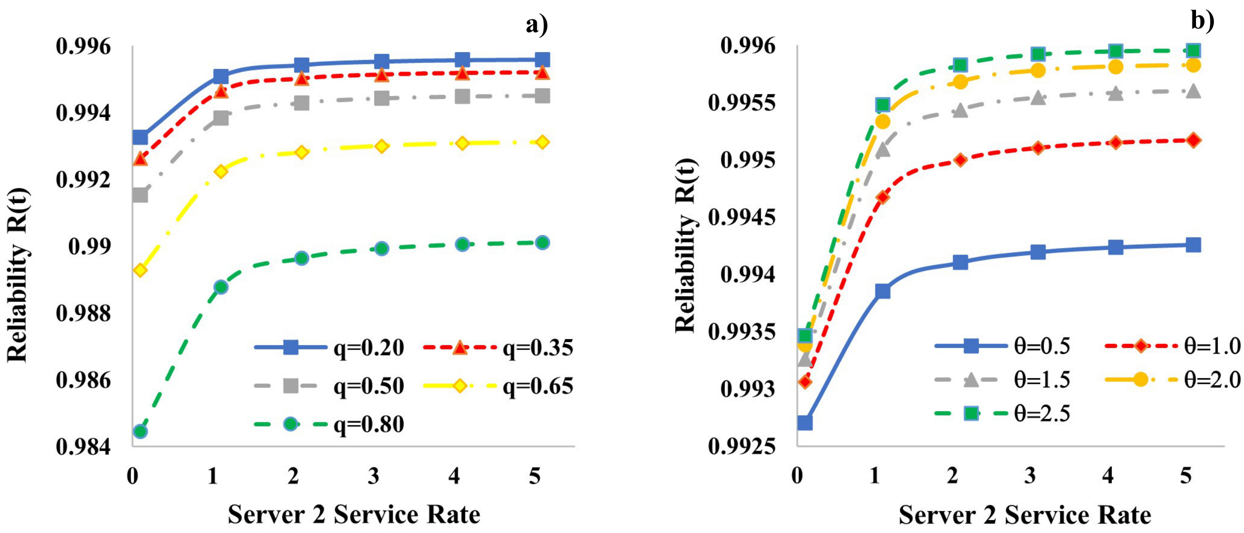

Figure 7. Reliability vs. server 2 service rate (μ_2) by varying a) q, and b) θ

Table 2. Effect of performance measures by varying time (t) and λ

Failure rate	Time (t)	E_f	E_s	A	P_b^1	P_b^2	P_{bd}	Wf
$\lambda = 0.1$	0.2	0.283353	3.618602	0.983332	0.133047	0.002732	0.017449	0.011332
	0.4	1.06207	3.663522	0.937525	0.321703	0.025434	0.060579	0.016034
	0.6	2.238555	3.691118	0.86832	0.514154	0.073727	0.119086	0.016502
	0.8	3.691069	3.710473	0.782878	0.707611	0.141741	0.18572	0.01656
$\lambda = 1.1$	0.2	0.538987	1.909377	0.968295	0.161907	0.007408	0.017855	0.012512
	0.4	1.932503	1.913327	0.886323	0.36624	0.04713	0.062156	0.017087
	0.6	3.696819	1.916656	0.78254	0.568887	0.113879	0.120869	0.017524
	0.8	5.524305	1.919488	0.675041	0.765288	0.196034	0.185968	0.017576

Table 3. Effect of performance measures by varying time (t) and q

Failure probability	Time (t)	E_f	E_s	A	P_b^1	P_b^2	P_{bd}	Wf
$q = 0.2$	0.2	0.283353	3.618602	0.983332	0.133047	0.002732	0.017449	0.011332
	0.4	1.06207	3.663522	0.937525	0.321703	0.025434	0.060579	0.016034
	0.6	2.238555	3.691118	0.86832	0.514154	0.073727	0.119086	0.016502
	0.8	3.691069	3.710473	0.782878	0.707611	0.141741	0.18572	0.01656
$q = 0.3$	0.2	0.290891	3.606405	0.982889	0.133978	0.002881	0.017555	0.01159
	0.4	1.091312	3.650212	0.935805	0.324953	0.026212	0.061168	0.016354
	0.6	2.296536	3.677072	0.86491	0.520042	0.075485	0.120402	0.016827
	0.8	3.778541	3.695876	0.777733	0.71599	0.144605	0.187856	0.016886

Table 4. Effect of performance measures by varying time (t) and β

Server 1 repair rate	Time (t)	E_f	E_s	A	P_b^1	P_b^2	P_{bd}	Wf
$\beta = 0.5$	0.2	0.283353	3.618602	0.983332	0.097053	0.001626	0.017449	0.011332
	0.4	1.06207	3.663522	0.937525	0.31608	0.012565	0.060579	0.016034
	0.6	2.238555	3.691118	0.86832	0.594464	0.039332	0.119086	0.016502
	0.8	3.691069	3.710473	0.782878	0.901392	0.084229	0.18572	0.01656
$\beta = 1.0$	0.2	0.283352	3.633343	0.983332	0.097053	0.00163	0.016915	0.003615
	0.4	1.061911	3.681542	0.937535	0.316079	0.012707	0.057097	0.005418
	0.6	2.237732	3.711994	0.868369	0.594452	0.040262	0.10928	0.005642
	0.8	3.688793	3.73385	0.783012	0.901327	0.08745	0.166141	0.00567

Table 5. Effect of performance measures by varying time (t) and μ_1

Service rate	Time (t)	E_f	E_s	A	P_b^1	P_b^2	P_{bd}	Wf
$\mu_1 = 1$	0.2	0.283353	3.618602	0.983332	0.133047	0.002732	0.017449	0.011332
	0.4	1.06207	3.663522	0.937525	0.321703	0.025434	0.060579	0.016034
	0.6	2.238555	3.691118	0.86832	0.514154	0.073727	0.119086	0.016502
	0.8	3.691069	3.710473	0.782878	0.707611	0.141741	0.18572	0.01656
$\mu_1 = 2$	0.2	0.270257	3.821925	0.984103	0.129058	0.002531	0.016859	0.00464
	0.4	0.98822	3.904142	0.941869	0.308692	0.02309	0.05739	0.005466
	0.6	2.064689	3.960702	0.878548	0.492363	0.066888	0.112589	0.005474
	0.8	3.399417	4.003103	0.800034	0.678294	0.129148	0.176211	0.005474

E_f, P_b^1, P_{bd}, Wf decrease whereas P_b^2, A, E_s increase with increases in θ. In Table 7, E_f, P_b^1, P_{bd}, Wf increase whereas P_b^2, A, E_s decrease with increasing α.

6.2. Numerical simulation of MTTF system

The MTTF is the key reliability metric for many fault-tolerant control systems which needs to be evaluated. We performed a numerical experiment to observe the effect of parameters on the MTTF system.

For the computational results of the MTTF, we used the following default parameters $\lambda = 0.02$, $\lambda_d = 0.01$, $\lambda_c = 0.1$, $\alpha_1 = 0.01$, $\alpha_2 = 0.02$, $\alpha_3 = 0.03$, $q = 0.01$, $p = 0.5$, $\overline{p} = 0.5$, $\mu_1 = 0.02$, $\mu_2 = 0.1$, $\mu_2' = 1.0$, $\theta = 0.8$, $\alpha = 0.01$, $\beta = 1.5$. The results are summarized in Figures 8–11 and Tables 8–9. Figures 8–10 illustrate the effect of the MTTF system with respect to standby failure rate α_1, α_2 and α_3, respectively. We see that MTTF decreases regularly with the increase in α_1, α_2, and α_3. As we increase λ, or λ_c, the MTTF decreases (Figures 8a, 9a and 10a and Figures 8b, 9b, and 10b, respectively).

Figure 11 represents the pattern of the MTTF system by varying repair parameter μ_2. An increasing trend in MTTF is seen for increasing values of μ_2. As we increase λ or λ_c, the MTTF decreases (Figures 11a, 11b, respectively).

In Table 8, we summarize the numerical results for MTTF by varying λ, μ_1, α, α_1, α_3, whereas in Table 9, we summarize the numerical results for MTTF by varying λ_c, μ_1, α, and α_1, α_3. From these tables, we can emphasize that the MTTF changes with respect to parameters very significantly.

7. Conclusion and managerial implications

In the present paper, we studied a fault-tolerant control system, when the system and its units are subject to failures due to various reasons. However, the server may show unreliable behaviour at some point in time. We have constructed a Markovian model based on the birth-death process to determine reliability, MTTF, and other operational characteristics. The system governing steady-state probabilities equations was solved by using the matrix approach based on the Cramer rule. We performed computational and sensitivity analyses of several performance measures for different parameter variability. The numerical illustration clearly shows that the performance of the FTC system is truly different from that FTC system without spare provision and standby switching failure. The sensitivity analysis reveals that reliability and MTTF can be improved by controlling suitable parameters. The following conclusions may be drawn from the results obtained.

- The system reliability of fault-tolerant systems is highly dependent on a number of factors including operational and standby unit failure and service rates, breakdown and service rates of servers, standby switching failure rate, and vacation rate.
- When the server vacation rate increases, the probability of server breakdown and switching failure decreases, which is a natural phenomenon in real-time systems.
- We can improve system reliability by preventing the failure of working and standby units through proper maintenance and by providing better service.
- As the service rate μ_1 and β increase, a decreasing pattern can be seen for some measures such as the expected number of failed units, probabilities of servers busy and probability of servers in

Table 6. Effect of performance measures by varying time (t) and θ

Vacation rate	Time (t)	E_f	E_s	A	P_b^1	P_b^2	P_{bd}	Wf
$\theta = 1.5$	0.2	0.283353	3.618602	0.983332	0.133047	0.002732	0.017449	0.011332
	0.4	1.06207	3.663522	0.937525	0.321703	0.025434	0.060579	0.016034
	0.6	2.238555	3.691118	0.86832	0.514154	0.073727	0.119086	0.016502
	0.8	3.691069	3.710473	0.782878	0.707611	0.141741	0.18572	0.01656
$\theta = 2.5$	0.2	0.283348	3.621254	0.983332	0.133046	0.004384	0.017382	0.007155
	0.4	1.061994	3.667488	0.93753	0.321683	0.03917	0.059334	0.009993
	0.6	2.237892	3.696546	0.868359	0.513916	0.109211	0.113926	0.010313
	0.8	3.688377	3.717364	0.783037	0.706776	0.202946	0.173452	0.010354

Table 7. Effect of performance measures by varying time (t) and α

Server 1 breakdown rate	Time (t)	E_f	E_s	A	P_b^1	P_b^2	P_{bd}	Wf
$\alpha = 1.0$	0.2	0.283353	3.618602	0.983332	0.097053	0.001626	0.017449	0.011332
	0.4	1.06207	3.663522	0.937525	0.31608	0.012565	0.060579	0.016034
	0.6	2.238555	3.691118	0.86832	0.594464	0.039332	0.119086	0.016502
	0.8	3.691069	3.710473	0.782878	0.901392	0.084229	0.18572	0.01656
$\alpha = 1.5$	0.2	0.28339	3.560406	0.983326	0.097053	0.001549	0.025363	0.019495
	0.4	1.062301	3.599192	0.937512	0.316088	0.011477	0.085748	0.028847
	0.6	2.239495	3.622426	0.868265	0.594532	0.034694	0.165026	0.029675
	0.8	3.692013	3.638559	0.782823	0.901658	0.072285	0.253174	0.029775

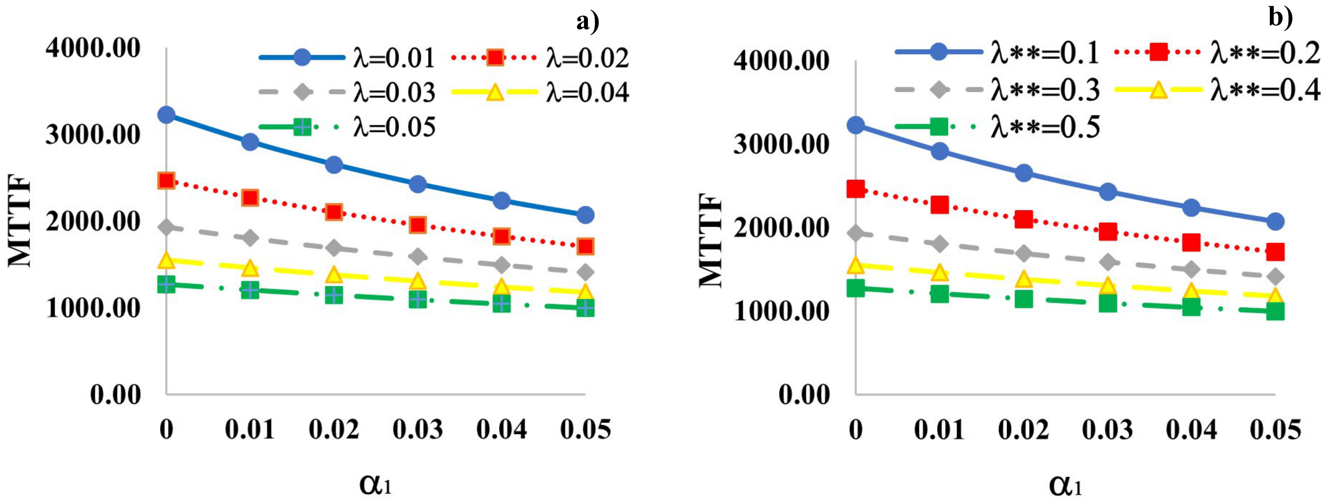

Figure 8. MTTF vs. α_1 by varying a) λ, and b) λ_c

Table 8. Effect of α, μ_1, α_1 and α_3 on MTTF system

α	$\lambda = 0.02$	$\lambda = 0.03$	$\lambda = 0.04$	μ_1	$\lambda = 0.02$	$\lambda = 0.03$	$\lambda = 0.04$
0.01	2269.18	1804.34	1462.15	0	44.70	32.14	24.16
0.02	2257.84	1795.63	1455.36	0.002	52.97	38.08	28.62
0.03	2246.60	1787.00	1448.62	0.004	63.88	45.93	34.53
0.04	2235.47	1778.45	1441.95	0.006	78.73	56.64	42.60
0.05	2224.46	1769.98	1435.33	0.008	99.69	71.81	54.05
0.06	2213.55	1761.60	1428.78	0.01	130.75	94.38	71.16
α_1	$\lambda = 0.02$	$\lambda = 0.03$	$\lambda = 0.04$	α_2	$\lambda = 0.02$	$\lambda = 0.03$	$\lambda = 0.04$
0	2463.85	1933.51	1551.83	0	2911.47	2268.25	1803.64
0.01	2269.18	1804.34	1462.15	0.01	2765.35	2163.28	1726.98
0.02	2099.90	1689.66	1381.31	0.02	2628.68	2064.87	1654.81
0.03	1951.53	1587.26	1308.11	0.03	2500.84	1972.53	1586.81
0.04	1820.56	1495.33	1241.53	0.04	2381.20	1885.83	1522.69
0.05	1704.24	1412.40	1180.77	0.05	2269.18	1804.34	1462.15

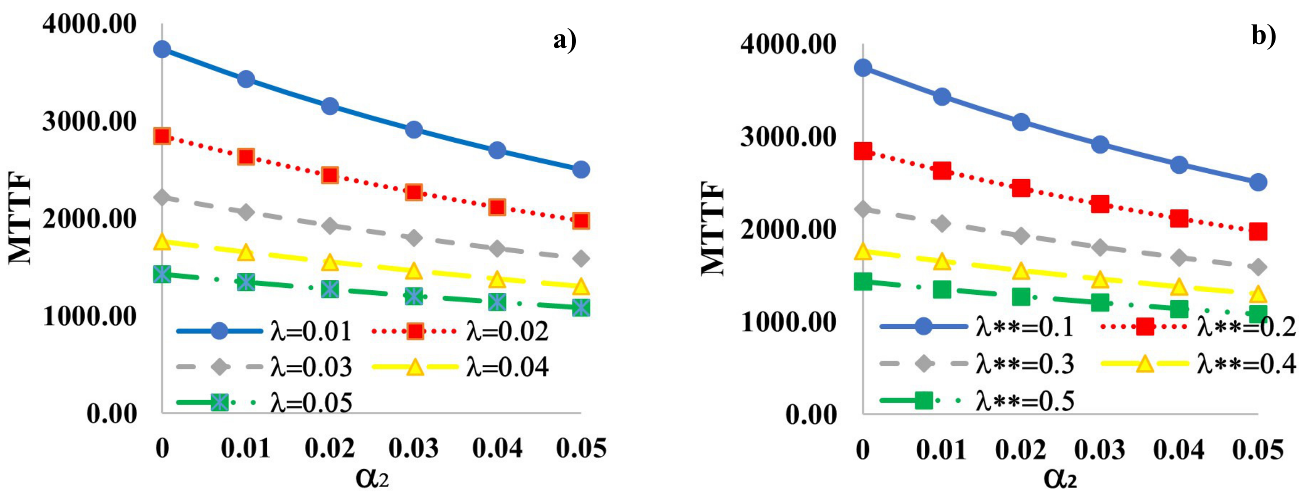

Figure 9. MTTF vs. α_2 by varying a) λ, and b) λ_c

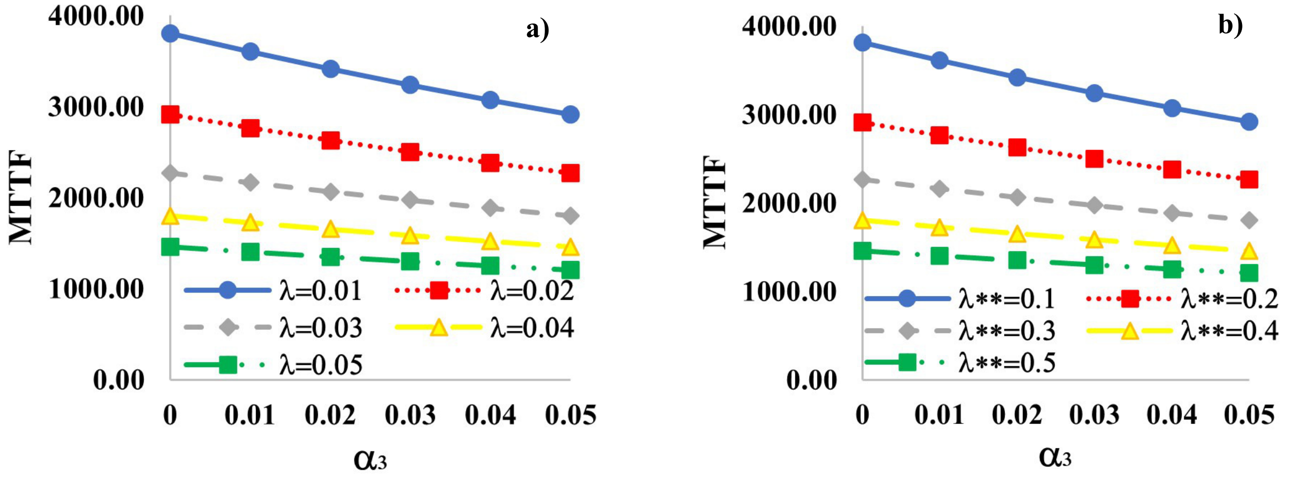

Figure 10. MTTF vs. α_3 by varying a) λ, and b) λ_c

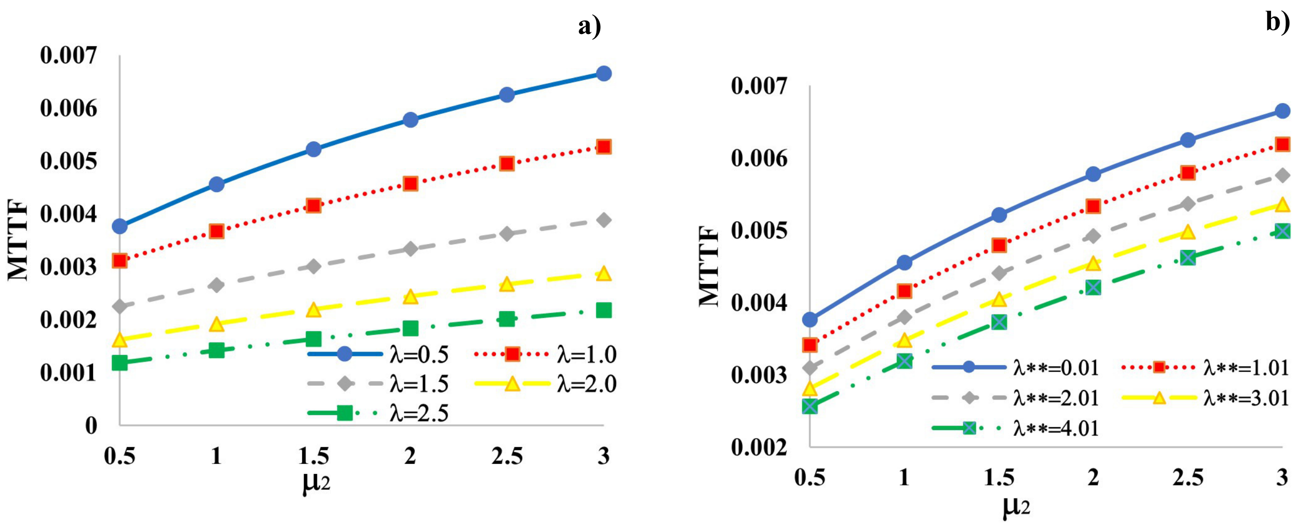

Figure 11. MTTF vs. μ_2 by varying a) λ, and b) λ_c

a breakdown state which reflects that availability of active servers in the system increases system efficiency and the system will work longer.

- With a rise in the failure rates of operating (standby) units and switching failure probability, the probabilities of servers being busy, servers in a breakdown state, and failure frequency are found to be raised, which matches with real-world experience.
- Even uncertain changes in some system parameters, such as failure and service rates of operational units, degraded failure rate, service rate of server in breakdown state, and actual service rate also have a significant impact on the MTTF system. The MTTF, on the other hand, is found to be highly sensitive to the unit failure rate and common cause failure rate.

The reliability indicator of FTC systems with warm standbys, server breakdown, and standbys switching failure has many practical uses in electronic industries, power plants, aircraft safety systems, security systems, etc. The current study may give useful direction to production engineers and system designers for improving the reliability of the FTC system. The research may be extended with consideration of the reboot process and working vacation. However, this situation is more realistic and will increase the complexity of the proposed work.

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Appendix

Since we have

$$D(s)=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

For brevity, we use

$$\Phi_n = M\lambda q^n,\ \ 1\le n\le S^{(k)}$$

$$\Psi_n = M\lambda q^n(1-q),\ \ 1\le n\le S^{(k)}-1$$

Here,

$$M_{11}=\begin{bmatrix} A_0 & A_1 & A_2 & A_3 & 0 \\ B_0 & B_1 & B_2 & 0 & B_4 \\ 0 & C_1 & C_2 & 0 & 0 \\ D_0 & 0 & 0 & D_3 & D_4 \\ 0 & E_1 & 0 & 0 & E_4 \end{bmatrix} \quad M_{12}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & B_6 & 0 & 0 & 0 \\ C_5 & 0 & C_7 & 0 & 0 \\ D_5 & 0 & 0 & 0 & 0 \\ E_5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{21}=\begin{bmatrix} 0 & 0 & F_2 & 0 & 0 \\ G_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ I_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad M_{22}=\begin{bmatrix} F_5 & 0 & 0 & 0 & 0 \\ 0 & G_6 & G_7 & G_8 & 0 \\ 0 & H_6 & H_7 & 0 & H_9 \\ 0 & I_6 & 0 & I_8 & I_9 \\ 0 & J_6 & J_7 & 0 & J_9 \end{bmatrix}$$

The submatrices of $D(s)$ are taken as:

$$A_0=\begin{bmatrix} (\lambda_0+M\lambda q+s+a) & -\lambda_0 & -\Psi_1 & \cdots & \cdots & -\Psi_{N-3} & -\Psi_{N-2} \\ -\mu_1 & (\lambda_1+\mu_1+M\lambda q+s+a) & -\lambda_1 & \cdots & \cdots & -\Psi_{N-4} & -\Psi_{N-3} \\ 0 & -\mu_1 & (\lambda_2+\mu_1+M\lambda q+s+a) & \cdots & \cdots & -\Psi_{N-5} & -\Psi_{N-4} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & (\lambda_{N-2}+\mu_1+M\lambda q+s+a) & -\lambda_{N-2} \\ 0 & 0 & \cdots & \cdots & \cdots & -\mu_1 & (\lambda_{N-1}+\mu_1+M\lambda q+s+a) \end{bmatrix}_{N\times N}$$

$$D_3=\begin{bmatrix} (\lambda_0+M\lambda q+s+\beta) & -\lambda_0 & -\Psi_1 & \cdots & \cdots & -\Psi_{N-3} & -\Psi_{N-2} \\ 0 & (\lambda_1+\mu_1+M\lambda q+s+\beta) & -\lambda_1 & \cdots & \cdots & -\Psi_{N-4} & -\Psi_{N-3} \\ 0 & 0 & (\lambda_2+\mu_1+M\lambda q+s+\beta) & \cdots & \cdots & -\Psi_{N-5} & -\Psi_{N-4} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & (\lambda_{N-2}+\mu_1+M\lambda q+s+\beta) & -\lambda_{N-2} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & (\lambda_{N-1}+\mu_1+M\lambda q+s+\beta) \end{bmatrix}_{N\times N}$$

$$A_2=\begin{bmatrix} -\Phi_{S(K)} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -\Phi_{S(K)-1} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -\Phi_{S(K)-2} & 0 & 0 & \ddots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots \\ -\Phi_{S(K)-N+2} & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ -\Phi_{S(K)-N+1} & 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}_{N\times(L-S^{(K)})} \quad D_5=\begin{bmatrix} -\Phi_{S(K)} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -\Phi_{S(K)-1} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -\Phi_{S(K)-2} & 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ -\Phi_{S(K)-N+2} & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ -\Phi_{S(K)-N+1} & 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}_{N\times(L-S^{(K)})}$$

$$A_1 = \begin{bmatrix} -\Psi_{N-1} & -\Psi_N & \cdots & \cdots & -\Psi_{S^{(K)}-2} & -\Psi_{S^{(K)}-1} \\ -\Psi_{N-2} & -\Psi_{N-1} & \ddots & \ldots & -\Psi_{S^{(K)}-3} & -\Psi_{S^{(K)}-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda_{N-1} & -\Psi_1 & \cdots & \cdots & -\Psi_{S^{(K)}-N+1} & -\Psi_{S^{(K)}-N} \end{bmatrix}_{N\times(S^{(K)}-N+1)}$$

$$D_4 = \begin{bmatrix} -\Psi_{N-1} & -\Psi_N & \cdots & \cdots & -\Psi_{S^{(K)}-2} & -\Psi_{S^{(K)}-1} \\ -\Psi_{N-2} & -\Psi_{N-1} & \ddots & \ldots & -\Psi_{S^{(K)}-3} & -\Psi_{S^{(K)}-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda_{N-1} & -\Psi_1 & \cdots & \cdots & -\Psi_{S^{(K)}-N+1} & -\Psi_{S^{(K)}-N} \end{bmatrix}_{N\times(S^{(K)}-N+1)}$$

$$B_1 = \begin{bmatrix} (\lambda_N+\lambda+\mu_1+a+s+M\lambda q) & -\lambda_N & \Psi_1 & -\Psi_2 & \ldots & -\Psi_{S^{(K)}-N-2} & -\Psi_{S^{(K)}-N-1} \\ -\mu_1 & (\lambda_{N+1}+\lambda+\mu_1+a+s+M\lambda q) & -\lambda_{N+1} & -\Psi_1 & \ldots & -\Psi_{S^{(K)}-N-3} & -\Psi_{S^{(K)}-N-2} \\ 0 & -\mu_1 & (\lambda_{N+2}+\lambda+\mu_1+a+s+M\lambda q) & -\lambda_{N+2} & \ldots & -\Psi_{S^{(K)}-N-4} & -\Psi_{S^{(K)}-N-3} \\ \vdots & \vdots & -\mu_1 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \ddots & \ddots & -\lambda_{S^{(K)}-2} & -\Psi_1 \\ 0 & 0 & \ldots & 0 & \ddots & (\lambda_{S^{(K)}-1}+\lambda+\mu_1+a+s+M\lambda q) & -\lambda_{S^{(K)}-1} \\ 0 & 0 & \ldots & 0 & 0 & -\mu_1 & (\lambda_{S^{(K)}}+\lambda+\mu_1+a+s) \end{bmatrix}_{(S^{(K)}-N+1)\times(S^{(K)}-N+1)}$$

$$E_4 = \begin{bmatrix} (\lambda_N+\beta+s+M\lambda q) & -\lambda_N & -\Psi_1 & -\Psi_2 & \ldots & -\Psi_{S^{(K)}-N-2} & -\Psi_{S^{(K)}-N-1} \\ 0 & (\lambda_{N+1}+\beta+s+M\lambda q) & -\lambda_{N+1} & -\Psi_1 & \ldots & -\Psi_{S^{(K)}-N-3} & -\Psi_{S^{(K)}-N-2} \\ 0 & 0 & (\lambda_{N+2}+\beta+s+M\lambda q) & -\lambda_{N+2} & \ldots & -\Psi_{S^{(K)}-N-4} & -\Psi_{S^{(K)}-N-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -\lambda_{S^{(K)}-2} & -\Psi_1 \\ 0 & 0 & 0 & 0 & \ldots & (\lambda_{S^{(K)}-1}+\beta+s+M\lambda q) & -\lambda_{S^{(K)}-1} \\ 0 & 0 & 0 & 0 & \ldots & 0 & (\lambda_{S^{(K)}}+\beta+s) \end{bmatrix}_{(S^{(K)}-N+1)\times(S^{(K)}-N+1)}$$

$$G_6 = \begin{bmatrix} (\lambda_N+\mu_1+\mu_2+s+M\lambda q) & -\lambda_N & -\Psi_1 & -\Psi_2 & \ldots & --\Psi_{S^{(K)}-N-2} & -\Psi_{S^{(K)}-N-1} \\ -(\mu_1+p\mu_2) & (\lambda_{N+1}+\mu_1+\mu_2+s+M\lambda q) & -\lambda_{N+1} & -\Psi_1 & \ldots & -\Psi_{S^{(K)}-N-3} & -\Psi_{S^{(K)}-N-2} \\ 0 & -(\mu_1+p\mu_2) & (\lambda_{N+2}+\mu_1+\mu_2+s+M\lambda q) & -\lambda_{N+2} & \ldots & -\Psi_{S^{(K)}-N-4} & -\Psi_{S^{(K)}-N-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \ddots & \ddots & -\lambda_{S^{(K)}-2} & -\Psi_1 \\ 0 & 0 & \ldots & 0 & \ddots & (\lambda_{S^{(K)}-1}+\mu_1+\mu_2+s+M\lambda q) & -\lambda_{S^{(K)}-1} \\ 0 & 0 & \ldots & 0 & 0 & -(\mu_1+p\mu_2) & (\lambda_{S^{(K)}}+\mu_1+\mu_2+s) \end{bmatrix}_{(S^{(K)}-N+1)\times(S^{(K)}-N+1)}$$

$$I_8 = \begin{bmatrix} (\lambda_N+\mu_2'+s+M\lambda q) & -\lambda_N & -\Psi_1 & -\Psi_2 & \ldots & -\Psi_{S^{(K)}-N-2} & -\Psi_{S^{(K)}-N-1} \\ 0 & (\lambda_{N+1}+\mu_2'+s+M\lambda q) & -\lambda_{N+1} & -\Psi_1 & \ldots & -\Psi_{S^{(K)}-N-3} & -\Psi_{S^{(K)}-N-2} \\ 0 & 0 & (\lambda_{N+2}+\mu_2'+s+M\lambda q) & -\lambda_{N+2} & \ldots & -\Psi_{S^{(K)}-N-4} & -\Psi_{S^{(K)}-N-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \ddots & \ddots & -\lambda_{S^{(K)}-2} & -\Psi_1 \\ 0 & 0 & \ldots & 0 & \ddots & (\lambda_{S^{(K)}-1}+\mu_2'+s+M\lambda q) & -\lambda_{S^{(K)}-1} \\ 0 & 0 & \ldots & 0 & 0 & 0 & (\lambda_{S^{(K)}}+s+\mu_2') \end{bmatrix}_{(S^{(K)}-N+1)\times(S^{(K)}-N+1)}$$

$$E_5 = \begin{bmatrix} -\Phi_{S^{(K)}-N} & 0 & 0 & \dots & \dots & 0 & 0 \\ -\Phi_{S^{(K)}-N-1} & 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -\Phi_2 & 0 & \dots & \dots & \dots & 0 & 0 \\ -\Phi_1 & 0 & \dots & \dots & \dots & 0 & 0 \\ -\lambda_{S^{(K)}} & 0 & \dots & \dots & \dots & 0 & 0 \end{bmatrix}_{(S^{(K)}-N+1)\times(L-S^{(K)})}$$

$$I_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\mu_2' & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\mu_2' & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & -\mu_2' & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & -\mu_2' & 0 \end{bmatrix}_{(S^{(K)}-N+1)\times(S^{(K)}-N+1)}$$

$$G_7 = \begin{bmatrix} -\Phi_{S^{(K)}-N} & 0 & 0 & \dots & \dots & 0 & 0 \\ -\Phi_{S^{(K)}-N-1} & 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -\Phi_2 & 0 & \vdots & \vdots & \vdots & 0 & 0 \\ -\Phi_1 & 0 & \vdots & \vdots & \vdots & 0 & 0 \\ -\lambda_{S^{(K)}} & 0 & \vdots & \vdots & \vdots & 0 & 0 \end{bmatrix}_{(S^{(K)}-N+1)\times(L-S^{(K)})}$$

$$I_9 = \begin{bmatrix} -\Phi_{S^{(K)}-N} & 0 & 0 & \vdots & \vdots & 0 & 0 \\ -\Phi_{S^{(K)}-N-1} & 0 & 0 & \vdots & \vdots & 0 & 0 \\ \vdots & 0 & 0 & \ddots & \vdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -\Phi_2 & 0 & \vdots & \vdots & \vdots & 0 & 0 \\ -\Phi_1 & 0 & \vdots & \vdots & \vdots & 0 & 0 \\ -\lambda_{S^{(K)}} & 0 & \vdots & \vdots & \vdots & 0 & 0 \end{bmatrix}_{(S^{(K)}-N+1)\times(L-S^{(K)})}$$

$$F_5 = \begin{bmatrix} (\lambda_{S^{(K)}+1}+s+\beta) & -\lambda_{S^{(K)}+1} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & (\lambda_{S^{(K)}+2}+s+\beta) & -\lambda_{S^{(K)}+2} & \cdots & \cdots & 0 & 0 \\ 0 & 0 & (\lambda_{S^{(K)}+3}+s+\beta) & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & (\lambda_{L-1}+s+\beta) & -\lambda_{L-1} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}_{(L-S^{(K)})\times(L-S^{(K)})}$$

$$J_9 = \begin{bmatrix} (\lambda_{S^{(K)}+1}+s+\mu_2') & -\lambda_{S^{(K)}+1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & (\lambda_{S^{(K)}+2}+s+\mu_2') & -\lambda_{S^{(K)}+2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & (\lambda_{L-1}+s+\mu_2') & -\lambda_{L-1} \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \end{bmatrix}_{(L-S^{(K)})\times(L-S^{(K)})}$$

$$C_2 = \begin{bmatrix} (\lambda_{S^{(K)}+1}+\theta+a+s+\mu_1) & -\lambda_{S^{(K)}+1} & 0 & \cdots & \cdots & 0 & 0 \\ -\mu_1 & (\lambda_{S^{(K)}+2}+\theta+a+s+\mu_1) & -\lambda_{S^{(K)}+2} & \cdots & \cdots & 0 & 0 \\ 0 & -\mu_1 & (\lambda_{S^{(K)}+3}+\theta+a+s+\mu_1) & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & -\lambda_{L-2} & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & (\lambda_{L-1}+\theta+a+s+\mu_1) & -\lambda_{L-1} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}_{(L-S^{(K)})\times(L-S^{(K)})}$$

$$H_7 = \begin{bmatrix} (\lambda_{S^{(K)}+1}+\mu_1+\mu_2+s) & -\lambda_{S^{(K)}+1} & 0 & \cdots & \cdots & 0 & 0 \\ -(\mu_1+p\mu_2) & (\lambda_{S^{(K)}+2}+\mu_1+\mu_2+s) & -\lambda_{S^{(K)}+2} & \cdots & \cdots & 0 & 0 \\ 0 & -(\mu_1+p\mu_2) & (\lambda_{S^{(K)}+3}+\mu_1+\mu_2+s) & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & (\lambda_{L-1}+\mu_1+\mu_2+s) & -\lambda_{L-1} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}_{(L-S^{(K)})\times(L-S^{(K)})}$$

$$B_2 = \begin{bmatrix} -\Phi_{S^{(K)}-N} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -\Phi_{S^{(K)}-N-1} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -\Phi 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ -\Phi_1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ -\lambda_{S^{(K)}} & 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}_{(S^{(K)}-N+1)\times(L-S^{(K)})}$$

$$J_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\mu_2' & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\mu_2' & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & -\mu_2' & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \end{bmatrix}_{(L-S^{(K)})\times(L-S^{(K)})}$$

$$B_6 = \operatorname{diag}\big(-\theta, -\theta, -\theta, \ldots, -\theta\big)_{(S^{(K)}-N+1)\times(S^{(K)}-N+1)}$$

$$C_7 = \operatorname{diag}\big(-\theta, -\theta, -\theta, \ldots, -\theta,\ 0\big)_{(L-S^{(K)})\times(L-S^{(K)})}$$

$$A_3 = \operatorname{diag}[-\alpha, -\alpha, -\alpha, \ldots, -\alpha]_{N\times N}$$

$$B_4 = \operatorname{diag}\big(-\alpha, -\alpha, -\alpha, \ldots, -\alpha\big)_{(S^{(K)}-N+1)\times(S^{(K)}-N+1)}$$

$$C_5 = \operatorname{diag}\big(-\alpha, -\alpha, -\alpha, \ldots, -\alpha,\ 0\big)_{(L-S^{(K)})\times(L-S^{(K)})}$$

$$D_0 = \operatorname{diag}\big(-\beta, -\beta, -\beta, \ldots, -\beta\big)_{N\times N}$$

$$F_2 = \operatorname{diag}\big(-\beta, -\beta, -\beta, \ldots, -\beta,\ 0\big)_{(L-S^{(K)})\times(L-S^{(K)})}$$

$$E_1 = \operatorname{diag}\big(-\beta, -\beta, -\beta, \ldots, -\beta\big)_{(S^{(K)}-N+1)\times(S^{(K)}-N+1)}$$

$$G_8 = \operatorname{diag}\big(-\overline{p}\mu_2, -\overline{p}\mu_2, -\overline{p}\mu_2, \ldots, -\overline{p}\mu_2\big)_{(S^{(K)}-N+1)\times(S^{(K)}-N+1)}$$

$$H_9 = \operatorname{diag}\big(-\overline{p}\mu_2, -\overline{p}\mu_2, -\overline{p}\mu_2, \ldots, -\overline{p}\mu_2,\ 0\big)_{(L-S^{(K)})\times(L-S^{(K)})}$$

$$B_0 = \big(b_{ij}\big)_{(S^{(K)}-N+1)\times N} = \begin{cases} -\mu_1,\ i=1, j=N \\ 0,\ \text{otherwise} \end{cases}$$

$$C_1 = \big(c_{ij}\big)_{(L-S^{(K)})\times(S^{(K)}-N+1)} = \begin{cases} -\mu_1, \quad i=1, j=S^{(k)}-N+1 \\ 0,\ \text{otherwise} \end{cases}$$

$$G_0 = \left(g_{ij}\right)_{\left(S^{(K)}-N+1\right)\times N} = \begin{cases} -(\mu_1{+}p\mu_2), \ i=1, j=N-1 \\ 0, \ \text{otherwise} \end{cases}$$

$$H_6 = \left(h_{ij}\right)_{\left(L-S^{(K)}\right)\times\left(S^{(K)}-N+1\right)} = \begin{cases} -(\mu_1{+}p\mu_2), \quad i=1, j=S^{(k)}-N+1 \\ 0, \ \text{otherwise} \end{cases}$$

$$I_0 = \left(I_{ij}\right)_{\left(S^{(K)}-N+1\right)\times N} = \begin{cases} -\mu_2', \ i=1, j=N-1 \\ 0, \ \text{otherwise} \end{cases}$$

$$J_6 = \left(J_{ij}\right)_{\left(L-S^{(K)}\right)\times\left(S^{(K)}-N+1\right)} = \begin{cases} -\mu_2', \qquad\qquad i=1, j=S^{(k)}-N+1 \\ 0, \ \text{otherwise} \end{cases}$$

Table 9. Effect of α, μ_1, α_2 and α_3 on MTTF system

α	$\lambda_c = 0.1$	$\lambda_c = 0.2$	$\lambda_c = 0.3$	μ_1	$\lambda_c = 0.1$	$\lambda_c = 0.2$	$\lambda_c = 0.3$
0.01	191.98	130.75	94.38	0	65.99	44.70	32.18
0.02	190.89	130.05	93.90	0.005	104.37	70.71	50.92
0.03	189.81	129.35	93.42	0.01	191.98	130.75	94.51
0.04	188.75	128.66	92.95	0.015	477.12	331.75	243.20
0.05	187.69	127.98	92.48	0.02	2912.96	2269.18	1804.02
0.06	186.65	127.31	92.02	0.025	10788.40	4836.64	2754.29
α_1	$\lambda_c = 0.1$	$\lambda_c = 0.2$	$\lambda_c = 0.3$	α_3	$\lambda_c = 0.1$	$\lambda_c = 0.2$	$\lambda_c = 0.3$
0	3226.44	2463.85	1933.16	0	3811.38	2911.47	2267.21
0.01	2915.58	2269.18	1804.02	0.01	3608.93	2765.35	2162.43
0.02	2653.05	2099.90	1689.37	0.02	3418.44	2628.68	2064.19
0.03	2428.83	1951.53	1587.00	0.03	3239.69	2500.84	1971.99
0.04	2235.47	1820.56	1495.10	0.04	3072.25	2381.20	1885.41
0.05	2067.28	1704.24	1412.19	0.05	2915.58	2269.18	1804.02