B A D A N I A   O P E R A C Y J N E   I   D E C Y Z J E

Nr 3–4 2004

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PROCESS DENSITY FUNCTIONS IN THE PROBLEM OF THE IDENTIFICATION OF A BARRIER IN THE FUNCTIONING OF A CERTAIN INVENTORY STORAGE AND ISSUE SYSTEM

The subject matter of the investigation is a barrier to the functioning of a certain inventory system in the case where the storage input is a non-aggregated dynamic-parameter process. The authors derive a system of differential equations satisfied by probability distribution density functions for the intermediate states of subsystem L. The system of equations expresses relations between the densities and the parameters of the functioning of the transport subsystem and the parameters of the products supply process.

Key words: *inventory system, barrier, transport, system of differential equations*

1. Introduction

The quantitative analysis of barriers to the functioning of actual inventory systems is of essential importance in the process of the rationalization of such systems (cf., e.g., [1], [2], [4]–[6]).

The subject matter of our investigation is a barrier existing in the following inventory storage and issue system.

A receiver E (e.g., a power plant), whose operation depends on the constant supply of a units of a product (e.g., coal), is supplied continuously (e.g., by conveyor belt, pipeline, power supply line) with a production flow $y(t)$, generated by a production subsystem $\widetilde{P}$. Random changes of the process $y(t)$ and unplanned interruptions (fail-

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ures) of the transport subsystem $\widetilde{T}$ are factors that result in a decrease of the efficiency of the system. The efficiency may be increased – and at the same time the probability of interruptions in the supply of the product to the receiver E may be reduced – by placing a storage container M with a certain volume V near the receiver E. The product flow $y(t)$ is stored in subsystem M if the storage level is less than V and $y(t) > a$. If the instantaneous contents of M are equal to V and $y(t) > a$, the flow $y(t)$ is reduced to the level of a. If M is empty and $y(t) < a$, the situation is disadvantageous to the receiver E; the determination of the probability of such an occurrence is of essential practical importance.

Subsystem $\widetilde{P}$ may be exemplified by a group of excavators. The input $w(t)$ to subsystem M, which describes the process of the supply of the product to M, will be considered in a nonaggregated variant: $w(t) = y(t)v(t)$; the process $w(t)$ explicitly reflects both subsystem $\widetilde{P}$ (the product flow $y(t)$) and subsystem $\widetilde{T}$ (the process $v(t)$ describes the functioning of the transport subsystem).

Let us consider a variant where the production $y(t)$ is supplied from subsystem $\widetilde{P}$ to subsystem M in a continuous way, e.g., by means of a system of conveyor belts arranged in a series. The product $y(t)$ is then relayed from one conveyor ($\widetilde{T}_1$) to the next one ($\widetilde{T}_2$) by means of the so-called relay L. The relay, in addition to directing the product flow from conveyor $\widetilde{T}_1$ to conveyor $\widetilde{T}_2$, by virtue of its capacity absorbs irregularities in the supply of the product $y(t)$, preventing the occurrence of the so-called barrier. We will define a barrier as a random event occurring if the contents of the relay L are equal to its volume V_1 and the level of the product $y(t)$ exceeds the flow capacity d of the relay. The occurrence of a barrier within a certain time interval will cause an interruption in the functioning of conveyors $\widetilde{T}_1$ and $\widetilde{T}_2$, and consequently a stoppage of the entire transport subsystem $\widetilde{T}$ working with subsystems $\widetilde{P}$ and M.

The determination of the probability of a barrier occurring in the system under consideration as well as the determination of the mean quantity of the product that spills over from the relay L as a result of the occurrence of such a barrier is of great practical importance in the analysis of the efficiency of an inventory storage and issue system.

The functioning of the relay L may be described by a vector stochastic process $(z(t), y(t), v(t))$, whose coordinates are defined as follows:

$v(t)$ – a process describing the functioning of the transport subsystem $\widetilde{T}$;

$y(t)$ – a process describing the quantity of production generated by the production subsystem $\widetilde{P}$, conveyed by means of the transport subsystem to the receiver E;

$z(t)$ – a process describing the filling level of relay L.

Below we will assume that $y(t)$ is a homogeneous, continuous and separable Markov process (cf., e.g., [3]) with a finite number of different states $y_1, y_2, ..., y_n$, which are non-negative real numbers, and $\pi_{ij}^{(l)}$ denotes the intensity of the transition of this process from state y_i to state y_j within time T_l, $l = 1, 2, ..., m$. The intensities $\pi_{ij}^{(l)}$ will be termed the parameters of the process $y(t)$. In order to simplify the notation we will denote the time period T_l and the set of instants (moments) making up the period T_l by the same symbol.

As is easy to note, the filling process $z(t)$ in the interval $[\alpha_1, \alpha_2)$ of the stability of the process $y(t)$ satisfies the condition

$$z(t) = h[z(t_1) + (y(t_1) - d)(t - t_1)], \quad \alpha_1 \le t_1 < t < \alpha_2,$$

where

$$h(r) = \begin{cases} 0, & r \le 0, \\ r, & 0 < r < V_1, \\ V_1, & r \ge V_1. \end{cases}$$

In order to explicitly incorporate the behaviour of the transport subsystem $\widetilde{T}$ in the analysis of the functioning of the subsystem L, we will introduce a process $v(t)$ defined by the following formula:

$$v(t) = \begin{cases} 1, & \text{if subsystem } \widetilde{T} \text{ is in the working state,} \\ 0, & \text{if subsystem } \widetilde{T} \text{ is not working (is in a failure state).} \end{cases}$$

Thus, the filling level $z(t)$ of the relay is controlled by the process $w(t)$ = $y(t)v(t)$, where $y(t)$ describes the quantity of the production of subsystem $\widetilde{P}$ relayed to subsystem E. Assume that the processes $y(t)$ and $v(t)$ are independent and that $v(t)$ is a continuous, homogeneous and separable Markov process with intensities $\pi_{10}^* = \pi_1^*$ (the intensity of transition of subsystem $\widetilde{T}$ from the working state to the interruption (failure) state), $\pi_{01}^* = \pi_0^*$ (the intensity of transition of subsystem $\widetilde{T}$ from the interruption (failure) state to the working state).

In order to also reflect changes over time of the parameters π_{10}^*, π_{01}^* of the controlling process $v(t)$, we will analyse the functioning of subsystem L during m consecutive time periods $T_1, T_2, ..., T_m$. Therefore, let the period T_l correspond to intensities $\pi_{10}^{*l} = \pi_1^{*l}$, $\pi_{01}^{*l} = \pi_0^{*l}$ $(l = 1, 2, ..., m)$.

The purpose of [2] and subsequent papers is to obtain a quantitative identification of the barrier to the functioning of an inventory storage and issue system. In paper [2], we analyse the intermediate states of subsystem L defined by a random event of the form: $0 < z(t) < V_1$.

This paper is a continuation of investigations reported in paper [2], where the notion of a barrier in the functioning of an inventory storage and issue system with nonaggregated dynamic-parameter input is defined. The paper presented the general operating principles of the system and the conditional distributions for intermediate states of the subsystem L. These distributions will be used below to derive a system of equations satisfied by the probability density functions defined in [2].

2. System of equations for the density functions

In order to obtain relations satisfied by the density functions $f_k^{ul}(z,t)$ (cf., equation (1) in [2]), we will use the notations, terminology and results given in [2]. We will also assume that the supply of the product to subsystem M and the process describing the functioning of the transport subsystem are Markov processes (cf. e.g. [3]).

Taking into account equations (1), (12), (18)–(32) from [1], for $x_k > 0$, $t \in T_l$, $t+\tau \in T_l$, $0 < \alpha < V_1$, we get

$$
\begin{aligned}
Q_k^{1l}(\alpha,t+\tau) &= \int_0^{\alpha} f_k^{1l}(z,t+\tau)dz \\
= \sum_i \left\{ \int_0^{V_1} q_{ik}^{11l}(z,\alpha;\tau,t) f_i^{1l}(z,t)dz + Q_i^{1l}(\{V_1\},t) q_{ik}^{11l}(V_1,\alpha;\tau,t) + Q_i^{1l}(\{0\},t) q_{ik}^{11l}(0,\alpha;\tau,t) \right\} \\
+ \int_0^{V_1} q_{kk}^{01l}(z,\alpha;\tau,t) f_k^{0l}(z,t)dz + Q_k^{0l}(\{V_1\},t) q_{kk}^{01l}(V_1,\alpha;\tau,t) + Q_k^{0l}(\{0\},t) q_{kk}^{01l}(0,\alpha;\tau,t) \\
&= B_{k,l}^{+1} + B_{k,l}^{01} + B_{k,l}^{-1},
\end{aligned} \tag{1}
$$

where

$$
\begin{aligned}
& B_{k,l}^{-1} \\
& \sum_{\substack{i\neq k \\ x_i<0}} \left\{ \int_0^{V_1} q_{ik}^{11l}(z,\alpha;\tau,t) f_i^{1l}(z,t)dz + Q_i^{1l}(\{V_1\},t) q_{ik}^{11l}(V_1,\alpha;\tau,t) + Q_i^{1l}(\{0\},t) q_{ik}^{11l}(0,\alpha;\tau,t) \right\} \\
& = \sum_{\substack{i\neq k \\ x_i<0}} \left\{ \int_0^{-\tau x_i} \left[(1-\pi_1^{*l}\tau) \frac{\pi_{ik}^{(l)} z}{-x_i} + o^{(l)}(\tau;z) \right] f_i^{1l}(z,t)dz + \int_{-\tau x_i}^{\alpha-\tau x_k} [(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau + o^{(l)}(\tau)] f_i^{1l}(z,t)dz \right. \\
& \left. + \int_{\alpha-\tau x_k}^{\alpha-\tau x_i} \left[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)} \left(\tau - \frac{\alpha - z - \tau x_k}{x_i - x_k} \right) + o^{(l)}(\tau;\alpha,z) \right] f_i^{1l}(z,t)dz + \int_{\alpha-\tau x_i}^{V_1} o^{(l)}(\tau) f_i^{1l}(z,t)dz \right\},
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
&B^{01}_{k,l}\\
&= \int_0^{V_1} q^{11l}_{kk}(z,\alpha;\tau,t) f^{1l}_k(z,t)dz + Q^{1l}_k(\{V_1\},t)q^{11l}_{kk}(V_1,\alpha;\tau,t) + Q^{1l}_k(\{0\},t)q^{11l}_{kk}(0,\alpha;\tau,t)\\
&+ \int_0^{V_1} q^{01l}_{kk}(z,\alpha;\tau,t) f^{0l}_k(z,t)dz + Q^{0l}_k(\{V_1\},t)q^{01l}_{kk}(V_1,\alpha;\tau,t) + Q^{0l}_k(\{0\},t)q^{01l}_{kk}(0,\alpha;\tau,t)\\
&\quad = \int_0^{\alpha-\tau x_k} [(1-\pi_1^{*l}\tau)(1-\pi_k^{(l)}\tau) + o^{(l)}(\tau)] f^{1l}_k(z,t)dz + \int_{\alpha-\tau x_k}^{V_1} o^{(l)}(\tau) f^{1l}_k(z,t)dz\\
&\quad + Q^{1l}_k(\{V_1\},t)o^{(l)}(\tau) + Q^{1l}_k(\{0\},t)[(1-\pi_1^{*l}\tau)(1-\pi_k^{(l)}\tau) + o^{(l)}(\tau)]\\
&\quad \int_0^{\tau d}\left[(1-\pi_k^{(l)}\tau)\pi_0^{*l}\frac{z}{a} f^{0l}_k(z,t) + o^{(l)}(\tau,z)\right] f^{0l}_k(z,t)dz\\
&\quad + \int_{\tau d}^{\alpha-\tau x_k} [(1-\pi_k^{(l)}\tau)\pi_0^{*l}\tau + o^{(l)}(\tau)] f^{0l}_k(z,t)dz + \int_{\alpha+\tau d}^{V_1} o^{(l)}(\tau) f^{0l}_k(z,t)dz\\
&\quad + \int_{\alpha-\tau x_k}^{\alpha+\tau d}\left[(1-\pi_k^{(l)}\tau)\pi_0^{*l}\left(\tau + \frac{\alpha - z - \tau x_k}{d + x_k}\right) + o^{(l)}(\tau;\alpha,z)\right] f^{0l}_k(z,t)dz\\
&\quad + Q^{0l}_k(\{V_1\},t)o^{(l)}(\tau;z) + Q^{0l}_k(\{0\},t)o^{(l)}(\tau),
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
&B^{+1}_{k,l}\\
&= \sum_{\substack{i\neq k\\ x_i \geq 0}} \left\{ \int_0^{V_1} q^{11l}_{ik}(z,\alpha;\tau,t) f^{1l}_i(z,t)dz + Q^{1l}_i(\{V_1\},t)q^{11l}_{ik}(V_1,\alpha;\tau,t) + Q^{1l}_i(\{0\},t)q^{11l}_{ik}(0,\alpha;\tau,t)\right\}\\
&\quad = \sum_{\substack{i\neq k\\ x_i > x_k}} \left\{ \int_0^{\alpha-\tau x_i} [(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau + o^{(l)}(\tau)] f^{1l}_i(z,t)dz \right.\\
&\quad + \int_{\alpha-\tau x_i}^{\alpha-\tau x_k}\left[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\frac{\alpha - z - \tau x_k}{x_i - x_k} + o^{(l)}(\tau;\alpha,z)\right] f^{1l}_{ik}(z,t)dz\\
&+ \left.\int_{\alpha-\tau x_k}^{V_1} o^{(l)}(\tau) f^{1l}_i(z,t)dz + [(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau + o^{(l)}(\tau)]Q^{1l}_i(\{0\},t) + o^{(l)}(\tau)Q^{1l}_i(\{V_1\},t)\right\}\\
&\quad + \sum_{\substack{i\neq k\\ x_i < x_k}} \left\{ \int_0^{\alpha-\tau x_k} [(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau + o^{(l)}(\tau)] f^{1l}_i(z,t)dz \right.\\
&+ \int_{\alpha-\tau x_k}^{\alpha-\tau x_i}\left[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\left(\tau - \frac{\alpha - z - \tau x_k}{x_i - x_k}\right) + o^{(l)}(\tau;\alpha,z)\right] f^{1l}_i(z,t)dz + o^{(l)}(\tau;\alpha,z)\\
&+ \left.\int_{\alpha-\tau x_i}^{V_1} o^{(l)}(\tau) f^{1l}_i(z,t)dz + [(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau + o^{(l)}(\tau)]\, Q^{1l}_i(\{0\},t) + o^{(l)}(\tau)Q^{1l}_i(\{V_1\},t)\right\}.
\end{aligned}
\tag{4}
$$

By differentiating both sides of equation (1) with respect to α, taking into account relations (2)–(4), we get

$$f_k^{1l}(\alpha,t+\tau)=\frac{d}{d\alpha}B_{k,l}^{+1}+\frac{d}{d\alpha}B_{k,l}^{01}+\frac{d}{d\alpha}B_{k,l}^{-1}, \tag{5}$$

where

$$\frac{d}{d\alpha}B_{k,l}^{-1}=\sum_{\substack{i\neq k\\ x_i<0}}\Big\{[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau+o^{(l)}(\tau)]f_i^{1l}(\alpha-\tau x_k,t)-o^{(l)}(\tau)f_i^{1l}(\alpha-\tau x_i,t)$$
$$\left.+\frac{d}{d\alpha}\int_{\alpha-\tau x_k}^{\alpha-\tau x_i}o^{(l)}(\tau;\alpha,z)f_i^{1l}(z,t)dz-\frac{1-\pi_1^{*l}\tau}{x_i-x_k}\pi_{ik}^{(l)}\int_{\alpha-\tau x_k}^{\alpha-\tau x_i}f_i^{1l}(z,t)dz-(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau f_i^{1l}(\alpha-\tau x_k,t)\right\},$$

$$\frac{d}{d\alpha}B_{k,l}^{01}=[(1-\pi_1^{*l}\tau)(1-\pi_k^{(l)})+o^{(l)}(\tau)]f_k^{1l}(\alpha-\tau x_k,t)-o^{(l)}(\tau)f_k^{1l}(\alpha-\tau x_k,t)$$
$$+[(1-\pi_k^{(l)}\tau)\pi_0^{*l}\tau+o^{(l)}(\tau)]f_k^{0l}(\alpha-\tau x_k,t),$$

$$\frac{d}{d\alpha}B_{k,l}^{+1}=\sum_{\substack{i\neq k\\ x_i>x_k}}\left\{\frac{1-\pi_1^{*l}\tau}{x_i-x_k}\int_{\alpha-\tau x_i}^{\alpha-\tau x_k}f_i^{1l}(z,t)dz+[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau+o^{(l)}(\tau)]f_i^{1l}(\alpha-\tau x_i,t)\right.$$
$$\left.+\frac{d}{d\alpha}\int_{\alpha-\tau x_i}^{\alpha-\tau x_k}o^{(l)}(\tau;\alpha,z)f_i^{1l}(z,t)dz-(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau f_i^{1l}(\alpha-\tau x_i,t)-o^{(l)}(\tau)f_i^{1l}(\alpha-\tau x_k,t)\right\}$$
$$\sum_{\substack{i\neq k\\ x_i<x_k}}\left\{\frac{1-\pi_1^{*l}\tau}{x_k-x_i}\pi_{ik}^{(l)}\int_{\alpha-\tau x_k}^{\alpha-\tau x_i}f_i^{1l}(z,t)dz+[(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau+o^{(l)}(\tau)]f_k^{1l}(\alpha-\tau x_k,t)\right.$$
$$\left.+\frac{d}{d\alpha}\int_{\alpha-\tau x_k}^{\alpha-\tau x_i}o^{(l)}(\tau;\alpha,z)f_i^{1l}(z,t)dz-(1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau f_i^{1l}(\alpha-\tau x_k,t)-o^{(l)}(\tau)f_i^{1l}(\alpha-\tau x_i,t)\right\}.$$

Lets us expand the functions $f_k^{1l}(\alpha-\tau x_k,t),\ \ f_k^{0l}(\alpha-\tau x_k,t),$ which occur in (5), using the Taylor formula into a second degree term, move the expression $f_k^{1l}(\alpha,t)$ to the left-hand side of equation (5), divide both sides of the obtained equation by τ, and go to the limit for $\tau\to 0$. As a result of these operations, we obtain

$$\frac{\partial f^{1l}(\alpha,t)}{\partial t}=-x_k\frac{\partial f_k^{1l}(\alpha,t)}{\partial\alpha}-(\pi_k^{(l)}+\pi_1^{*l})f_k^{1l}(\alpha,t)+\sum_{i\neq k}\pi_{ik}^{(l)}f_i^{1l}(\alpha,t)+\pi_0^{*l}f_k^{0l}(\alpha,t), \tag{6}$$
$$0<\alpha<V_1,t\in T_l,\ k=1,2,...,n.$$

Relation (6) is derived similarly for $x_k \le 0$. Relations satisfied by the density functions $f_k^{0l}(\alpha, t)$ can be obtained analogously. Thus, the density functions $f_k^{1l}(z, t)$, $f_k^{0l}(z, t)$ satisfy the following system of differential equations:

$$\begin{gathered}
\frac{\partial f_k^{1l}(z,t)}{\partial t} = -x_k \frac{\partial f_k^{1l}(z,t)}{\partial z} - (\pi_k^{(l)} + \pi_1^{*l}) f_k^{1l}(z,t) + \sum_{i \ne k} \pi_{ik}^{(l)} f_i^{1l}(z,t) + \pi_0^{*l} f_k^{0l}(z,t), \\
0 < z < V_1,\ t \in T_l,\ k = 1, 2, ..., n, \\
\frac{\partial f_k^{0l}(z,t)}{\partial t} = a \frac{\partial f_k^{0l}(z,t)}{\partial z} - (\pi_k^{(l)} + \pi_0^{*l}) f_k^{0l}(z,t) + \sum_{i \ne k} \pi_{ik}^{(l)} f_i^{0l}(z,t) + \pi_1^{*l} f_k^{1l}(z,t), \\
0 < z < V_1,\ t \in T_l,\ k = 1, 2, ..., n.
\end{gathered} \tag{7}$$

The relations in (7) will be used in the authors' further work for the identification of a barrier in the functioning of the system under consideration.

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Funkcje gęstości procesu w zagadnieniu identyfikacji bariery funkcjonowania pewnego systemu gromadzenia i wydawania zapasów

Przedmiotem badania jest bariera działania pewnego systemu gospodarki zapasami w przypadku, gdy wejście magazynu–zbiornika jest procesem niezagregowanym o dynamicznych parametrach. Wprowadzono układ równań różniczkowych, który spełniają funkcje gęstości rozkładów prawdopodobieństwa stanów pośrednich podsystemu *L*. Układ ten wyraża związki między gęstościami i parametrami funkcjonowania podsystemu transportowego i parametrami procesu podaży produktu.

Słowa kluczowe: *zapasy, bariera, transport, układ równań różniczkowych*