B A D A N I A   O P E R A C Y J N E   I   D E C Y Z J E

Nr 4 2003

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CERTAINTY AND UNCERTAINTY VERSUS PRECISION AND VAGUENESS

In this paper it is argued that uncertainty and vagueness are two distinct empirical phenomena and they must be explored by means of two distinct theories: probability theory and fuzzy sets theory respectively. The assertions of the first theory can be verified to be true or false in some model, on the contrary, the typical expressions of fuzzy sets theory are not interpreted in any domain, they rather form a kind of interpretation.

1. Introduction

Problems addressed in this paper are based on the assumption of the so-called reistic point of view on the world about which we know something with certainty and we conjecture about the things when we are not certain.

The result of observation and thinking, conceived as an information about the world, must be casted into linguistic form in order to be accessible for analysis as well as to be useful for people in their activity. It is argued that uncertainty and vagueness (or by other words, unsharpeness and impreciseness) are empirical phenomena and they should be treated by two completely distinct theories: probability theory and fuzzy sets theory.

It is argued moreover that classical **logic** offeres the tool for systematic representation of certain information. Probability theory, in its widest sense called also **stochastics,** is the only formal tool to tame uncertainty. **Fuzzy sets** theory, in turn, could be considered as a suitable formal tool (language) for expressing the meaning of unsharp concepts or notions.

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The basic methodological assumption which helps to understand the important difference between uncertainty and vagueness consists in separating an observed physical world from its representation by means of some system of symbols.

The observed world is as it is, it is neither certain nor uncertain.

Uncertainty is characteristic of observers, because of their ignorance and limited ability to understand and to foresee events occurring in the world. Things that science at its current level of development cannot predict are called contingent or random. Probability, or more generally stochastics, provides tools to tame all the kinds of chance regularities as well as to describe all kinds of uncertainty. After Laplace one can rightly say that perfect intelligence would have no need of probability, it is however indispensable for a mortal men.

On the other hand, vagueness is a property of signs of representational systems, but not a property of a represented system. It should not be confused with uncertainty. Apparently, these two empirical phenomena have something in common. In both situations an observer proclaims: I do not know. But the two are very different kinds of not knowing they belong to different worlds, i.e., they have different ontological status.

2. Certainty

The aim of science is, on the one hand, to make statements that inform us about the world, and on the other hand, to help us to live happily.

Information which belongs to the world of words and which can be proved or derived by means of valid logical arguments is called certain information or **knowledge**. Apart from that, the knowledge, unlike information, is sometimes required to possess an ability to be created within a system. Logic provides tools for developing such systems, particularly in the form of a formal or formalised theories. The idea of a formal theory can be presented easily by means of a very simple example.

Suppose that we have a priori information about some fragment of reality. Let us express this information in the form of two assertions (in the language of the first order logical calculus):

A1. $\forall x : \Pi(x, x)$,

A2. $\forall x\ \forall y\ \forall z : \Pi(x, z) \wedge \Pi(y, z) \Rightarrow \Pi(x, z)$.

The symbols used in these expressions have no fixed meaning.

Let us supplement these two expressions, treated as specific axioms, by the system of logical axioms (see [8, 9]):

L1. $\alpha \Rightarrow (\beta \Rightarrow \alpha)$,

L2. $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \beta))$,

L3. $(\neg\beta \Rightarrow \neg\alpha) \Rightarrow ((\neg\beta \Rightarrow \alpha) \Rightarrow \beta)$,
L4. $\forall\, x\ \alpha\,(x) \Rightarrow \alpha\,(x \mid t)$
L5. $\forall x\ (\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \forall x\ \beta)$.

These five axioms jointly with two basic inference rules (substitution rule and *modus ponens* rule) form an engine or *machine for creating* (producing) new pieces of information (this means additional information to those given by A1 and A2). These new pieces of information take the form of assertions called theorems.

For instance, one can easily prove the following assertions (called theorems):
T1. $\forall y\ \forall z : \Pi(y, z) \Rightarrow \Pi(z, y)$,
T2. $\forall x\ \forall y\ \forall z : \Pi(x, y) \wedge \Pi(y, z) \Rightarrow \Pi(x, z)$.

One can, however, raise the question:

what is this theory (set of theorems) about?

The shortest answer is : about nothing.

Any formal theory conveys some information about a fragment of reality only after the interpretation.

The formal theory formulated above by means of seven axioms (A1, A2, L1, ..., L5) can be interpreted in various domains, conceived as fragments of reality. Interpreted theorems inform us about this reality.

As an illustration let us consider a simple example.

Suppose that the fragment of reality consists of three artificial things, which are denoted here by the following three signs:

$$\nabla,\ \Delta,\ \mathrm{O}.$$

Between these three entities there is the following symmetric binary relation:

$$s\ (\mathrm{O}, \nabla) = \text{false},\ \ s\ (\nabla, \Delta) = \text{true},\ \ s\ (\nabla, \mathrm{O}) = \text{false},$$

which can be read, for example, as "is similar".

Suppose that predicate symbol Π is interpreted as the relation s defined above, then one can easily check that both axioms A1 and A2 are the true assertions about the world under consideration. This means that all theorems which can be proved within this theory are surely true statements about our world of three things connected by relation s.

The other approach to construction theory consists in taking a concrete domain, and **next** trying to formalise the knowledge about it.

Suppose, for example, that the problem consists in ordering cups of coffee according to their sweetness.

For some pairs of cups we can definitely decide which of them is sweeter. For some other pairs, we cannot say whether one is sweeter than the other.

There is probably a tolerance within which we allow a cup of coffee to move before we notice any difference. The relation of indifference can be defined in terms of ternary relation of betweenness as follows:

$$s(x, y) \Leftrightarrow B(x, y, x).$$

Axioms of the relation B are the following (see [6]):

A1. $B(x, y, z) \Rightarrow B(z, y, x)$,

A2. $B(x, y, z) \vee B(x, z, y) \vee B(y, x, z)$,

A3. $(B(x, y, u) \wedge B(y, z, u) \wedge \neg B(x, y, z)) \Rightarrow s(u, y) \wedge s(u, z)$,

A4. $\neg s(u, v) \Rightarrow (B(x, u, v) \wedge B(u, v, y) \Rightarrow B(x, u, y))$,

A5. $B(x, y, z) \wedge B(y, x, z) \Rightarrow (s(x, y) \vee (s(z, x) \wedge s(z, y)))$,

A6. $s(x, y) \Rightarrow B(x, y, z)$.

These axioms are sufficient and necessary for the existence of a function f defined on the set of all cups of coffee such that for some $\varepsilon > 0$ the following holds:

$$s(x, y) = \begin{cases} \text{true, if } |f(x) - f(y)| < \varepsilon \\ \text{false, otherwise} \end{cases}$$

This means that for some threshold ε two cups x and y, are indistinguishable if the absolute difference $| f(x) - f(x) |$, say in sweetness, is less than ε.

One should note that indistinguishability in this case is defined as a usual, crisp binary relation in the yes-no terms.

It seems natural that one desires to define indistinguishability as a graded relation, i.e., as a function taking on values from the unit interval.

It turns out, however, (see [3]) that in this case it is impossible to create a formal theory in a purely syntactic form. Admitting, in our understanding, the graduality of the reality we must use fuzzy sets concepts as a formal tool to formulate theories in a semantic form.

3. Uncertainty

Plato in his *Republic* had already distinguished between certain information, i.e., knowledge, and uncertain information called opinion or belief.

Certain knowledge is acquired by tools provided by logic. The ability to obtain this kind of information is also called the **art of thinking**.

Patterned after this expression, J. Bernoulli had written the book under the title **the art of conjecturing** or **stochastics**, intended to provide tools for making belief also a subject of an exact science. The art of conjecturing takes over where the art of thinking has left off. One way of making up an exact science from conjecturing is by attaching numbers to all our uncertainties. Uncertainties are however of different kinds.

Usually, one distinguishes between the kind of uncertainty that characterises our general knowledge of the world, and the kind of uncertainty that we discover in gambling. As a consequence, one distinguishes between two kinds of probabilities: **epistemic** probability and **aleatory** probability.

The former is dedicated to assessing the degree of belief in propositions, and the latter is concerned with stochastic laws of chance processes.

The best known theory for aleatory type of uncertainty is the Kolmogorov theory.

This theory, however, does not form axiomatization of an intuitive notion of uncertainty (see [6]). The uncertainty that people encounter in their everyday world cannot be formalised based on the repeatability of experiments. Usually an "experiment" is conducted only once. There are proposed a number of reasonable alternatives to the so-called *qualitative probability* structures. One of the best known is the Luce structure. In this structure, one assumes the existence of a non-empty set X and algebra of subsets of this set along with a weak order $\geq$ defined on the algebra, which satisfies the six axioms, quaranteeing the existence of a *unique* probability representation.

The probability, however, is understood in this settling as an additive measure.

The are known various formalisations admitting non-additive reprezentation. The best known is Dempster-Shafer theory of lower and upper probabilities (see [10]).

4. Unsharpness

One of the basic functions of language is to convey information about world.

The results of thinking processes as well as conjecturing processes become available for analysis and for communication only after their casting into linguistic form.

One of the modes of conveying information is to give appropriate definitions.

Most definitions in natural languages are made by examples. As a consequence of this, almost all words are vague.

According to M. Black and N. Rescher a word is vague when its (denotational) meaning is not fixed by sharp boundaries but spread over a range of possibilities, so that its **applicability** in a particular case may be dubious.

Unsharp or vague words are therefore characterised by the existence of a "grey area" where the applicability of the word is in doubt. In Zadeh's understanding, such words are called fuzzy.

The meaning of vague word can be precisely defined by means of **fuzzy set**. Fuzzy set, or more precisely fuzzy subset of a given set U, is defined as a mapping

$$\mu_X : U \to [0, 1],$$

where X stands for a "fuzzy" word.

For example, the meaning of the imprecise word "young" might be defined very precisely by the following fuzzy set

$$\mu_{\text{young}}(x) = \exp(-x),\ x \ge 0,$$

where x stands for an age.

The value $\mu_X(x)$ is interpreted as a **grade of applicability** of word X to a particular object $x \in X$. Alternatively, $\mu_X(x)$ can be conceived as a perceived psychological distance between an object x and the **ideal** prototype of a word X.

It is worth noting the essential difference between apparently similar phrases: "fuzzy word" which in Zadeh's terminology means nothing else but vague word, and "fuzzy set". Fuzzy word, in Black's terminology is termed as a vague word, so that some words may be fuzzy and the others are not fuzzy. Otherwise, fuzzy set is a sharp, proper name of some precisely defined mathematical object, so that the term "fuzzy set" is not fuzzy.

5. Confronting uncertainty with vagueness

For methodological convenience it is useful to make distinction between an observed world and its representational system, i.e., the world of words, whose typical example is language.

A simple example will make this assertion quite clear. Before rolling a die *I do not know* which **number** of spots will result.

This kind of uncertainty is called aleatory uncertainty. It pertains to the objective facts of real world.

On the other hand, before rolling a die, or even after the rolling *I do knot know* weather or not is the die **fair** ? This is epistemic uncertainty.

Suppose now that the die is cased, looking at it, and seeing the spots, *I do not know*, for example, whether or not there resulted a **small** number of spots. This pertains to the meaning of words from the world of signs, in this case the meaning of the word "small".

Fuzzy sets theory offers formal tools to quantify the applicability of words to particular objects.

From the above discussion it should be clear enough that probability theory (broadly considered) and fuzzy sets theory are quite different formalisms invented to convey quite different information. In other words, one can also say that these are different tools invented to cope with different (incomparable) problems.

For brevity, some distinct features of uncertainty and unsharpness are summarised in the table.

Table

Features of uncertainty and unsharpness

Uncertainty	Unsharpness
exists because of a lack of biunivocal correspondence between causes and consequences	exists because of a lack of sharp definitions
there are limits for certainty	there are no limits for sharpening definitions
pertains to the WORLD	pertains to WORDS about world
referres to reasoning and prediction	refers to classification and discrimination
it is my defect because of my ignorance	it is my doubt in applicability of words because of our (or your) carelessness in naming things
it is quantified by grades of certainty called probability; probability is warranted by evidence	it is quantified by grades of applicability called membership grade ; applicability is warranted by convention

The most important difference between uncertainty formalized by some probability theory, and vagueness formalized by fuzzy sets theory lies in the possibility to verify the truthfulness of expressions belonging to these theories.

Probability theories, particularly these theories which are formally formulated have different models, which means that they can be interpreted in domains where their expressions can be verified as being either true or false.

On the contrary, fuzzy sets theory, inspite of its formal definition, cannot be interpreted in a logical sense. Let us consider, for illustration, the following three assertions:

1) she is very attractive,
2) she is over 30 years old,
3) she is young.

From my point of view the first sentence is true, and for Henry it is false, as for him she is rather repellent. Similarly, the third sentence, containing the vague word “young”, is not verifiable, it is neither true nor false. There is no sense to consider this sentence with the truth value from the unit interval. From my point of view, the sentence is true, as for my son, aged 20, she is old.

6. Conditional information

The difference between certain and uncertain information is particularly apparent when one tries to formalize conditional information.

As a matter of fact, all information is inherently conditional, because all information has context. Context is nothing else but just another word for condition.

Conditional information is expressed by conditional statements of the following type:

if A, then B.

Within the classical logic statements of such type of certain conditional information are formalized by implication.

$$A \Rightarrow B,$$

where A and B are binary-valued assertions.

Thruth value of this (material) implication is defined as follows:

$$t\,(A \Rightarrow B) = \begin{cases} \text{false, if } t\,(A) = \text{true, } t\,(B) = \text{false,} \\ \text{true, otherwise.} \end{cases}$$

In classical logic material implication $A \Rightarrow B$ can be expressed equivalently in several other ways:

$$\neg A \vee B, \quad A \wedge B = A, \quad A \vee B = B, \quad \neg B \wedge A = 0,$$

where 0 represents the assertion whose truth value is “false”.

In the case of uncertain conditional information it would seem natural to expect some formal aid from the probability theory.

Unfortunately, in probability theory there are only a few proposals, which are still debatable, for definition of conditional information of the type:

“if event A, then event B”.

Symbolically it is denoted by $(A \mid B)$.

Within the traditional probability theory a **conditional probability** is offered and not a probability of **conditional event**. The conditional probability of event A, given event B, is defined as follows:

$$P\,(A \mid B) = \frac{P\,(A \cap B)}{P\,(B)}\,,$$

provided $P\,(B) > 0$.

One needs, however, the probability (not conditional!) of conditional event $(A|B)$, that is, one needs to define $P((A|B))$.

As is known, probability measure P is defined on the Boolean space of events, so that in order to enable calculation of $P((A|B))$, one needs to define conditional event $(A|B)$ as a Boolean element, i.e., by means of Boolean operations $\wedge$, $\vee$, and $\neg$.

It turns out, however, that this is impossible.

For that reason, various extensions of ordinal Boolean operations are proposed (see [1]).

The problem with conditional information becomes more complicated, even almost insuperable, if not only uncertainty but also vagueness is introduced into conditional statements of the type: "if — , then — ".

Suppose *A* and *B* are two vague terms which are modelled by two fuzzy sets:

$$\mu_A : U \to [0, 1] \text{ and } \mu_B : U \to [0, 1].$$

Conditional statement

$$\text{if } a \text{ is } A, \text{ then } b \text{ is } B$$

within the fuzzy sets theory is defined as a fuzzy relation

$$\mu_R : U \times U \to [0, 1]$$

defined as follows :

$$\mu_R (x, y) = I (\mu_A (x), \mu_B (y))$$

where *I* is an operator of the so-called fuzzy implication.

For example, this operator can be defined as follows:

$$I (x, y) = \min \{1, 1 - x + y\}.$$

There are many other possibilities to define fuzzy implication.

7. Concluding remarks

It is a common practice in using natural languages that the same words have very different meanings, which are usually easily recognisable. In some cases, however, they create sources for confusion.

It is a pity that, for example, the word "uncertainty" is usually used to denote the ignorance concerning verifiable facts from the real world, and the same word "uncertainty" is also used to denote the undecidedness concerning the application of a given word as a proper name for a given object (usually belonging to the real world).

In order to avoid confusion, some peoples prefer to use the longer phrases: "probabilistic uncertainty" and "non-probabilistic uncertainty". The former is formalized within the framework of probabilistic theories, and the latter can be treated as the synonim for fuzziness.

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Pewność i niepewność kontra precyzja i nieostrość

Przedstawiono, niepodzielany przez większość autorów zajmujących się zbiorami rozmytymi, pogląd, że niepewność i nieostrość są to dwa istotnie różne zjawiska empiryczne i dlatego muszą być wyjaśnione lub tylko opisywane za pomocą różnych teorii: teorii prawdopodobieństwa i teorii zbiorów rozmytych. Stwierdzenia pierwszej z tych dwóch teorii są weryfikowalne w pewnym modelu, to znaczy, że stwierdzenia te mogą być prawdziwe lub fałszywe. Typowe wyrażenia zbiorów rozmytych zaś nie są interpretowalne, w sensie interpretacji semantycznej, w żadnej dziedzinie, one same stanowią raczej pewien rodzaj interpretacji. Wyrażenia takie nie są więc ani prawdziwe, ani fałszywe.