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Analysis of a single server queue in a multi-phase random environment with working vacations and customers' impatience

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Abstract

In this paper, we analyze an $M/M/1$ queueing system under both single and multiple working vacation policies, multi-phase random environment, waiting server, balking and reneging. When the system is in operative phase $j = 1, 2, \dots, K$, customers are served one by one. Whenever the system becomes empty, the server waits a random amount of time before taking a vacation, causing the system to move to working vacation phase 0 at which new arrivals are served at a lower rate. Using the probability generating function method, we obtain the distribution for the steady-state probabilities of the system. Then, we derive important performance measures of the queueing system. Finally, some numerical examples are illustrated to show the impact of system parameters on performance measures of the queueing system.

Keywords: queueing models, multi-phase random environment, working vacation policies, impatient customers, probability generating function

1. Introduction

Queueing theory has received considerable attention due to its importance in a variety of applications, including service systems, telecommunication networks, production and manufacturing systems, airlines, healthcare, etc. For a notable research work in this area, we refer the reader, for instance, to [6, 19, 24, 25, 26, 27, 33, 42]. In recent past, queueing systems operating in a random environment have been widely studied, because of their significant applications in complex modern communication networks. Yechiali and Naor [46] and Yechiali [45] analyzed an infinite-buffer single server Markovian queueing model in a 2-phase random environment, where arrival and service rates depend on the environmental phase. Their investigation are considered to be the pioneer works on queueing systems in a random environment. Neuts [36] generalized their study to an $M/G/1$ queue. Then, in Neuts [37], the author presented an

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analysis of an infinite-buffer multiserver Markovian queueing model in a random environment, using the matrix analytic approach. In addition, the $M/G/1$ queueing model in a two-phase random environment was carried out by Boxma and Kurkova [18], and Huang and Lee [28]. Cordeiro and Kharoufeh [21] considered an unreliable $M/M/1$ retrial queueing system whose arrival, service, failure, repair, and retrial rates are all subject to random environment. An infinite-capacity single-server Markovian queue in random environment with disasters was investigated in [40]. Then Jiang et al. [30] extended their work to the $M/G/1$ queueing model. Kim and Kim [32] dealt with a single server queue with Markov modulated service rates and impatient customers. Later, Li and Liu [34] studied a discrete-time $Geo/G/1$ queueing model with vacations in random environment. Recently, Jiang and Liu [29] have discussed an $GI/M/1$ queueing model in a multi-phase service environment with disasters and working breakdowns. After that, a performance study of an $M/M/1$ queueing system with vacations operating in a multi-phase random environment was presented by Li and Liu [35]. Yu and Liu [47] established the analysis of queues in a random environment with customers' impatience. For the infinite-server queue in a random environment, excellent surveys were given by O'Cinneide and Purdue [38], Baykal-Gursoy and Xiao [7], and D'Auria [22].

In previous decades, queueing models with customers' impatience, namely balking (on arrival, a customer decides whether or not to enter the queue depending on the queue length) and reneging (a customer may quit the system without getting a service, because of the long waiting time), have been the focus of diverse research studies due to their large applicability in different real-world situations, including network service, healthcare, and production and manufacturing system (see, e.g., [1, 2, 8, 9, 11, 20, 23, 48]). Further, vacation/working vacation queues with customers' impatience have been extensively studied, because of their versatility and applicability. Vacation (V) policy represents the case where the server is unavailable to serve the new arrivals during this period ([3, 4, 10, 12, 13, 31]). While in working vacation (WV) period, the server serves new arrivals with a slow service rate ([14, 16, 17, 43, 44]). In addition, vacation queues with waiting server and impatient customers reflect real-world situations, especially when dealing with human behavior. In this context, once the system gets empty, the server waits a random period of time before going on vacation or working vacation period ([5], [15], and [41]).

Inspired by these applications and works by Li and Liu [35] and Yu and Liu [47], we develop in this paper the queueing model presented by [35], in which an $M/M/1$ queueing model with vacations operating in a multi-phase random environment was studied, and we are extending this model to the $M/M/1$ queue operating in a multi-phase random environment with both multiple and single working vacation policies, waiting server, balking and reneging. We use probability generating functions (PGFs), to obtain the steady-state solution of the queueing system. This approach is an important tool for presenting the solution of difference equations set and solving probability problems. Thus, we can without difficulty obtain closed-form expressions for the steady-state distributions of the queueing model and derive various system characteristics. The approach used is highly efficient in producing computational results for the suggested queueing system as well as more complex models. In addition, for our system, we carry out a numerical analysis to show the impact of the system parameters on the system performance.

The sections of this paper are the following. The model description is given in Section 2. In addition, a practical application of the suggested model is presented. Section 3 is devoted to obtaining the stationary distribution of the proposed queueing model. Then, some practical cases are provided. In Section 4, we derive useful performance measures. Section 5 gives some numerical examples to show the impact of system parameters on different system performances. In Section 6, we conclude the paper.

2. The model

We consider an $M/M/1$ queueing system with single and multiple working vacations, multi-phase random environment, waiting server, balking and reneging. The assumption of the queueing model can be given as follows.

- Customers arrive into the system one by one according to a Poisson process with an arrival rate λ. The service discipline is FCFS and there is an infinite space for customers to wait.
- During normal busy period (phase $j = \overline{1, K}$), the service times are i.i.d. exponential random variables with rate μ_j, $j = \overline{1, K}$.
- Whenever the system becomes empty (there are no customers in the system), the server waits a random period of time before going on working vacation period (phase 0). This waiting time follows an exponential distribution with parameter ϖ. The working vacation times are i.i.d. exponentially distributed with rate ϕ.
- Two policies are considered:
  - multiple working vacation: if the server returns from a working vacation to find an empty queue, another working vacation begins; otherwise, the system jumps from phase 0 to some service phase j with probability σ_j, $j = 1, 2, \ldots, K$, where $\sigma_j > 0$ and $\sum_{j=1}^{K} \sigma_j = 1$,
  - single working vacation: if the system is still empty after the working vacation ends, the server switches to the busy period and stays there, waiting for a new arrival.

  Let δ be the indicator function, so

$$\delta = \begin{cases} 1 & \text{for the single working vacation model (SWV)} \\ 0 & \text{for the multiple working vacation model (MWV)} \end{cases}$$

- During the working vacation period (phase 0), the server serves customers at a lower rate rather than staying inactive, the service time during this phase is assumed to be exponentially distributed with parameter μ_0, with $\mu_0 < \mu_j$, $j = 1, 2, \ldots, K$.
- Note that the system cannot move directly from one service phase to another service phase. That is, if the system becomes empty, it should first move to phase 0.
- On arrival, a customer either decides to join the queue with probability θ, if the number of customers in the system is larger than or equal to one, or balk with probability $1 - \theta$.
- During working vacation period, a customer activates an impatience timer T, which is exponentially distributed with parameter ξ. If the customer's service has not been completed before the customer's timer expires, the customer may abandon the queue. The customers timers are independent and identically distributed random variables and independent of the number of waiting customers.
- The inter-arrival times, waiting server, working vacation periods and service times are mutually independent.

2.1. Motivation and practical application

The motivation for dealing with the proposed queueing system in a multi-phase random environment comes from diverse areas, particularly in manufacturing systems at which when there is no raw material (customer) to be processed, the operator (server) waits a certain period of time before going on a working vacation even though there are no customers in the system.

For better server usage and optimal maintenance cost, working vacation is integrated in such a way that the server can also serve new arrivals at a slower rate. In other words, the operator keeps on working but with a slower rate than before. Here, when the server returns from working vacation and finds raw materials existing, he serves them immediately; otherwise, two possible cases are taken into account: (a) he goes on another working vacation (multiple working vacation), (b) he remains there waiting for new arrivals (single working vacation). Moreover, the new service rate may differ from the previous service rate. In such a system, the intensity of the service can vary depending on the type of work being handled. During the working vacation period, new arrivals may get impatient and leave the system if they have not completed their service before their impatience timers expire. Further, new arrivals decide whether to join the queue or balk based on the information concerning the system size.

3. Steady-state solution

Let $L(t)$ be the number of customers in the system at time t and $J(t)$ be the state of the server at time t such that

$$J(t) = \begin{cases} 1 & \text{when the server is on a phase, } j = 1, \ldots, K \\ 0 & \text{the sever is on phase } 0 \end{cases}$$

Clearly, the process $\{(J(t); L(t)) : t \geq 0\}$ is a continuous-time Markov process with state space $\Omega = \{(j; n) : j = \overline{0, K},\ n = 0, 1, \ldots\}$. Let $P_{j,n} = \lim_{t \to \infty} P\{J(t) = j; L(t) = n\}$, $j = \overline{0, K}$, $n = 0, 1, \ldots,$ $(j; n) \in \Omega$ be the system state probabilities.

The balance equations of the queueing model are

$$(\lambda_0 + \delta\phi) P_{0,0} = \varpi \sum_{j=1}^{K} P_{j,0} + (\mu_0 + \xi) P_{0,1},\ j = 0,\ n = 0 \tag{1}$$

$$(\theta\lambda_0 + \phi + \mu_0 + \xi) P_{0,1} = \lambda_0 P_{0,0} + (\mu_0 + 2\xi) P_{0,2},\ j = 0,\ \ n = 1 \tag{2}$$

$$(\theta\lambda_0 + \phi + \mu_0 + n\xi) P_{0,n} = \theta\lambda_0 P_{0,n-1} + (\mu_0 + (n+1)\xi) P_{0,n+1},\ j = 0,\ \ n \geq 2 \tag{3}$$

$$(\lambda_j + \varpi) P_{j,0} = \delta\sigma_j \phi P_{0,0} + \mu_j P_{j,1},\ \ j = \overline{1, K},\ \ n = 0 \tag{4}$$

$$(\theta\lambda_j + \mu_j) P_{j,1} = \sigma_j \phi P_{0,1} + \lambda_j P_{j,0} + \mu_j P_{j,2},\ \ j = \overline{1, K},\ \ n = 1 \tag{5}$$

$$(\theta\lambda_j + \mu_j) P_{j,n} = \sigma_j \phi P_{0,n} + \theta\lambda_j P_{j,n-1} + \mu_j P_{j,n+1},\ \ j = \overline{1, K},\ \ n \geq 2 \tag{6}$$

The normalization condition is defined as

$$\sum_{n=0}^{\infty} \sum_{j=0}^{K} P_{j,n} = 1 \tag{7}$$

Theorem 1. Under the stability condition $\theta\lambda_j < \mu_j$, $j = \overline{1, K}$, we have.

1. The steady-state-probability $P_{0,.}$ given as

$$P_{0,.} = \frac{\varpi\varrho + \phi(1 - \delta)}{\phi} P_{0,0}$$

2. The steady-state-probability $P_{j,.}$ given as

$$P_{j,.} = \sigma_j \Psi_j P_{0,0},\ \ j = \overline{1, K}$$

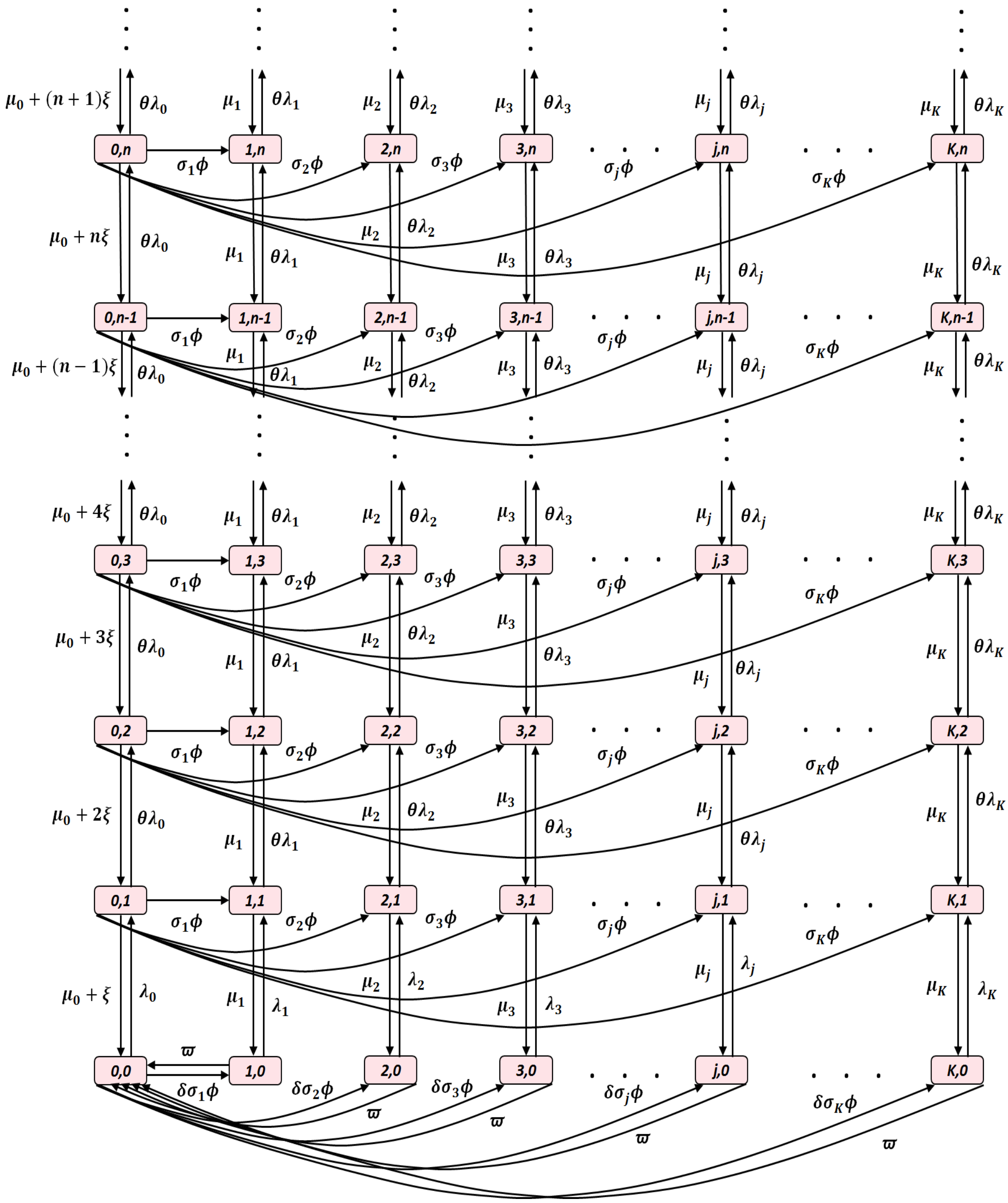

Figure 1. State-transition-rate diagram

where

$$P_{0,0} = \left(\frac{\varpi\varrho + \phi(1-\delta)}{\phi} + \sum_{j=1}^{K} \sigma_j \Psi_j \right)^{-1} \tag{8}$$

with

$$\Psi_j = \frac{(\bar{\theta}\lambda_j + \mu_j)(\xi+\phi)\varrho + \phi(\mu_0 + \theta\lambda_0) + (\theta\lambda_0 - \mu_0)(\varpi\varrho + \phi(1-\delta))}{(\mu_j - \theta\lambda_j)(\xi+\phi)}$$

Proof. The proof of this theorem is based on the probability generating functions technique (PGFs). Let

$$G_j(z) = \sum_{n=0}^{\infty} z^n P_{j,n}, \;\; j = \overline{0, K} \tag{9}$$

and

$$G_j'(z) = \frac{d}{dz} G_j(z), \;\; j = \overline{0, K}$$

Multiplying equations (1)-(3) by z^n and summing all possible values of n we find

$$\begin{aligned} &\xi z(1-z)G_0'(z) - \Big[(1-z)(\theta\lambda_0 z - \mu_0) + \phi z\Big] G_0(z) \\ &= -\varpi z \sum_{j=1}^{K} P_{j,0} - \Big[z(1-\delta)\phi - (1-z)(\mu_0 + \bar{\theta}\lambda_0 z)\Big] P_{0,0} \end{aligned} \tag{10}$$

Similarly, we get from equations (4)-(6)

$$\begin{aligned} &\Big[\theta\lambda_j z - \mu_j\Big](1-z)G_j(z) - \sigma_j \phi z G_0(z) \\ &= -\sigma_j \phi z(1-\delta)P_{0,0} - \Big[(\bar{\theta}\lambda_j z + \mu_j)(1-z) + \varpi z\Big] P_{j,0}, \quad j = \overline{1, K} \end{aligned} \tag{11}$$

Taking $z = 1$ in equations (10) and (11), respectively, we get

$$\phi G_0(1) = \varpi \sum_{j=1}^{K} P_{j,0} + \phi(1-\delta)P_{0,0} \tag{12}$$

and

$$\sigma_j \phi G_0(1) = \varpi P_{j,0} + \sigma_j \phi(1-\delta) P_{0,0}, \;\; j = \overline{1, K} \tag{13}$$

Now, for $z \neq 1$, equation (10) can be written as follows:

$$G_0'(z) - \left[\frac{\theta\lambda_0}{\xi} - \frac{\mu_0}{\xi z} + \frac{\phi}{\xi(1-z)}\right] G_0(z) = \frac{-\varpi}{\xi(1-z)} \sum_{j=1}^{K} P_{j,0} + \left[\frac{\bar{\theta}\lambda_0}{\xi} + \frac{\mu_0}{\xi z} - \frac{\phi(1-\delta)}{\xi(1-z)}\right] P_{0,0} \tag{14}$$

By multiplying both sides of equation (14) by $e^{-\frac{\theta\lambda_0}{\xi}z}(1-z)^{\frac{\phi}{\xi}} z^{\frac{\mu_0}{\xi}}$, we find

$$\begin{aligned} &\frac{d}{dz}\left(e^{-\frac{\theta\lambda_0}{\xi}z} z^{\frac{\mu_0}{\xi}} (1-z)^{\frac{\phi}{\xi}} G_0(z)\right) = -\frac{\varpi}{\xi} e^{-\frac{\theta\lambda_0}{\xi}z} z^{\frac{\mu_0}{\xi}} (1-z)^{\frac{\phi}{\xi}-1} \sum_{j=1}^{K} P_{j,0} \\ &+ \left[\frac{\bar{\theta}\lambda_0}{\xi} e^{-\frac{\theta\lambda_0}{\xi}z} z^{\frac{\mu_0}{\xi}} (1-z)^{\frac{\phi}{\xi}} + \frac{\mu_0}{\xi} e^{-\frac{\theta\lambda_0}{\xi}z} z^{\frac{\mu_0}{\xi}-1} (1-z)^{\frac{\phi}{\xi}} - \frac{\phi(1-\delta)}{\xi} e^{-\frac{\theta\lambda_0}{\xi}z} z^{\frac{\mu_0}{\xi}} (1-z)^{\frac{\phi}{\xi}-1}\right] P_{0,0} \end{aligned} \tag{15}$$

Integrating equation (15) from 0 to z, yields

$$G_0(z)=\frac{e^{\frac{\theta\lambda_0}{\xi}z}}{(1-z)^{\frac{\phi}{\xi}}z^{\frac{\mu_0}{\xi}}}\left\{-\varpi\sum_{j=1}^{K}P_{j,0}K_1(z)-\Big[\phi(1-\delta)K_1(z)-\mu_0K_2(z)-\bar{\theta}\lambda_0K_3(z)\Big]P_{0,0}\right\} \quad (16)$$

where

$$K_1(z)=\frac{1}{\xi}\int_0^z e^{-\frac{\theta\lambda_0}{\xi}s}(1-s)^{\frac{\phi}{\xi}-1}s^{\frac{\mu_0}{\xi}}\,ds,\quad K_2(z)=\frac{1}{\xi}\int_0^z e^{-\frac{\theta\lambda_0}{\xi}s}(1-s)^{\frac{\phi}{\xi}}s^{\frac{\mu_0}{\xi}-1}\,ds$$

and

$$K_3(z)=\frac{1}{\xi}\int_0^z e^{-\frac{\theta\lambda_0}{\xi}s}(1-s)^{\frac{\phi}{\xi}}s^{\frac{\mu_0}{\xi}}\,ds$$

Since $P_{0,.}=G_0(1)=\sum_{n=0}^{\infty}P_{0,n}>0$ and $z=1$ is the root of the denominator of the right hand side of equation (16), we get

$$\sum_{j=1}^{K}P_{j,0}=\varrho P_{0,0} \quad (17)$$

with

$$\varrho=\frac{\mu_0K_2(1)+\bar{\theta}\lambda_0K_3(1)-\phi(1-\delta)K_1(1)}{\varpi K_1(1)}$$

By substituting equation (17) into equation (16), we obtain

$$G_0(z)=\frac{e^{\frac{\theta\lambda_0}{\xi}z}}{(1-z)^{\frac{\phi}{\xi}}z^{\frac{\mu_0}{\xi}}}\left\{-\Big(\varpi\varrho+\phi(1-\delta)\Big)K_1(z)+\mu_0K_2(z)+\bar{\theta}\lambda_0K_3(z)\right\}P_{0,0} \quad (18)$$

Next, by substituting equation (17) in equation (12), we find $P_{0,.}$, i.e. the probability that the server is in working vacation period

$$P_{0,.}=\frac{\varpi\varrho+\phi(1-\delta)}{\phi}P_{0,0} \quad (19)$$

Equation (11) can be written as

$$G_j(z)=\frac{\sigma_j\phi z(G_0(z)-G_0(1))-(1-z)[\bar{\theta}\lambda_jz+\mu_j]P_{j,0}}{(1-z)(\theta\lambda_jz-\mu_j)} \quad (20)$$

Now, we need to define $G_j(z)$ in terms of $P_{0,0}$. To this end, we have to express $P_{j,0}$ in terms of $P_{0,0}$. From equation (13), using equation (19), we have

$$P_{j,0}=\sigma_j\varrho P_{0,0},\ \ j=\overline{1,K} \quad (21)$$

Substituting equation (21) into equation (20), we obtain the following.

$$G_j(z)=\frac{\sigma_j\phi z(G_0(z)-G_0(1))-(1-z)[\bar{\theta}\lambda_jz+\mu_j]\sigma_j\varrho P_{0,0}}{(1-z)(\theta\lambda_jz-\mu_j)} \quad (22)$$

From equation (22), applying the l'Hospital rule, we get

$$G_j(1)=\frac{\sigma_j\phi G_0'(1)+(\bar{\theta}\lambda_j+\mu_j)\sigma_j\varrho P_{0,0}}{\mu_j-\theta\lambda_j} \quad (23)$$

Then, from equation (10), applying the l'Hospital rule, we find the following:

$$G_0'(1) = \frac{(\theta\lambda_0 - \mu_0)G_0(1) + (\mu_0 + \bar{\theta}\lambda_0)P_{0,0}}{\xi + \phi} \tag{24}$$

Next, substituting equation (19) into equation (24), we get

$$G_0'(1) = \frac{(\theta\lambda_0 - \mu_0)(\varpi\varrho + \phi(1-\delta)) + \phi(\mu_0 + \bar{\theta}\lambda_0)}{\phi(\xi + \phi)} P_{0,0} \tag{25}$$

Since $P_{j,.} = G_j(1) = \sum_{n=0}^{\infty} P_{j,n} > 0$, by substituting equation (25) into equation (23) we obtain $P_{j,.}$, the probability that the server is busy during phase j, $j = \overline{1, K}$ is

$$P_{j,.} = \sigma_j \Psi_j P_{0,0}, \;\; j = \overline{1, K} \tag{26}$$

where

$$\Psi_j = \frac{(\bar{\theta}\lambda_j + \mu_j)(\xi + \phi)\varrho + \phi(\mu_0 + \bar{\theta}\lambda_0) + (\theta\lambda_0 - \mu_0)(\varpi\varrho + \phi(1-\delta))}{(\mu_j - \theta\lambda_j)(\xi + \phi)}$$

Finally, by substituting equations (19) and (26) into equation (8), we get

$$P_{0,0} = \left(\frac{\varpi\varrho + \phi(1-\delta)}{\phi} + \sum_{j=1}^{K} \sigma_j \Psi_j \right)^{-1}$$

□

3.1. Some particular cases

Case 1. $\delta = 0$, $\varpi \to +\infty$, $\theta^{'} = 0$, $\xi = 0$, and $\mu_0 = 0$. The steady-state-probabilities of the system are

$$P_{0,.} = \frac{\lambda_0 + \phi}{\lambda_0 \phi \Theta}$$

$$P_{j,.} = \frac{\sigma_j(\lambda_0 + \phi)}{\phi(\mu_j - \lambda_j)\Theta}, \;\; j = \overline{1, K}$$

where

$$\Theta = \frac{\lambda_0 + \phi}{\lambda_0 \phi} + \sum_{j=1}^{K} \frac{\sigma_j(\lambda_0 + \phi)}{\phi(\mu_j - \lambda_j)}, \;\; j = \overline{1, K}$$

The obtained results match with that given in [35].

Case 2. $K = 1$; $\lambda_0 = \lambda_j = \lambda$, $\mu_j = \mu$, $j = \overline{1, K}$, $\delta = 1$, $\theta' = 0$, and $\mu_0 = 0$. The steady-state probabilities are

$$P_{0,.} = \frac{(\phi + \xi)(\mu(\varpi C - \xi\pi_{0,0}) - \lambda\varpi C)}{\phi C(\phi\mu + \xi(\mu - \lambda))}$$

and

$$P_{1,.} = \frac{\lambda\phi\varpi C + \xi\mu(\phi + \xi)P_{0,0}}{\varpi C(\mu\phi + \xi(\mu - \lambda))}$$

where

$$P_{0,0} = \frac{\phi\varpi C(\phi + \xi)(\mu - \lambda)}{(\mu\phi\varpi + \mu\varpi\xi - \lambda\varpi\xi + \mu\phi^2 + \mu\phi\xi)\xi}$$

and

$$C = \int_0^1 e^{\frac{-\lambda}{\xi}s}(1-s)^{\frac{\phi}{\xi}-1}ds$$

These results coincide with [39][equations (17) and (18)].

Case 3. $K = 1$; homogeneous arrival rates $\lambda_0 = \lambda_j = \lambda$, homogeneous service rates $\mu_j = \mu$, $j = \overline{1, K}$, $\delta = 0$, $\varpi \to +\infty$, $\theta' = 0$, and $\mu_0 = 0$. The steady-state probabilities are as follows:

$$P_{1,.} = \frac{\lambda\phi}{\mu\phi + \xi(\mu - \lambda)}$$

and

$$P_{0,.} = \frac{(\phi + \xi)(\mu - \lambda)}{\mu\phi + \xi(\mu - \lambda)}$$

which coincide with [3][equation (2.17)].

Case 4. $K = 1$; $\lambda_0 = \lambda_j = \lambda$, $\mu_j = \mu$, $j = \overline{1, K}$, $\delta = 1$, $\varpi \to +\infty$, $\theta' = 0$, and $\mu_0 = 0$. The steady-state probabilities are as:

$$P_{0,.} = \frac{\xi}{\phi C(1)} P_{0,0}$$

and

$$P_{1,.} = \frac{1}{\mu - \lambda}\left(\frac{\lambda\xi}{\phi + \xi} + \frac{\phi\mu C(1)}{\lambda}\right) P_{0,0}$$

where

$$P_{0,0} = (\mu - \lambda)\left(\frac{\lambda\xi}{(\xi + \phi)} + \frac{(\mu - \lambda)}{\phi} + \frac{\phi\mu}{\lambda}\right)^{-1}$$

and

$$C = \int_0^1 (1-s)^{\frac{\phi}{\xi}-1} e^{\frac{-\lambda}{\xi}s}ds$$

The obtained results match with [3][equations (5.8) and (5.12)].

Remark 1. In the current study, we assumed that a real system (manufacturing system) is modeled by a an infinite buffer queue with arrival rate λ_j and service rate μ_j. Customers join the system with some probability θ (the probability of joining the system) and quit the system after getting the service. Customers may abandon the system due to the absence of the server. Thus, the queueing model during a normal busy period $(j = \overline{1, K})$ is considered as a classical $M/M/1$ queue with balking. It is well known that for this queue, for the steady-state condition to exist, we should have $\theta\lambda_j < \beta\mu_j$. Otherwise, we loose the stable state. On the other hand, during the working vacation period, even if $\beta\mu_j < \theta\lambda_j$, the stationary queue length distribution exists.

4. System performance measures

Performance measures are significant features of any queueing system. Once steady-state probabilities are known, various measures of system characteristics can be derived.

- The mean system size

$$\mathbb{E}(L) = \mathbb{E}(L_0) + \sum_{j=1}^{K} \mathbb{E}(L_j)$$

where L_0 is the system size when the server is in working vacation period and L_j represents the system size when the server is in operative phase j for $j = \overline{1, K}$. Then, the mean system size when the server is in working vacation period is given as

$$\mathbb{E}(L_0) = \lim_{z \to 1} G_0'(z) = G_0'(1)$$

From equation (25), we have

$$\mathbb{E}(L_0) = \frac{(\theta\lambda_0 - \mu_0)(\varpi\varrho + \phi(1-\delta)) + \phi(\mu_0 + \bar{\theta}\lambda_0)}{(\xi + \phi)\phi} P_{0,0}$$

The mean system size when the server is in operative phase j, $j = \overline{1, K}$ is given as

$$\mathbb{E}(L_j) = G_j'(1) = \lim_{z \to 1} G_j'(z)$$

From equation (22), we get

$$\mathbb{E}(L_j) = \frac{\sigma_j \phi}{2(\mu_j - \theta\lambda_j)} G_0''(1) + \frac{\mu_j \sigma_j \phi}{(\mu_j - \theta\lambda_j)^2} G_0'(1) + \frac{\mu_j \lambda_j \sigma_j \varrho}{(\mu_j - \theta\lambda_j)^2} P_{0,0} \tag{27}$$

where $G_0''(1)$ is obtained by differentiating twice $G_0(z)$ at $z = 1$. Thus, using equation (24), we find

$$G_0''(1) = \frac{2(\theta\lambda_0 - \xi - \mu_0 - \phi)}{\phi + 2\xi} G_0'(1) + \frac{2\theta\lambda_0}{\phi + 2\xi} G_0(1) + \frac{2\bar{\theta}\lambda_0}{\phi + 2\xi} P_{0,0} \tag{28}$$

Via equations (19) and (27)-(28), we have

$$\begin{aligned}
\mathbb{E}(L_j) = & \left[\frac{\sigma_j \mu_j \phi}{(\mu_j - \theta\lambda_j)^2} + \frac{\sigma_j \phi(\theta\lambda_0 - \xi - \mu_0 - \phi)}{(\mu_j - \theta\lambda_j)(2\xi + \phi)}\right] \mathbb{E}(L_0) \\
& + \left[\frac{\mu_j \lambda_j \sigma_j \varrho}{(\mu_j - \theta\lambda_j)^2} + \frac{\sigma_j \theta\lambda_0(\varpi\varrho + \phi(1-\delta)) + \sigma_j \bar{\theta}\lambda_0}{(\mu_j - \theta\lambda_j)(2\xi + \phi)}\right] P_{0,0}
\end{aligned}$$

- The mean queue size

$$\begin{aligned}
\mathbb{E}(Q) &= \sum_{n=1}^{\infty} (n-1) P_{0,n} + \sum_{j=1}^{K} \sum_{n=1}^{\infty} (n-1) P_{j,n} \\
&= \sum_{n=1}^{\infty} n P_{0,n} + \sum_{j=1}^{K} \sum_{n=1}^{\infty} n P_{j,n} - \left[\sum_{n=0}^{\infty} P_{0,n} + \sum_{j=1}^{K} \sum_{n=0}^{\infty} P_{j,n} - P_{0,0} - \sum_{j=1}^{K} P_{j,0}\right] \\
&= \mathbb{E}(L) - \left[1 - (1 + \varrho) P_{0,0}\right]
\end{aligned}$$

- The probability that the server is on working vacation period. From (19), we have

$$P_{wv} = P_{0,.} = \frac{\varpi\varrho + \phi(1-\delta)}{\phi} P_{0,0}$$

- The probability that the server is in busy period. From (26), we get

$$P_b = \sum_{j=1}^{K} P_{j,.} = \sum_{j=1}^{K} \sigma_j \Psi_j P_{0,0}$$

- The probability that the server is idle during busy period. From (17), we obtain

$$P_{id} = \sum_{j=1}^{K} P_{j,0} = \varrho P_{0,0}$$

- The probability that the server is working (serving customers) during busy period

$$P_s = 1 - P_{wv} - P_{id}$$

- The mean expected number of customers served

$$\begin{aligned}\mathbb{E}(CS) &= \mu_0 \sum_{n=1}^{\infty} P_{0,n} + \sum_{j=1}^{K}\sum_{n=1}^{\infty} \mu_j P_{j,n} \\ &= \mu_0 \Big(\sum_{n=0}^{\infty} P_{0,n} - P_{0,n} \Big) + \sum_{j=1}^{K} \mu_j \sum_{n=0}^{\infty} \Big(P_{j,n} - P_{j,0} \Big) \\ &= \mu_0 (P_{wv} - P_{0,0}) + \sum_{j=1}^{K} \mu_j \sigma_j (\Psi_j - \varrho) P_{0,0}\end{aligned}$$

5. Numerical analysis

To analyze the system parameter impact on the system performance, numerical calculus are carried out and few ones are presented in the form of figures and tables. For the whole analysis we have chosen: $K = 3, \sigma_1 = 0.3, \sigma_2 = 0.4$, and $\sigma_3 = 0.3$. Then, the following cases are considered:

- Table 1: $\xi = 1.0, \bar{\theta} = 0.4, \mu_0 = 1.2, \mu_1 = 2.3, \mu_2 = 2.1, \mu_3 = 2.6, \phi = 1.7, \varpi = 1.4$.
- Table 2: $\xi = 0.8, \bar{\theta} = 0.5, \lambda_0 = 1.4, \lambda_1 = 2.1, \lambda_2 = 1.9, \lambda_3 = 2.3, \phi = 4.2, \varpi = 1.1$.
- Table 3: $\xi = 1.6, \bar{\theta} = 0.4, \lambda_1 = 2.1, \lambda_2 = 1.9, \lambda_3 = 2.3, \mu_1 = 2.3, \mu_2 = 2.0, \mu_3 = 2.5, \phi = 2.2, \varpi = 1.3$.
- Figure 2: $\xi = 1.0, \bar{\theta} = 0.6, \mu_0 = 1.2, \mu_1 = 2.3, \mu_2 = 2.0, \mu_3 = 2.6, \phi = 1.7, \varpi = 1.4$.
- Figure 3: $\xi = 1.0, \bar{\theta} = 0.4, \lambda_0 = 1.8, \lambda_1 = 2.1, \lambda_2 = 1.7, \lambda_3 = 2.3, \phi = 1.7, \varpi = 1.4$.
- Table 4 and Figures 4-5: $\lambda_0 = 1.9, \lambda_1 = 2.1, \lambda_2 = 1.9, \lambda_3 = 2.3, \mu_0 = 1.2, \mu_1 = 2.3, \mu_2 = 2.0, \mu_3 = 2.5, \phi = 1.9, \varpi = 3.2$.
- Table 5 and Figures 6-7 : $\lambda_0 = 1.9, \lambda_1 = 2.1, \lambda_2 = 1.9, \lambda_3 = 2.3, \mu_0 = 1.2, \mu_1 = 2.3, \mu_2 = 2.0, \mu_3 = 2.5, \bar{\theta} = 0.6, \xi = 2.0$.

Table 1. Impact of λ_1, λ_2 and λ_3 on system performance

			MWV				SWV			
λ_1	λ_2	λ_3	$\mathbb{E}(L)$	$\mathbb{E}(CS)$	P_{id}	P_s	$\mathbb{E}(L)$	$\mathbb{E}(CS)$	P_{id}	P_s
1.9	1.7	2.3	0.9380	1.0665	0.1312	0.4048	1.1147	1.2780	0.2607	0.5246
		2.5	0.9958	1.0978	0.1286	0.4170	1.1931	1.3201	0.2524	0.5397
	2.0	2.3	1.0632	1.1153	0.1250	0.4329	1.2719	1.3352	0.2426	0.5577
		2.5	1.1159	1.1442	0.1226	0.4440	1.3403	1.3728	0.2354	0.5708
2.1	1.7	2.3	0.9911	1.0922	0.1285	0.4172	1.1834	1.3104	0.2524	0.5397
		2.5	1.0466	1.1224	0.1259	0.4289	1.2572	1.3502	0.2446	0.5539
	2.0	2.3	1.1114	1.1388	0.1226	0.4442	1.3313	1.3637	0.2354	0.5707
		2.5	1.1621	1.1668	0.1202	0.4548	1.3960	1.3994	0.2286	0.5831

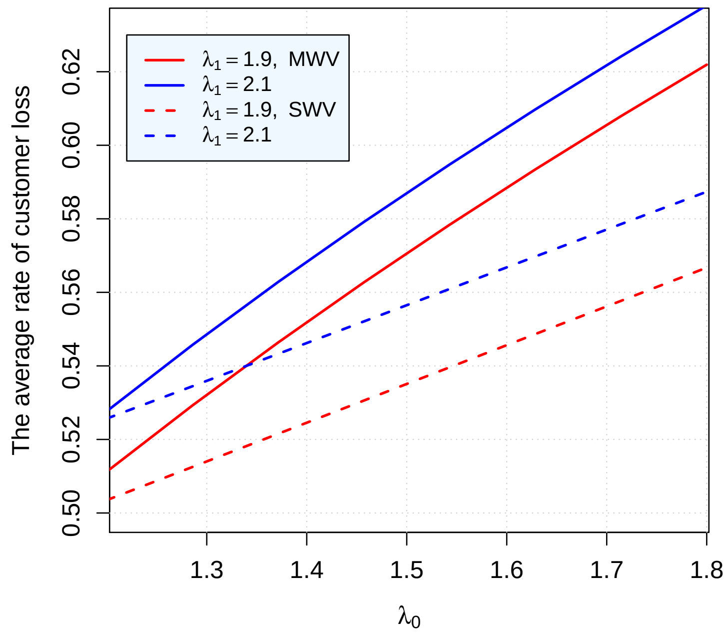

Figure 2. Effect of λ_0 and λ_1 on R_{ren} when $\lambda_2 = 1.7, \lambda_3 = 2.3$ in MWV and SWV

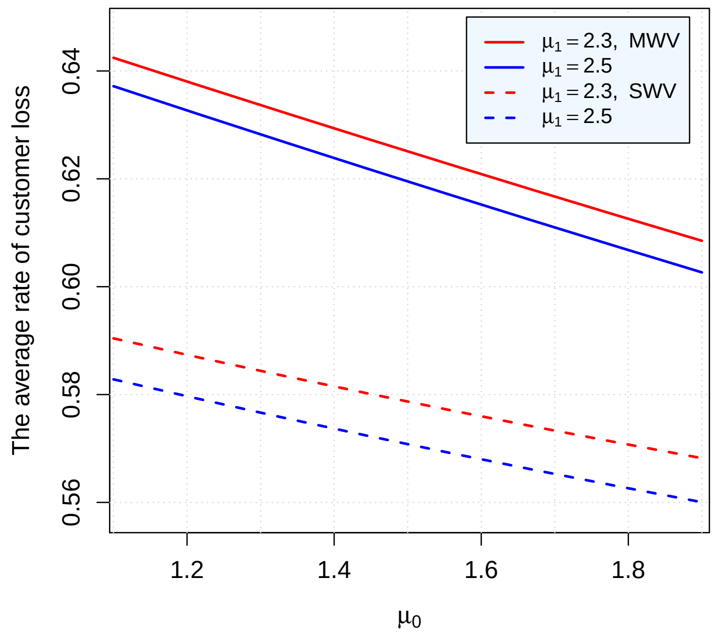

Figure 3. Effect of μ_0 and μ_1 on R_{ren} when $\mu_2 = 2.0, \lambda_3 = 2.6$ in MWV and SWV

5.1. Discussion

1. From Tables 1-5, for both single and multiple working vacations, we have.

   - With the increasing of λ_j, $j = \overline{0, 3}$, $(E(L))$ and $(E(Q))$ significantly increase, as intuitively expected. This implies an increases in (P_s) and a decrease in (P_{id}) which results in the increasing

Table 2. Impact of μ_1, μ_2 and μ_3 on system performance

			MWV				SWV			
μ_1	μ_2	μ_3	$\mathbb{E}(L)$	$\mathbb{E}(CS)$	P_{id}	P_s	$\mathbb{E}(L)$	$\mathbb{E}(CS)$	P_{id}	P_s
2.3	2.0	2.4	1.0392	1.2535	0.1931	0.5458	1.1578	1.3646	0.3082	0.6111
		2.6	1.0026	1.2559	0.1975	0.5354	1.1171	1.3703	0.3164	0.6007
	2.2	2.4	0.9809	1.2651	0.2005	0.5285	1.0959	1.3819	0.3211	0.5948
		2.6	0.9415	1.2678	0.2052	0.5173	1.0517	1.3884	0.3300	0.5836
2.5	2.0	2.4	1.0081	1.2579	0.1973	0.5360	1.1240	1.3723	0.3157	0.6016
		2.6	0.9700	1.2605	0.2019	0.5251	1.0814	1.3784	0.3243	0.5907
	2.2	2.4	0.9473	1.2699	0.2050	0.5179	1.0591	1.3904	0.3293	0.5845
		2.6	0.9062	1.2728	0.2100	0.5062	1.0128	1.3972	0.3386	0.5727

Table 3. Impact of μ_0 and λ_0 on system performance

		MWV					SWV				
μ_0	λ_0	$\mathbb{E}(L)$	$\mathbb{E}(Q)$	$\mathbb{E}(CS)$	P_s	P_{wv}	$\mathbb{E}(L)$	$\mathbb{E}(Q)$	$\mathbb{E}(CS)$	P_s	P_{wv}
1.1	1.2	1.1686	0.6080	1.1585	0.4813	0.3844	1.3949	0.7619	1.3773	0.6025	0.1477
	1.5	1.2649	0.6633	1.2440	0.5173	0.3400	1.4203	0.7749	1.3986	0.6094	0.1451
	1.8	1.3417	0.7091	1.3090	0.5448	0.3066	1.4456	0.7887	1.4185	0.6160	0.1426
1.5	1.2	1.1346	0.5880	1.1603	0.4691	0.3998	1.3870	0.7578	1.3825	0.6004	0.1484
	1.5	1.2323	0.6436	1.2503	0.5060	0.3540	1.4108	0.7696	1.4051	0.6071	0.1459
	1.8	1.3105	0.6896	1.3190	0.5344	0.3193	1.4346	0.7823	1.4261	0.6134	0.1436
1.9	1.2	1.1022	0.5691	1.1618	0.4574	0.4145	1.3800	0.7541	1.3872	0.5986	0.1491
	1.5	1.2012	0.6249	1.2563	0.4952	0.3675	1.4023	0.7649	1.4109	0.6050	0.1467
	1.8	1.2805	0.6710	1.3287	0.5244	0.3316	1.4247	0.7765	1.4331	0.6111	0.1445

Table 4. Impact of ξ and $\bar{\theta}$ on system performance

		MWV				SWV			
ξ	$\bar{\theta}$	$\mathbb{E}(L)$	R_{ren}	P_s	P_{wv}	$\mathbb{E}(L)$	R_{ren}	P_s	P_{wv}
1.4	0.3	1.8039	0.6227	0.5576	0.3711	1.9050	0.5625	0.6177	0.2399
	0.6	0.7811	0.9223	0.3900	0.5146	0.8393	0.8611	0.4609	0.3382
	0.9	0.4420	1.1094	0.2878	0.6043	0.4811	1.0545	0.3614	0.4007
2.4	0.3	1.6506	0.7117	0.5169	0.4147	1.8240	0.6094	0.6000	0.2510
	0.6	0.7079	1.0057	0.3555	0.5558	0.7989	0.9075	0.4459	0.3477
	0.9	0.4086	1.1800	0.2617	0.6396	0.4629	1.0953	0.3504	0.4076
3.4	0.3	1.5300	0.7878	0.4827	0.4520	1.7691	0.6446	0.5872	0.2590
	0.6	0.6492	1.0751	0.3272	0.5902	0.7696	0.9428	0.4349	0.3546
	0.9	0.3782	1.2388	0.2402	0.6690	0.4471	1.1269	0.3421	0.4128

Table 5. Impact of ϕ and ϖ on system performance

		MWV				SWV			
ϕ	ϖ	$\mathbb{E}(L_0)$	P_{id}	P_s	P_{wv}	$\mathbb{E}(L_0)$	P_{id}	P_s	P_{wv}
1.8	1.2	0.1150	0.1472	0.5215	0.3313	0.0551	0.2382	0.6030	0.1588
	1.7	0.1329	0.1201	0.4972	0.3828	0.0705	0.2149	0.5821	0.2030
	2.2	0.1452	0.1014	0.4804	0.4182	0.0831	0.1958	0.5650	0.2393
2.1	1.2	0.1015	0.1525	0.5387	0.3088	0.0460	0.2451	0.6148	0.1401
	1.7	0.1179	0.1251	0.5162	0.3588	0.0593	0.2229	0.5966	0.1805
	2.2	0.1293	0.1060	0.5005	0.3935	0.0704	0.2044	0.5814	0.2142
2.4	1.2	0.0907	0.1567	0.5524	0.2909	0.0391	0.2508	0.6238	0.1254
	1.7	0.1059	0.1291	0.5314	0.3395	0.0507	0.2296	0.6078	0.1626
	2.2	0.1165	0.1098	0.5167	0.3735	0.0605	0.2117	0.5942	0.1941

of the mean number of customers served $(E(CS))$.

- With the increasing of μ_j, $j = \overline{0,3}$, the mean system size $(E(L))$ decreases, as it should. This yields to the decreasing of (P_s) and (P_{id}). Therefore, the mean number of customers served significantly increases. Further, obviously, the probability that the server is in working vacation

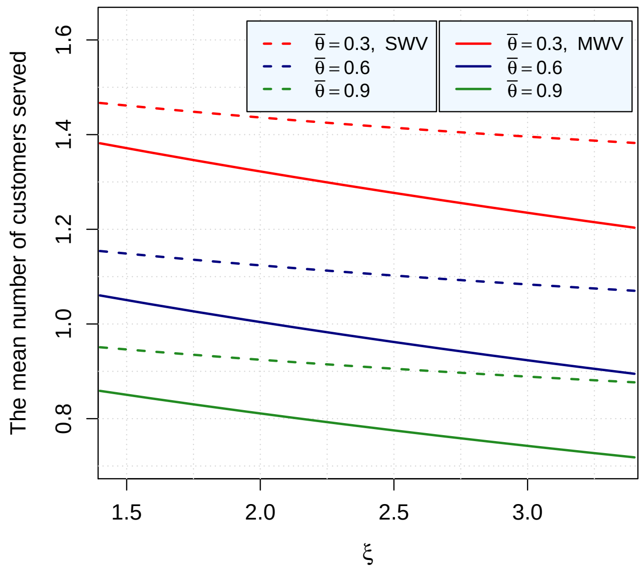

Figure 4. Effect of ξ and $\bar{\theta}$ on $\mathbb{E}(CS)$ in MWV and SWV

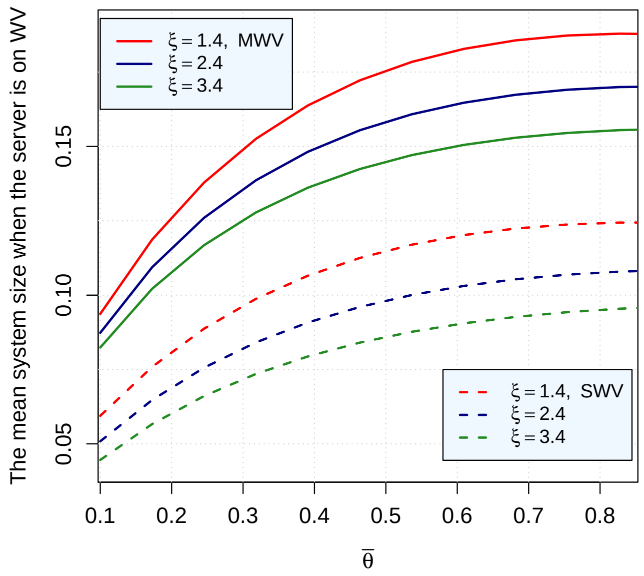

Figure 5. Effect of $\bar{\theta}$ and ξ on $\mathbb{E}(L_0)$ in MWV and SWV

period increases with μ_0.

- As the vacation rate ϕ (resp. waiting server rate ϖ) increases, both (P_{wv}) and $(E(L_0))$ decrease (resp. increase). This results in the increasing (resp. decreasing) of (P_s) and (P_{id}). This is quite reasonable, the larger the vacation rate ϕ (resp. the waiting server rate ϖ) the greater (resp. the smaller) (P_s) and (P_{id}).
- When ξ and the probability of balking $\bar{\theta}$ increase, the system characteristics (R_{ren}) and (P_{wv}) increase. While $(E(L))$ and (P_s) decrease. Obviously, the greater the impatience rate and the balking probability, the smaller the mean system size and the probability that the server is serving customers during regular busy period and the larger the average reneging rate.

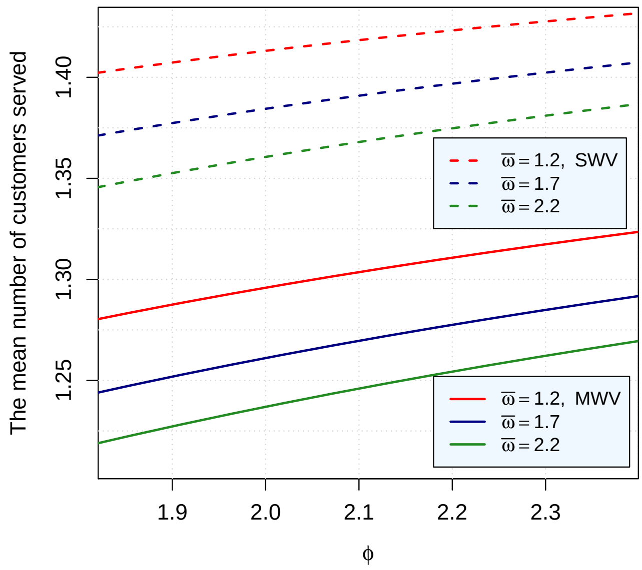

Figure 6. Effect of ϕ and ϖ on $\mathbb{E}(CS)$ in MWV and SWV

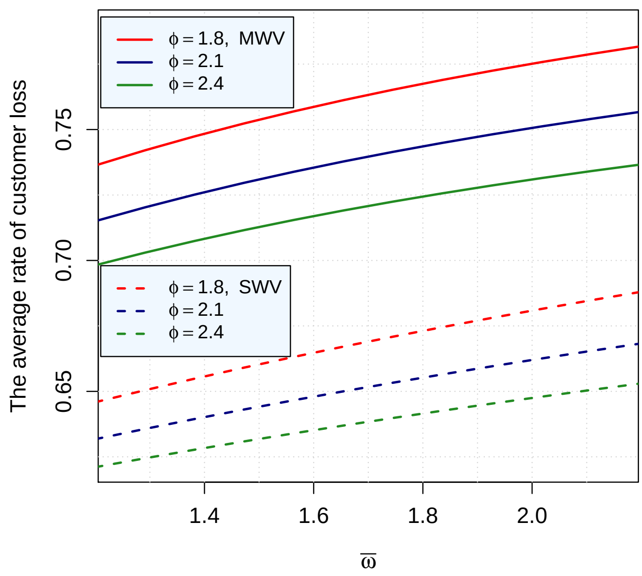

Figure 7. Effect of ϖ and ϕ on R_{ren} in MWV and SWV

2. From both single and multiple working vacations, we have.

   - From Figures 2-3, the average reneging rate (R_{ren}) increases with λ_j and decreases with μ_j, $j = 0, 1$. Obviously, the larger the average arrival rate (resp. the service rate), the higher (resp. the smaller) the system size and the bigger (resp. the lower) the average rate of reneging.
   - From Figures 4-5, the increasing of ξ as well as $\bar{\theta}$ implies a decrease in the mean number of customers served, which is quite reasonable; the larger the impatience rate (either balking or reneging) the higher the mean number of balked and reneged customers and the smaller the mean number of customers served. In addition, the increasing of balking probability implies a

decrease in the probability of busy period which results in the increasing of the mean system size in the working vacation period $(E(L_0))$. While this system characteristic decreases with ξ, as it should be.

- From Figures 6-7, when ϕ increases the server rapidly switches to the normal busy period at which the customers are served with a high rate. Therefore, the average rate of reneging decreases and the mean number of customers served monotonically increases. Nevertheless, the increasing of the waiting server rate ϖ implies a decrease in the probability of busy period. Thus, the server goes faster on a working vacation wherein the customers are served with a smaller rate. In addition, during working vacation period (phase 0) customers may get impatient and leave the system. This implies an increase in the average rate of reneging and a decrease in the mean number of customers served.

3. From Tables 1-5 and Figures 2-7, we can conclude that $(E(L))$, $(E(Q))$, (P_s), (P_{id}) as well as $(E(CS))$ in multiple working vacation (MWV) policy are less than those in the single working vacation (SWV) policy. While, $(E(L_0))$, (P_{wv}), and (R_{ren}) in the SWV policy are smaller than those in the MWV policy.

In conclusion, we can say that the single working vacation model has better performance measures than the multiple working vacations model. The results obtained perfectly agree with our expected intuition.

6. Conclusion

In the present work, we analyzed the phenomenon of impatience (balking and reneging) in an infinite-space single server Markovian queue under both single and multiple working vacation policies, multi-phase random environment, and waiting server. We aimed to establish theoretical foundations for applications and obtained explicit computational expressions for system characteristics. Our analysis approach was based on the use of probability generation functions (PGFs). By solving these equations, we have been able to determine useful performance measures. Then, using the R program, we carried out a numerical analysis to show the impact of different system parameters on system performances demonstrating the applicability of the theoretical results obtained. Due to the diverse potential applications of the queueing model discussed in this paper, we expect that the results can be applied to more convenient queuing systems.

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