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Characterisation of some generalised continuous distributions by doubly truncated moments

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Abstract

The characterisation of probability distribution plays an important role in statistical studies. There are various methods of characterisation available in the literature. The characterisation using truncated moments limits the observations; hence, researchers may save time and cost. In this paper, the characterisation of three general forms of continuous distributions based on doubly truncated moments has been studied. The results are given simply and explicitly. Further, the results have been applied to some well-known continuous distributions.

Keywords: *truncated moments, characterisation, probability distribution, Weibull, power function, Fréchet , Pareto, Lindley*

1. Introduction and motivation

The characterisation of probability distributions is an incredibly important aspect of distribution theory. The characterisation is a certain distributional or statistical property of a statistic or statistics that uniquely determines the associated stochastic model. Thus, the characterisation of probability models plays a vital role in statistical studies, in the fields of natural and allied sciences.

A probability distribution can be characterised through various methods, for instance, the characterisation of distributions by truncated moments was studied by Laurent [30], Glänzel et al. [17], Glänzel [15, 16], Ahsanullah et al. [1, 5], Kilany [28] and others. Gupta and Gupta [18], Huang and Su [21] characterised the distributions by moments of residual life whereas Hamedani [20] characterised various distributions based on infinite divisibility. The characterisation through conditional expectation is investigated by several authors, for example, see Khan and Abu-Salih [25], Balasubramanian and Beg [10], Franco and Ruiz [12, 13], Balasubramanian and Dey [11], Su and Huang [38], Khan and Abouammoh [24], Khan and Alzaid [27], Gupta and Ahsanullah [19], Noor and Athar [36], Ahsanullah et al. [3]. For

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a detailed review on the characterisation of distributions using other statistical properties, one may refer to Galambos and Kotz [14], Kotz and Shanbhag [29], Khan and Ali [26], Nagaraja [35], Wesolowski and Ahsanullah [39], López-Blázquez and Moreno-Rebello [31], Ali [6], Ali and Khan [7], Athar et al. [9], Ahsanullah and Hamedani [2], Muhammad and Yahaya [34], Javed et al. [22], Ahsanullah and Shakil [4] and references therein.

Recently, the characterisation of the class of probability distributions via truncated moments has developed significant interest from several authors. For instance, Muhammad and Liu [33] characterised the Marshall-Olkin-G family of distributions by truncated moments whereas Athar and Abdel-Aty [8] characterised a class of continuous distributions based on left and right truncated moments. Further, Shakil and Ahsanullah [37] characterised the Gaussian distributions. In contrast, Metiri et al. [32] studied the characterisation of X-Lindley distribution using the relation between the truncated moment and the failure rate/reverse failure rate function.

There are not many papers available on the characterisation of the general class of probability distributions by doubly truncated moments. The motive of this paper is to develop unified characterisation results for some general classes of distributions when truncation is from both sides, that is left and right. The obtained results can then be applied to some well-known continuous distributions. Thus, the earlier results obtained for different distributions can be unified here.

The second section of this paper presents the main results and is expository in nature. Subsections present and explore the particular cases of the main results. A demonstration of the application of the characterisation results is presented in Section 3. We discuss the characterisation of some well-known continuous distributions based on the main results in Section 2. The whole study is then concluded in Section 4.

2. Characterisation theorems

First, we shall present two propositions, which are to be used for establishing the main results.

Proposition 1. Let $X \in (\alpha, \beta)$ be a continuous random variable with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$, such that $F(\alpha) = 0$ and $F(\beta) = 1$. Further, suppose that $f'(x)$ and $E(T^k(X)|x \le X \le y)$ exist, where $T^k(X)$ is a continuous and differentiable function and $k = 1, 2, \ldots$ If

$$E\big(T^k(X)|x \le X \le y\big) = g(x, y)\eta(x, y) \tag{1}$$

where $g(x, y)$ is differentiable function of $x, y \in (\alpha, \beta)$ and $\eta(x, y) = \dfrac{f(y)}{F(y) - F(x)}$, then

$$f(y) = K_1 \exp\left(\int \frac{T^k(y) - \frac{\partial}{\partial y} g(x, y)}{g(x, y)} dy \right) \tag{2}$$

The constant K_1 can be determined using the relation

$$\int_{\alpha}^{\beta} f(y)dy = 1 \tag{3}$$

Proof. We know that

$$E\big(T^k(X)|x \le X \le y\big) = \frac{1}{F(y) - F(x)} \int_{\alpha}^{y} T^k(u)f(u)du$$

Therefore,

$$\frac{1}{F(y) - F(x)} \int_{\alpha}^{y} T^k(u)f(u)du = \frac{g(x,\, y)f(y)}{F(y) - F(x)}$$

which implies

$$\int_{\alpha}^{y} T^k(u)f(u)du = g(x,\, y)f(y) \tag{4}$$

Differentiating both the sides of (4) w.r.t. y, then we have

$$T^k(y)f(y) = g(x,\, y)f'(y) + f(y)\frac{\partial}{\partial y}g(x,\, y)$$

which in simplification gives

$$\frac{f'(y)}{f(y)} = \frac{T^k(y) - \frac{\partial}{\partial y}g(x,\, y)}{g(x,\, y)} \tag{5}$$

Now expression (2) can be obtained by integrating both the sides of (5) w.r.t. y.
This completes the proof. □

Proposition 2. Under the conditions as stated in Proposition 1. If

$$E\big(T^k(X)|x \le X \le y\big) = g(x,\, y)\nu(x,\, y) \tag{6}$$

where $\nu(x,\, y) = \dfrac{f(x)}{F(y) - F(x)}$, then

$$f(x) = K_2 \, \exp\left(- \int \frac{T^k(x) + \frac{\partial}{\partial x}g(x,\, y)}{g(x,\, y)} dy \right) \tag{7}$$

The constant K_2 can be determined using the relation

$$\int_{\alpha}^{\beta} f(x)dx = 1$$

Proof. In view of (6), we have

$$\int_{x}^{\beta} T^k(u) f(u) du = g(x, y) f(x) \tag{8}$$

Differentiating both sides of (8) w.r.t. x, we get

$$\frac{f'(x)}{f(x)} = -\frac{T^k(x) + \frac{\partial}{\partial x} g(x, y)}{g(x, y)} \tag{9}$$

Now integrate (9) w.r.t. x to get (7).

Thus, the Proposition is established. □

Theorem 1. Suppose an absolutely continuous (w.r.t. Lebesgue measure) random variable X has cdf $F(x)$ and pdf $f(x)$ with $F(\alpha) = 0$ and $F(\beta) = 1$. Further, if $f'(x)$ and $E(T^k(X)|x \leq X \leq y)$ exist for every x and y, $\alpha < x < y < \beta$, then for a continuous and twice differentiable function $h(.)$

$$E\big(T^k(X)|x \leq X \leq y\big) = g(x, y)\eta(x, y) \tag{10}$$

where

$$\eta(x, y) = \frac{f(y)}{F(y) - F(x)}$$

and

$$g(x, y) = \frac{1}{ah'(y)} \Big(T^k(x)\, \mathrm{e}^{-a\{h(x)-h(y)\}} - T^k(y) + k \int_{x}^{y} T^{k-1}(u) T'(u)\, \mathrm{e}^{-a\{h(u)-h(y)\}} du \Big) \tag{11}$$

if and only if

$$F(y) = \mathrm{e}^{-a\, h(y)}, \ y \in (\alpha, \beta);\ a \neq 0. \tag{12}$$

Proof. First, we shall prove that (12) implies (10).

In view of (4), we have

$$g(x, y) = \frac{\int_{x}^{y} T^k(u) F'(u) du}{f(y)}$$

Integrating above expression by parts and note that $F'(y) = f(y) = -a\, \mathrm{e}^{-a\, h(y)} h'(y)$, we get

$$g(x, y) = \frac{T^k(y)\, \mathrm{e}^{-ah(y)} - T^k(x)\, \mathrm{e}^{-ah(x)} - k \int_{x}^{y} T^{k-1}(u) T'(u)\, \mathrm{e}^{-ah(u)} du}{-a\, \mathrm{e}^{-a\, h(y)} h'(y)}$$

which after simplification yields (10).

To prove (10) implies (12), we have

$$g(x, y) = \frac{T^k(x)\, \mathrm{e}^{-a\{h(x)-h(y)\}}}{ah'(y)} - \frac{T^k(y)}{ah'(y)} + \frac{k}{ah'(y)} \int_{x}^{y} T^{k-1}(u) T'(u)\, \mathrm{e}^{-a\{h(u)-h(y)\}} du$$

Let

$$g(x,\,y)=A(x,\,y)-B(y)+C(x,\,y) \tag{13}$$

where

$$A(x,\,y)=\frac{T^k(x)\,\mathrm{e}^{-a\{h(x)-h(y)\}}}{ah'(y)}$$

implies

$$\frac{\partial}{\partial\,y}A(x,\,y)=T^k(x)\,\mathrm{e}^{-a\{h(x)-h(y)\}}\left(1-\frac{h''(y)}{a\,(h'(y))^2}\right)$$

$$B(y)=\frac{T^k(y)}{ah'(y)}$$

implies

$$\frac{\partial}{\partial\,y}B(y)=\frac{k\,T^{k-1}(y)T'(y)}{a\,h'(y)}-\frac{T^k(y)h''(y)}{a(h'(y))^2}$$

and

$$C(x,\,y)=\frac{k}{ah'(y)}\int\limits_x^y T^{k-1}(u)T'(u)\,\mathrm{e}^{-a(h(u)-h(y))}du$$

implies

$$\frac{\partial}{\partial\,y}C(x,\,y)=\frac{k\,T^{k-1}(y)T'(y)}{ah'(y)}+\left(1-\frac{h''(y)}{a(h'(y))^2}\right)\left(ah'(y)g(x,\,y)+T^k(y)\right)$$

$$-\left(1-\frac{h''(y)}{a(h'(y))^2}\right)T^k(x)\,\mathrm{e}^{-a\{h(x)-h(y)\}}$$

Therefore, from (13), we have

$$\frac{\partial}{\partial y}\,g(x,\,y)=\frac{\partial}{\partial y}\,A(x,\,y)-\frac{\partial}{\partial y}B(y)+\frac{\partial}{\partial y}\,C(x,\,y)$$

$$=T^k(y)+g(x,\,y)\left(ah'(y)-\frac{h''(y)}{h'(y)}\right)$$

Now in view of (5), we have

$$\frac{f'(y)}{f(y)}=\frac{T^k(y)\;-\;\dfrac{\partial}{\partial y}\,g(x,\,y)}{g(x,\,y)}=\frac{h''(y)}{h'(y)}-ah'(y) \tag{14}$$

Now, integrating both the sides of (14) w.r.t. y,

$$\int\frac{f'(y)}{f(y)}dy=\int\frac{h''(y)}{h'(y)}dy-a\int\;h'(y)dy$$

which implies

$$\log f(y)=\log K_3+\log h'(y)+\log\,\mathrm{e}^{-ah(y)}$$

or

$$f(y) = K_3\, h'(y)\, \mathrm{e}^{-ah(y)}$$

where K_3 is the constant of integration. Now, $\int\limits_{\alpha}^{\beta} f(y)dy = 1$ gives $K_3 = -a$.

Therefore,

$$f(y) = -a\, h'(y)\, \mathrm{e}^{-ah(y)}$$

This gives

$$F(y) = \mathrm{e}^{-ah(y)},\ \ a \neq 0$$

□

Corollary 1. Let Y be a continuous r.v. with cdf $F(y)$ and pdf $f(y)$, $\alpha < y < \beta$. If $E(T^k(Y)|Y \leq y)$ exists, then

$$E\big(T^k(Y)|Y \leq y\big) = g(y)\phi(y) \tag{15}$$

where $\phi(y) = \frac{f(y)}{F(y}$ and

$$g(y) = \frac{-T^k(y) + k\int\limits_{\alpha}^{y} T^{k-1}(u)T'(u)\, \mathrm{e}^{-a\{h(u)-h(y)\}}du}{a\, h'(y)} \tag{16}$$

if and only if

$$F(y) = \mathrm{e}^{-ah(y)},\ \ a \neq 0 \tag{17}$$

Proof. The corollary can be proved by letting $x \to \alpha$ in (10). □

Theorem 2. Under the conditions as stated in Theorem 1,

$$E\big(T^k(X)|x \leq X \leq y\big) = g(x,\, y)\nu(x,\, y) \tag{18}$$

where

$$\nu(x,\, y) = \frac{f(x)}{F(y) - F(x)}$$

and

$$g(x,\, y) = \frac{1}{ah'(x)}\Big(T^k(x) - T^k(y)\, \mathrm{e}^{-a\{h(y)-h(x)\}} + k\int\limits_{x}^{y} T^{k-1}(u)T'(u)\, \mathrm{e}^{-a\{h(u)-h(x)\}}du\Big) \tag{19}$$

if and only if

$$F(x) = 1 - \mathrm{e}^{-a\,h(x)},\ \ x \in (\alpha, \beta);\ a \neq 0 \tag{20}$$

Proof. In view of (8), we have

$$g(x,\, y) = \frac{\int\limits_{x}^{y} T^k(u)F'(u)du}{f(x)}$$

Thus, the necessary part can be proved by integrating the above expression by parts and noting that $F'(x) = f(x) = a\,\mathrm{e}^{-a\,h(x)}h'(x)$.

To prove the sufficiency part, let

$$g(x,\, y) = L(x) - M(x,\, y) + N(x,\, y)$$

where

$$L(x) = \frac{T^k(x)}{ah'(x)}$$

implies

$$\frac{\partial}{\partial x}L(x) = \frac{k\,T^{k-1}(x)T'(x)}{ah'(x)} - \frac{T^k(x)\,h''(x)}{a\,(h'(x))^2}$$

$$M(x,\, y) = \frac{T^k(y)\,\mathrm{e}^{-a[h(y)-h(x)]}}{ah'(x)}$$

implies

$$\frac{\partial}{\partial x}M(x,\, y) = T^k(y)\,\mathrm{e}^{-a[h(y)-h(x)]}\Big(1 - \frac{h''(x)}{a(h'(x))^2}\Big)$$

and

$$N(x,\, y) = \frac{k}{a}\,\frac{\int\limits_x^y T^{k-1}(u)\,T'(u)\,\mathrm{e}^{-a[h(u)-h(x)]}du}{h'(x)}$$

implies

$$\frac{\partial}{\partial x}N(x,\, y) = -\frac{k\,T^{k-1}(x)T'(x)}{ah'(x)} + \Big(1 - \frac{h''(x)}{a(h'(x))^2}\Big)\big(ah'(x)g(x,\, y) - T^k(x)\big)$$

$$+\Big(1 - \frac{h''(x)}{a(h'(x))^2}\Big)\,T^k(y)\,\mathrm{e}^{-a[h(y)-h(x)]}$$

therefore,

$$\frac{\partial}{\partial x}g(x,\, y) = -T^k(x) + g(x,\, y)\Big(ah'(x) - \frac{h''(x)}{h'(x)}\Big)$$

Now using (9), we get

$$\frac{f'(x)}{f(x)} = -\frac{T^k(x) + \frac{\partial}{\partial x}g(x,\, y)}{g(x,\, y)} = -ah'(x) + \frac{h''(x)}{h'(x)} \tag{21}$$

which gives

$$F(x) = 1 - \mathrm{e}^{-a\,h(x)},\ x \in (\alpha,\, \beta),\, a > 0$$

□

Corollary 2. Let X be a continuous r.v. with cdf $F(x)$ and pdf $f(x)$, $\alpha < x < \beta$. If $E(T^k(X)|X \geq x)$ exists, then for $k = 1, 2, \ldots$

$$E\big(T^k(X)|X \geq x\big) = g(x)\tau(x) \tag{22}$$

where $\tau(x) = \frac{f(x)}{1-F(x)}$ and

$$g(x) = \frac{T^k(x) + k\int\limits_x^\beta T^{k-1}(u)T'(u)\,\mathrm{e}^{-a\{h(u)-h(x)\}}du}{a\,h'(x)} \tag{23}$$

if and only if

$$F(x) = 1 - \mathrm{e}^{-ah(x)},\ a \neq 0 \tag{24}$$

Proof. The corollary can be proved by letting $y \to \beta$ in (18). □

Theorem 3. Let $X : \Omega \to (\alpha, \beta)$, where α and β may be finite or infinite, be a continuous random variable with absolutely continuous cdf $F(x)$ and pdf $f(x)$, such that $F(\alpha) = 0$ and $F(\beta) = 1$ and $f'(x)$ exist. Further, suppose that $h(x)$ be an increasing and twice differentiable function of x such that $h(x) \to 1$ as $x \to \alpha^+$ and $h(x) \to \infty$ as $x \to \beta^-$. If $E\big(T^k(X)|x \leq X \leq y\big)$ exists, then

$$E\big(T^k(X)|x \leq X \leq y\big) = g(x, y)\nu(x, y) \tag{25}$$

where

$$\nu(x, y) = \frac{f(x)}{F(y) - F(x)}$$

and

$$g(x, y) = \frac{h(x)}{\lambda\,h'(x)}\left(T^k(x) - T^k(y)\left(\frac{h(y)}{h(x)}\right)^{-\lambda} + k\int\limits_x^y T^{k-1}(u)T'(u)\left(\frac{h(u)}{h(x)}\right)^{-\lambda}du\right) \tag{26}$$

if and only if

$$F(x) = 1 - \big(h(x)\big)^{-\lambda},\ \lambda > 0 \tag{27}$$

Proof. Let X have cdf as given in (27). Thus, in view of (8), the expression for $g(x, y)$ as given in (26) can be obtained. Hence, the necessary part.

To prove the sufficiency part, let

$$g(x, y) = \frac{1}{\lambda}\big(S(x) - U(x, y) + V(x, y)\big)$$

where

$$S(x) = \frac{T^k(x)\,h(x)}{h'(x)}$$

implies

$$\frac{\partial}{\partial x}S(x) = T^k(x) + k\,T^{k-1}(x)T'(x)\,\frac{h(x)}{h'(x)} - \frac{T^k(x)\,h(x)\,h''(x)}{(h'(x))^2}$$

$$U(x, y) = T^k(y)\,\big(h(y)\big)^{-\lambda}\,\frac{\big(h(x)\big)^{\lambda+1}}{h'(x)}$$

implies

$$\frac{\partial}{\partial x}U(x,\,y) = (\lambda+1)T^k(y)\big(h(x)\big)^{\lambda}\big(h(y)\big)^{-\lambda} - T^k(y)\frac{\big(h(x)\big)^{\lambda+1}\big(h(y)\big)^{-\lambda}h''(x)}{\big(h'(x)\big)^2},$$

and

$$V(x,\,y) = \frac{k\,\big(h(x)\big)^{\lambda+1}}{h'(x)}\int\limits_x^y T^{k-1}(u)T'(u)\big(h(u)\big)^{-\lambda}du$$

implies

$$\frac{\partial}{\partial x}V(x,\,y) = -k\,T^{k-1}(x)T'(x)\,\frac{h(x)}{h'(x)}$$

$$+\left(\frac{\lambda\,h'(x)}{h(x)}g(x,\,y) + T^k(y)\Big(\frac{h(y)}{h(x)}\Big)^{-\lambda} - T^k(x)\right)\left((\lambda+1) - \frac{h(x)\,h''(x)}{\big(h'(x)\big)^2}\right).$$

Therefore,

$$\frac{\partial}{\partial x}\,g(x,\,y) = \frac{1}{\lambda}\left(\frac{\partial}{\partial x}\,S(x) - \frac{\partial}{\partial x}\,U(x,\,y) + \frac{\partial}{\partial x}\,V(x,\,y)\right)$$

$$= -T^k(x)\left((\lambda+1)\,\frac{h'(x)}{h(x)} - \frac{h''(x)}{h'(x)}\right)g(x,\,y)$$

Now, using (9), we get

$$\frac{f'(x)}{f(x)} = -\frac{T^k(x) + \frac{\partial}{\partial x}g(x,\,y)}{g(x,\,y)} = \frac{h''(x)}{h'(x)} - (\lambda+1)\,\frac{h'(x)}{h(x)} \tag{28}$$

which gives

$$F(x) = 1 - \big(h(x)\big)^{-\lambda},\ \ \lambda > 0$$

□

Corollary 3. Under the conditions as stated in Theorem 3,

$$E\big(T^k(X)|X \geq x\big) = g(x)\tau(x)$$

where

$$\tau(x) = \frac{f(x)}{1-F(x)}$$

and

$$g(x) = \frac{h(x)}{\lambda\,h'(x)}\left(T^k(x) + k\int\limits_x^\beta T^{k-1}(u)T'(u)\Big(\frac{h(u)}{h(x)}\Big)^{-\lambda}du\right)$$

Proof. The corollary can be proved by letting $y \to \beta$ in Theorem 3. □

3. Examples

Examples based on Theorem 1

In this subsection, characterisation of inverse Weibull, power function, and Fréchet distributions based on kth truncated moment of X are presented.

Inverse Weibull distribution

Corollary 4. Let X be a continuous random variable with cdf $F(x)$ and pdf $f(x)$. Further, suppose that $f'(x)$ and $E(X^k|x \leq X \leq y)$ exist for every $x, y \in (0, \infty)$. Then for $k = 1, 2, \ldots$

$$E[X^k|x \leq X \leq y] = g(x,\, y)\eta(x,\, y) \tag{29}$$

where

$$g(x,\, y) = \frac{y^{p+1}}{p\theta}\Big(y^k - x^k\,\mathrm{e}^{-\theta(x^{-p}-y^{-p})} - k\int\limits_x^y t^{k-1}\mathrm{e}^{-\theta(t^{-p}-y^{-p})}dt\Big)$$

and

$$\eta(x,\, y) = \frac{p\,\theta\,y^{-(p+1)}\mathrm{e}^{-\theta\,y^{-p}}}{\mathrm{e}^{-\theta\,y^{-p}} - \mathrm{e}^{-\theta x^{-p}}}$$

if and only if

$$F(y) = \mathrm{e}^{-\theta\,y^{-p}}, \theta, p > 0; y \in (0, \infty) \tag{30}$$

Proof. To prove necessary part, we compare (30) with (12) and get,

$$a = \theta \quad \text{and} \quad h(y) = y^{-p} \quad \Longrightarrow \quad h'(y) = -p\,y^{-(p+1)}, \quad \text{and} \quad h''(y) = p(p+1)\,y^{-(p+2)}$$

Thus, given (11)

$$g(x,\, y) = \frac{y^{p+1}}{p\,\theta}\Big(y^k - x^k\,\mathrm{e}^{-\theta(x^{-p}-y^{-p})} - k\int\limits_x^y t^{k-1}\mathrm{e}^{-\theta(t^{-p}-y^{-p})}dt\Big)$$

Hence the necessary part.

To prove sufficiency part, from (14) we have

$$\frac{f'(y)}{f(y)} = \frac{h''(y)}{h'(y)} - ah'(y)$$

This implies

$$f(y) = K_4\,y^{-(p+1)}\,\mathrm{e}^{-\theta\,y^{-p}}$$

where K_4 is a constant of integration. Now using the condition $\int\limits_0^\infty f(y)dy = 1$, we get

$$f(y) = \,p\,\theta\,y^{-(p+1)}\,\mathrm{e}^{-\theta\,y^{-p}}$$

Further,

$$\int_0^y f(t)dt = p\,\theta \int_0^y t^{-(p+1)}\,\mathrm{e}^{-\theta\,t^{-p}}dt$$

which gives

$$F(y) = \mathrm{e}^{-\theta\,y^{-p}}, p > 0, y \in (0, \infty)$$

□

Power function distribution

Corollary 5. Under the conditions as stated in Corollary 4 and for $x, y \in (0, \nu)$,

$$E[X^k | x \le X \le y] = g(x,\, y)\eta(x,\, y) \tag{31}$$

where

$$g(x,\, y) = \frac{y^{k+p} - x^{k+p}}{(k+p)y^{p-1}}$$

and

$$\eta(x,\, y) = \frac{p\,y^{p-1}}{y^p - x^p}$$

if and only if

$$F(y) = \nu^{-p}\,y^p,\ \nu, p > 0; 0 < y < \nu \tag{32}$$

Proof. First, we shall prove the necessary part.

On comparison of (32) with (12), we get $a = -p$ and , $h(y) = \log(y/\nu)$ implies $h'(y) = (1/y)$ and $h''(y) = (-1/y^2)$. Now in view of (11), we get (31). Hence the necessary part is true.

To prove sufficiency part, from (14) we have

$$\frac{f'(y)}{f(y)} = \frac{h''(y)}{h'(y)} - ah'(y) = \frac{p-1}{y}$$

This implies

$$f(y) = p\,\nu^{-p}\,y^{p-1}$$

and hence

$$F(y) = \nu^{-p}\,y^p,\ \ \nu, p > 0;\ 0 < y < \nu$$

□

Fréchet distribution

Corollary 6. For Fréchet distribution and conditions similar to Corollary 4 with $x, y \in (0, \infty)$

$$E[X^k | x \le X \le y] = g(x,\, y)\eta(x,\, y) \tag{33}$$

where

$$g(x,\, y) = \frac{y^{\lambda+1}}{\lambda\, \sigma^{\lambda}} \Big(y^k - x^k\, \mathrm{e}^{-\sigma^{\lambda}(x^{-\lambda}-y^{-\lambda})} - k \int\limits_x^y t^{k-1} \mathrm{e}^{-\sigma^{\lambda}(t^{-\lambda}-y^{-\lambda})} dt \Big)$$

and

$$\eta(x,\, y) = \frac{\lambda \sigma^{\lambda} y^{-(\lambda+1)} \mathrm{e}^{-\sigma^{\lambda} y^{-\lambda}}}{\mathrm{e}^{-\sigma^{\lambda} y^{-\lambda}} - \mathrm{e}^{-\sigma^{\lambda} x^{-\lambda}}}$$

if and only if

$$F(y) = \exp\Big(- \Big(\frac{\sigma}{y}\Big)^{\lambda} \Big), \; y > 0;\; \lambda,\, \sigma > 0 \tag{34}$$

Proof. Necessary part

By comparing (34) with (12), we get $a = 1$ and $h(y) = \left(\frac{\sigma}{y}\right)^{\lambda}$ implies $h'(y) = -\lambda\, \sigma^{\lambda}\, y^{-(\lambda+1)}$ and $h''(y) = \lambda(\lambda+1)\, \sigma^{\lambda}\, y^{-(\lambda+2)}$

Now on application of (11), we get (33).

To prove the sufficiency part, in view of (14), we have

$$\frac{f'(y)}{f(y)} = \frac{h''(y)}{h'(y)} - a h'(y) = -\frac{\lambda+1}{y} + \lambda\, \sigma^{\lambda}\, y^{-(\lambda+1)}$$

Integrating both the sides of above expression w.r.t. y, we get

$$\int \frac{f'(y)}{f(y)} dy = \lambda\, \sigma^{\lambda} \int y^{-(\lambda+1)} dy - (\lambda+1) \int \frac{1}{y} dy$$

or

$$f(y) = K_5\, y^{-(\lambda+1)}\, \mathrm{e}^{-\sigma^{\lambda}\, y^{-\lambda}}$$

where K_5 is a constant of integration. Now using the condition $\int\limits_0^\infty f(y) dy = 1$, we get

$$f(y) = \lambda\, \sigma^{\lambda}\, y^{-(\lambda+1)}\, \mathrm{e}^{-\sigma^{\lambda}\, y^{-\lambda}}$$

Further,

$$\int\limits_0^y f(t) dt = \lambda\, \sigma^{\lambda} \int\limits_0^y t^{-(\lambda+1)}\, \mathrm{e}^{-\sigma^{\lambda}\, t^{-\lambda}} dt$$

which gives

$$F(y) = \exp\Big(- \Big(\frac{\sigma}{y}\Big)^{\lambda}, \; y > 0;\; \lambda,\, \sigma > 0$$

□

Similarly, with proper choice of a and $h(y)$ several other distributions can be characterised using Theorem 1. For more distributions belonging to this class, one may refer to Khan and Abu-Salih [25], Ali and Khan [7].

Examples based on Theorem 2

In this subsection, Weibull, Pareto, Gumbel extreme value I, and Lindley distributions are characterised through kth truncated moment using Theorem 2 and considering $T(X) = X$.

Weibull distribution

Corollary 7. Suppose X be a continuous random variable and $f'(x)$ and $E(X^k|x \leq X \leq y)$, $x, y \in (0, \infty)$ exist. Then

$$E[X^k|x \leq X \leq y] = g(x,\, y)\nu(x,\, y), \tag{35}$$

where

$$g(x,\, y) = \frac{x^{1-p}}{p\,\theta}\left(x^k - y^k\, \mathrm{e}^{-\theta(y^p - x^p)} + k\int_x^y t^{k-1}\mathrm{e}^{-\theta(t^p - x^p)}dt\right) \text{ and } \nu(x,\, y) = \frac{p\,\theta x^{p-1}\,\mathrm{e}^{-\theta x^p}}{\mathrm{e}^{-\theta x^p} - \mathrm{e}^{-\theta\, y^p}}$$

if and only if

$$F(x) = 1 - \mathrm{e}^{-\theta x^p}, p > 0, x \in (0, \infty) \tag{36}$$

Proof. To prove necessary part, we compare (36) with (20) and get $a = \theta$ and $h(x) = x^p$ implies $h'(x) = p\,x^{p-1}$, $h''(x) = p(p-1)x^{p-2}$.

Thus, in view of (19)

$$g(x,\, y) = \frac{x^{1-p}}{p\theta}\left(x^k - y^k\, \mathrm{e}^{-\theta(y^p - x^p)} + k\int_x^y t^{k-1}\mathrm{e}^{-\theta(t^p - x^p)}dt\right)$$

Hence the necessary part is true.

To prove sufficiency part, from (21), we have

$$\frac{f'(x)}{f(x)} = -ah'(x) + \frac{h''(x)}{h'(x)} = -\theta\, p\, x^{p-1} + \frac{p-1}{x}$$

This implies

$$f(x) = C_1\, x^{p-1}\,\mathrm{e}^{-\theta x^p}$$

where C_1 is a constant of integration. Now using the condition $\int_0^\infty f(x)dx = 1$, we get

$$f(x) = \, p\,\theta x^{p-1}\,\mathrm{e}^{-\theta x^p}$$

Further,

$$\int_0^x f(t)dt = p\,\theta\int_0^x t^{p-1}\,\mathrm{e}^{-\theta\, t^p}dt$$

implies

$$F(x) = 1 - \mathrm{e}^{-\theta x^p}, p, \theta > 0, x \in (0, \infty)$$

□

Pareto distribution

Corollary 8. Under the conditions as stated in Corollary 7 and for $x, y \in (\alpha, \infty)$,

$$E[X^k|x \leq X \leq y] = g(x,\, y)\nu(x,\, y) \tag{37}$$

where

$$g(x,\, y) = \frac{1}{k-\theta}\left(\frac{y^{k-\theta} - x^{k-\theta}}{x^{-(\theta+1)}}\right)$$

and

$$\nu(x,\, y) = \frac{\theta x^{-(\theta+1)}}{x^{-\theta} - y^{-\theta}}$$

if and only if

$$F(x) = 1 - \alpha^{\theta}\, x^{-\theta}, \alpha < x < \infty;\; \alpha > 0,\, \theta > 1 \tag{38}$$

Proof. First, we shall prove the necessary part. On comparison of (38) with (20), we get

$$a = \theta \;\; \text{and} \;\; h(x) = \log\left(\frac{x}{\alpha}\right)$$

which implies

$$h'(x) = \frac{1}{x}, \quad h''(x) = -\frac{1}{x^2}$$

Now, in view of (19), we get the value $g(x,\, y)$, which leads to necessary part of the corollary.
To prove the sufficiency part, we use the relation (21)

$$\frac{f'(x)}{f(x)} = -ah'(x) + \frac{h''(x)}{h'(x)} = -\frac{\theta+1}{x}$$

This gives

$$f(x) = \theta\, \alpha^{\theta}\, x^{-(\theta+1)},\;\; \alpha < x < \infty;\; \theta,\, \alpha > 0$$

and hence

$$F(x) = 1 - \alpha^{\theta}\, x^{-\theta}$$

□

Gumbel extreme value I distribution

Corollary 9. For the Gumbel extreme value I distribution, suppose that $f'(x)$ and $E(X^k|x \leq X \leq y)$ exist for every $x, y \in (-\infty, \infty)$. Then for $k = 1, 2, \ldots$

$$E[X^k|x \leq X \leq y] = g(x,\, y)\nu(x,\, y) \tag{39}$$

where

$$g(x,\, y) = \frac{1}{e^x}\left(x^k - y^k\, e^{-(e^y - e^x)} + k\int_x^y t^{k-1}\, e^{-(e^t - e^x)}\, dt\right)$$

and

$$\nu(x,\, y) = \frac{\mathrm{e}^{x}\,\mathrm{e}^{-\mathrm{e}^{x}}}{\mathrm{e}^{-\mathrm{e}^{x}} - \mathrm{e}^{-\mathrm{e}^{y}}}$$

if and only if

$$F(x) = 1 - \mathrm{e}^{-\mathrm{e}^{x}},\ \ -\infty < x < \infty \tag{40}$$

Proof. To prove necessary part, we compare (40) with (20) and get $a = 1,\ \ h(x) = \mathrm{e}^{x}$. Thus, the value of $g(x,\, y)$ can be obtained in view of (19).

To prove the sufficiency part, in view of the relation (21), we have

$$\frac{f'(x)}{f(x)} = -ah'(x) + \frac{h''(x)}{h'(x)} = 1 - \mathrm{e}^{x}$$

This implies

$$f(x) = \mathrm{e}^{x}\,\mathrm{e}^{-\mathrm{e}^{x}},\ \ -\infty < x < \infty$$

and subsequently,

$$F(x) = 1 - \mathrm{e}^{-\mathrm{e}^{x}},\ \ -\infty < x < \infty$$

□

Lindley distribution

Corollary 10. Suppose that for one parameter Lindley distribution $f'(x)$ and $E(X^k|x \le X \le y)$, $x, y \in (0, \infty)$ exist. Then

$$\begin{aligned} E[X^k|x \le X \le y] = &\frac{1}{\Big((1+\theta+\theta x)\mathrm{e}^{-\theta x} - (1+\theta+\theta y)\mathrm{e}^{-\theta y}\Big\}} \\ &\times \Big((1+\theta+\theta x)x^k\mathrm{e}^{-\theta x} - (1+\theta+\theta y)y^k\mathrm{e}^{-\theta y} \\ &+ k\int_x^y t^{k-1}(1+\theta+\theta t)\mathrm{e}^{-\theta t}dt\Big) \end{aligned} \tag{41}$$

if and only if

$$F(x) = 1 - \left(\frac{1+\theta+\theta x}{1+\theta}\right)\mathrm{e}^{-\theta x},\ \ x > 0,\ \ \theta > 0 \tag{42}$$

Proof. The necessary part can be proved on the lines of Corollary 9 with $a = 1$ and $h(x) = \theta x - \ln(1+\theta+\theta x) + \ln(1+\theta)$.

To prove the sufficiency part, in view of the relation (21), we have

$$\frac{f'(x)}{f(x)} = -ah'(x) + \frac{h''(x)}{h'(x)} = \frac{1-\theta-\theta x}{x+1}$$

This implies

$$f(x) = \frac{\theta^2}{1+\theta}(1+x)\,\mathrm{e}^{-\theta x},\ \ x > 0,\ \ \theta > 0$$

and hence,

$$F(x) = 1 - \Big(\frac{1+\theta+\theta x}{1+\theta}\Big)\, \mathrm{e}^{-\theta x},\ \ x>0,\ \ \theta>0$$

□

Remark 1. If $y \to \infty$ and $k = 1$, then from (41) we get

$$E\big[X|X \geq x\big] = x + \frac{1}{(1+\theta+\theta x)\,\mathrm{e}^{-\theta x}} \int_x^\infty (1+\theta+\theta\, t)\mathrm{e}^{-\theta\, t} dt$$

$$= x + \frac{\Gamma[2, (1+\theta+\theta x)]}{\theta\,(1+\theta+\theta x)\,\mathrm{e}^{-(1+\theta+\theta x)}}$$

where $\Gamma(a, x) = \int_x^\infty t^{a-1}\,\mathrm{e}^{-t} dt$ is upper incomplete gamma function.

A similar result was also obtained by Kilany [28]. The characterisation results for XLindley and Xgamma distributions can also be obtained on similar lines of Corollary 10. Further, with proper choice of a and $h(x)$ several other distributions can be characterised using Theorem 2. For more distributions belonging to this class, one may refer to Khan and Abu-Salih [25], Ali and Khan [7].

Examples based on Theorem 3

In this subsection characterisation of Pareto and exponential distributions based on Theorem 3 are presented.

Pareto distribution

Corollary 11. Let a random variable $X : \Omega \to (1, \infty)$ have cdf $F(x)$ and pdf $f(x)$. Suppose that $f'(x)$ and $E(X^k|x \leq X \leq y)$ exist for every $x, y \in (1, \infty)$. Then

$$E[X^k|x \leq X \leq y] = g(x,\, y)\nu(x,\, y) \tag{43}$$

where

$$g(x,\, y) = \frac{1}{(k-\lambda)}\frac{y^{k-\lambda} - x^{k-\lambda}}{x^{-(\lambda+1)}}$$

and

$$\nu(x,\, y) = \frac{\lambda\, y^\lambda}{x\, y^\lambda - x^{\lambda+1}}$$

if and only if

$$F(x) = 1 - x^{-\lambda},\ \ x \in (1, \infty),\ \ \lambda > 0 \tag{44}$$

Proof. On comparison of (44) with (27), we get $h(x) = x$, which implies $h'(x) = 1$ and $h''(x) = 0$. Thus, the necessary part can be seen in view of (25). To prove sufficiency part, from (28), we have

$$\frac{f'(x)}{f(x)} = \frac{h''(x)}{h'(x)} - (\lambda+1)\frac{h'(x)}{h(x)} = -\frac{\lambda+1}{x}$$

This implies

$$F(x) = 1 - x^{-\lambda}, \ \lambda > 0; \ 1 < x < \infty$$

□

Exponential distribution

Corollary 12. Let a random variable $X \in (0, \infty)$ have pdf $f(x)$ and $f'(x)$, $E(X^k|x \le X \le y)$ exist for every $x, y \in (0, \infty)$. Then

$$E\big[X^k|x \le X \le y\big] = g(x,\, y)\nu(x,\, y) \tag{45}$$

where

$$g(x,\, y) = \frac{1}{\lambda}\Bigg(x^k - y^k\,\mathrm{e}^{-\lambda(y-x)} + k\int\limits_x^y t^{k-1}\,\mathrm{e}^{-\lambda(t-x)}\,dt\Bigg)$$

and

$$\nu(x,\, y) = \frac{\lambda\,\mathrm{e}^{-\lambda x}}{\mathrm{e}^{-\lambda x} - \mathrm{e}^{-\lambda y}}$$

if and only if

$$F(x) = 1 - \mathrm{e}^{-\lambda x}, \ x \in (0, \infty), \ \lambda > 0 \tag{46}$$

Proof. If we compare (46) with (27), we have $h(x) = \mathrm{e}^x$, which implies $h'(x) = \mathrm{e}^x$ and $h''(x) = \mathrm{e}^x$.

Now, the value of $g(x,\, y)$ can be obtained in view of (26) and hence the necessary part.

To prove sufficiency part, from (28), we have

$$\frac{f'(x)}{f(x)} = \frac{h''(x)}{h'(x)} - (\lambda + 1)\frac{h'(x)}{h(x)} = -\lambda$$

This implies

$$F(x) = 1 - \mathrm{e}^{-\lambda x}, \ \lambda > 0; \ 0 < x < \infty$$

□

Similarly, with the proper choice of $h(x)$ several other distributions can be characterised using Theorem 3. For more distributions belonging to this class, one may refer to Jin and Lee [23].

4. Conclusions

This study explores the importance of characterisations in recent years. We developed characterisation results for three general classes of continuous distributions using doubly truncated moments. Further, results are applied to well-known distributions like inverse Weibull, power function, Fréchet, Weibull, Pareto, the extreme value I, Lindley, and exponential. The results are obtained in a straightforward and precise manner. Our results are the unification of all earlier results obtained for particular distributions when the truncation point is considered either left or right. One may obtain more characterisation results for some other continuous distributions from our generalised results as consequences. The problem with product moments is still open to the readers.

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