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SENSITIVITY ANALYSIS OF GREY LINEAR PROGRAMMING FOR OPTIMISATION PROBLEMS

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Sensitivity analysis of parameters is usually more important than the optimal solution when it comes to linear programming. Nevertheless, in the analysis of traditional sensitivities for a coefficient, a range of changes is found to maintain the optimal solution. These changes can be functional constraints in the coefficients, such as good values or technical coefficients, of the objective function. When real-world problems are highly inaccurate due to limited data and limited information, the method of grey systems is used to perform the needed optimisation. Several algorithms for solving grey linear programming have been developed to entertain involved inaccuracies in the model parameters; these methods are complex and require much computational time. In this paper, the sensitivity of a series of grey linear programming problems is analysed by using the definitions and operators of grey numbers. Also, uncertainties in parameters are preserved in the solutions obtained from the sensitivity analysis. To evaluate the efficiency and importance of the developed method, an applied numerical example is solved.

Keywords: *sensitivity analysis, uncertainty, interval grey number, grey linear programming*

1. Introduction

Sensitivity analysis is an important issue in dealing with optimisation problems. It includes analysing the effect of changes in the cost vector, in the right-hand side vector on the optimal value of the objective function, and the validity of these effects. Many attempts have been made to investigate the behaviour of such a problem under the influence of changing inputs. During a decision-making process, the presence of uncertainty in the data and the situation a linear-programming problem usually confronts the

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decision-maker with conditions of doubt and uncertainty and makes it difficult for him to decide and choose the best option [4]. Therefore, to model a problem with linear programming (LP) techniques, although the coefficients are usually determined by experts with precision, they usually contain a level of uncertainty in the expressed values. For this reason, modelling real-life problems, each of which is seen as an uncertain system, with inaccurate data by using interval, fuzzy, or grey parameters would be more appropriate [29]. For example, fuzzy sets are utilised to define vague information, interval numbers are introduced to describe the boundary information and grey numbers are employed to characterise information that is partially known with limited observations [46]. In the stochastic approach, imprecise parameters are considered as random variables following some known probability distributions. On the contrary, in fuzzy approaches, the uncertainty is viewed either as a fuzzy set with an appropriate membership function or as a fuzzy number. Generally, it is practically difficult for a researcher to obtain either an appropriate membership function or a reasonable probability distribution in an uncertain decision-making situation. To overcome such a practical difficulty, some scholars have used intervals to specify imprecise parameters, from which reasonable results are obtained [47]. As a novel approach to describe and depict a realistic uncertainty and to deal with real-life decision-making problems, grey system theory employs grey numbers to represent the uncertainty of information [33]. This theory possesses the advantages of all such theories as fuzzy mathematics, probability and statistics, and interval numbers also can appropriately describe the spirit of practical problems [48].

One approach to the study of uncertain systems is fuzzy set theory, where the fuzzy LP formulation was first proposed by Zimmerman [52].

A preliminary review of the sensitivity analysis of fuzzy LP is made by Hammaker et al. [10] and followed by others [13, 16, 22]. Another uncertainty approach to the study of uncertainty problems is the Grey Syst. Theory, first proposed by Julong Deng in 1982. The Grey linear programming (GLP) problem is a model developed for analysing grey systems (GS) that exists in an inaccurate environment [21]. Linear programming with grey parameters has been widely used in modelling real issues in resource management planning, economics, geography, industry, etc. [1, 14, 15, 18, 19, 40, 45, 49]. Researchers have proposed several methods to solve GLP [7]. The first of such methods uses the concept of grey-number whitening to solve the GLP problem [21, 27, 28, 34, 41, 42, 43]. The second is the method that seeks to find an answer to the problem of GLP based on the use of the concepts of grey matrix inverses [25, 26]. The existence of many procedural steps and the lack of criteria for stopping after finding the answer are the two main drawbacks of this method. The third method uses the display of grey numbers in the form of intervals and the ranking of intervals to solve the GLP problem [34, 41]. The fourth is the method presented by Nasseri et al. [39]. In this case, an attempt is made to solve the GLP problem directly without the need to whitenise parameters. The disadvantage of this method is that it is used only to solve the GLP problem with a grey objective function. Today, due to the unstable economic, social, geographical, and climatic conditions of the world, it seems necessary to analyse the sensitivity of the results of the GLP problem for decision-makers to make the right decisions in the world. Therefore, to deal with GLP problems, due to changes in some parameters, in terms of calculations, time and complexity, we have to conduct the sensitivity analysis of the GLP under consideration.

The different sections of this article are arranged as follows. In Section 2, the preliminaries and basic concepts of grey system theory (GST) are introduced. Section 3 introduces GLP and its types of modelling. In Section 4, the sensitivity analysis of GLP and related algorithms are presented. In Section 5, an example is shown to illustrate the performance of the algorithms, and finally, in Section 6, the results of this study are presented.

2. Basic concepts

This section is devoted to the concepts related to grey systems (GS), interval grey numbers, grey number arithmetic, and comparison of interval grey numbers used during this study [24, 29, 35, 51].

Information about real-life systems is not entirely known or completely unknown, but a mixture of some known and unknown information. Due to incomplete, inaccurate, and approximate information, there is a level of uncertainty in the understanding and description of the underlying system. Such systems are called grey systems [12]. The grey systems theory (GST) is one of the most important scientific achievements about how to make use of inaccurate information; it represents a new way to study problems that involve high levels of uncertainty due to limited availability of data and limited amounts of information when fuzzy system theory and statistics and probability cannot be effectively employed [30]. Many researchers have turned to grey systems theory as an appropriate approach to dealing with uncertainties in real-world problems [2, 9, 10, 32, 37, 38].

Definition 2.1. An interval grey number (IGN) $\otimes x$ is defined as follows:

$$\otimes x \in \left[\underline{x}, \overline{x}\right] = \left\{\underline{x} \leq t \leq \overline{x}\right\}, \ \underline{x} < \overline{x} \tag{1}$$

where t is a piece of information, $\underline{x}$ and $\overline{x}$ the lower and upper bounds of the information [3].

Definition 2.2. Given an IGN $\otimes x \in [\underline{x}, \overline{x}]$, its centre $(\otimes \hat{x})$ and width $(\otimes x_W)$ are defined as follows [39]:

$$\otimes \hat{x} = \frac{\underline{x} + \overline{x}}{2}, \quad \otimes x_W = \frac{\overline{x} - \underline{x}}{2} \tag{2}$$

Definition 2.3. For any IGN $\otimes x \in [\underline{x}, \overline{x}]$, the expression $\ell(\otimes x) = \overline{x} - \underline{x}$ is called the length of the IGN [39].

Definition 2.4. By using the centre and length of an IGN $\otimes x \in [\underline{x}, \overline{x}]$, $\underline{x} < \overline{x}$, the greyness of $\otimes x$ is defined as follows [31]:

$$g^{\circ}(\otimes x) = \frac{\ell(\otimes x)}{\otimes \hat{x}}, \quad \otimes \hat{x} \neq 0 \tag{3}$$

An interval representation means that any value within the interval is a possible value. However, we may know that the possible value can only be one of a finite number of values within the interval. For this situation, an interval representation cannot help. Intervals can be considered as a special case of grey numbers where we know the scope of the underlying number but do not know its exact position inside the continuous scope [50].

In grey systems theory, there is often a need for one to compare grey numbers to make the right decision. Darvishi et al. [10] study the comparison of grey numbers in more detail in various ways. One method of comparing IGNs is to use the centre and the degree of greyness. When two grey numbers are completely overlapping interval grey numbers, many of the existing methods such as grey possibility degree approaches, surveyed previously fail to recognise the ordering of these numbers. Many of these methods cannot identify positive and negative interval grey numbers. Due to the simultaneous use of the centre and the degree of greyness, such a comparison of IGNs performs better than other ranking methods and distinguishing between IGNs.

Definition 2.5. Comparison of IGNs

Suppose that $\otimes x = [\underline{x}, \overline{x}]$ and $\otimes y = [\underline{y}, \overline{y}]$ are two IGNs. If the centres and the greynesses of the given numbers are employed, the following relations hold [32].

If $\otimes \hat{x} < \otimes \hat{y} \Rightarrow \otimes x <_G \otimes y$;

If $\otimes \hat{x} = \otimes \hat{y}$, thus

If $g^{\circ}(\otimes x) = g^{\circ}(\otimes y) \Rightarrow \otimes x =_G \otimes y$;

If $g^{\circ}(\otimes x) < g^{\circ}(\otimes y) \Rightarrow \otimes x >_G \otimes y$;
If $g^{\circ}(\otimes x) > g^{\circ}(\otimes y) \Rightarrow \otimes x <_G \otimes y$.

Theorem 2.1. A grey number $\otimes x = [\underline{x}, \overline{x}]$ is said to be non-negative, $\otimes \hat{x} \geq 0$ [39].

Definition 2.6. Let $\otimes x_1 \in [\underline{x}_1, \overline{x}_1]$ and $\otimes x_2 \in [\underline{x}_2, \overline{x}_2]$ be two IGNs. The following operations can be defined [24]:

$$\begin{aligned}
&\otimes x_1 + \otimes x_2 = [\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2] \\
&\otimes x_1 - \otimes x_2 = \otimes x_1 + (-\otimes x_2) = [\underline{x}_1 - \overline{x}_2, \overline{x}_1 - \underline{x}_2] \\
&\otimes x_1 \times \otimes x_2 = \left[\min\{\underline{x}_1 \underline{x}_2, \overline{x}_1 \overline{x}_2, \overline{x}_1 \underline{x}_2, \underline{x}_1 \overline{x}_2\}, \max\{\underline{x}_1 \underline{x}_2, \overline{x}_1 \overline{x}_2, \overline{x}_1 \underline{x}_2, \underline{x}_1 \overline{x}_2\}\right] \\
&\frac{\otimes x_1}{\otimes x_2} = \otimes x_1 \times \otimes x_2^{-1} = \left[\min\left\{\frac{\underline{x}_1}{\underline{x}_2}, \frac{\underline{x}_1}{\overline{x}_2}, \frac{\overline{x}_1}{\underline{x}_2}, \frac{\overline{x}_1}{\overline{x}_2}\right\}, \max\left\{\frac{\underline{x}_1}{\underline{x}_2}, \frac{\underline{x}_1}{\overline{x}_2}, \frac{\overline{x}_1}{\underline{x}_2}, \frac{\overline{x}_1}{\overline{x}_2}\right\}\right] \quad 0 \notin [\underline{x}_2, \overline{x}_2]
\end{aligned} \tag{4}$$

The following section introduces the GLP problem, its different types and related concepts.

3. Grey linear programming (GLP)

The general form of LP in matrix form is given as follows:

$$\begin{aligned}
&\max(\min) Z = CX \\
&\text{s.t.} \\
&AX (\leq = \geq) b \\
&X \geq \circ
\end{aligned} \tag{5}$$

where all parameters of A, b, and C are exact numbers. However, in many real-world situations, the data and information of the problem of concern are not clear, not accurate, or not conclusive. In these situations, one should use some types of modelling that can handle inaccurate conditions [20]. Due to the inherent uncertainty existing in real-life problems, various methods, such as LP with interval parameters, fuzzy linear programming (FLP), or GLP problem, have been developed to deal with these problems. In the meantime, the GLP problem that is appropriate for dealing with inaccurate conditions provides better results. The general problem of GLP is given as follows [5, 30]:

$$
\begin{aligned}
&\max \otimes z =_G \sum_{j=1}^{n} \otimes c_j \otimes x_j \\
&\text{subject to} \\
&\sum_{j=1}^{n} \otimes a_{ij} \otimes x_j \leq_G \otimes b_i,\ i = 1, 2, ..., m \\
&\otimes x_j \geq_G \otimes \circ,\ \ j = 1, 2, ..., n
\end{aligned}
\tag{6}
$$

A GLP problem can be stated in a more convenient form by using matrix notation as follow.

$$
\begin{aligned}
&\max \otimes z =_G \otimes C \otimes X \\
&\text{subject to} \\
&\otimes A \otimes X \leq_G \otimes b \\
&\otimes X \geq_G \otimes 0
\end{aligned}
\tag{7}
$$

so that

$$
\begin{aligned}
&\otimes X = \left[\otimes x_1, \otimes x_2, \ldots, \otimes x_n\right]^T \\
&\otimes C = \left[\otimes c_1 \otimes c_2 \ldots \otimes c_n\right]^T \\
&\otimes b = \left[\otimes b_1 \otimes b_2 \ldots \otimes b_m\right]^T \\
&\otimes A = \begin{bmatrix} \otimes a_{11} \otimes a_{12} \ldots \otimes a_{1n} \\ \otimes a_{21} \otimes a_{22} \ldots \otimes a_{2n} \\ \ldots \\ \otimes a_{m1} \otimes a_{m2} \ldots \otimes a_{mn} \end{bmatrix}
\end{aligned}
\tag{8}
$$

where $\otimes X$ is the vector of grey decision variables. Vector $\otimes C$ is the grey cost coefficients vector, $\otimes b$ the grey right-hand-side vector, and $\otimes A$ the grey technological coefficients. All the relevant vectors satisfy

$$
\begin{aligned}
&\otimes Z \in \left[\underline{Z}, \overline{Z}\right] && \underline{Z}, \overline{Z} \in R \\
&\otimes x_j \in \left[\underline{x}_j, \overline{x}_j\right] && \underline{x}_j, \overline{x}_j \in R,\ j = 1, 2, ..., n \\
&\otimes c_j \in \left[\underline{c}_j, \overline{c}_j\right] && \underline{c}_j, \overline{c}_j \in R,\ j = 1, 2, ..., n \\
&\otimes b_i \in \left[\underline{b}_i, \overline{b}_i\right] && \underline{b}_i, \overline{b}_i \in R,\ i = 1, 2, ..., m \\
&\otimes a_{ij} \in \left[\underline{a}_{ij}, \overline{a}_{ij}\right] && \underline{a}_{ij}, \overline{a}_{ij} \in R,\ i = 1, 2, ..., m,\ j = 1, 2, ..., n
\end{aligned}
\tag{9}
$$

Other different models of the GLP problem can be introduced as follows:
1. The GLP problem with only the cost vector being grey.

$$\begin{aligned} &\max \otimes z =_G \sum_{j=1}^{n} \otimes c_j x_j \\ &\text{subject to} \\ &\sum_{j=1}^{n} a_{ij} x_j \le b_i,\ i = 1, 2, ..., m \\ &x_j \ge \circ,\ = 1, 2, ..., n \end{aligned} \tag{10}$$

2. The GLP with only the right-hand-side vector being grey.

$$\begin{aligned} &\max z = \sum_{j=1}^{n} c_j x_j \\ &\text{subject to} \\ &\sum_{j=1}^{n} a_{ij} x_j \le_G \otimes b_i,\ i = 1, 2, ..., m \\ &x_j \ge 0,\ j = 1, 2, ..., n \end{aligned} \tag{11}$$

3. The GLP with only the technological coefficients matrix being grey.

$$\begin{aligned} &\max z = \sum_{j=1}^{n} c_j x_j \\ &\text{subject to} \\ &\sum_{j=1}^{n} \otimes a_{ij} x_j \le_G b_i,\ i = 1, 2, ..., m \\ &x_j \ge 0,\ j = 1, 2, ..., n \end{aligned} \tag{12}$$

By combining these models in different ways, other types of GLP can be obtained, where some of the coefficients and decision variables of the problem include grey numbers. According to different models of grey linear programming, researchers have proposed different methods to solve them [5, 23, 44].

Definition 3.1. Any vector $\otimes x$ of IGNs that satisfy the constraints of the GLP in equation (7) is called a feasible solution.

Definition 3.2. Assume that Q is the set of all feasible solutions to the GLP in equation (7). Then $\otimes x_0 \in Q$ is said to be an optimal solvable solution to the GLP, if for all $\otimes x \in Q$, $\otimes C \otimes x \leq_G \otimes C \otimes x_0$.

Definition 3.3. For any real number x, we have $\otimes x = [x, x]$.

Remark 3.1. Consider the following GLP

$$\begin{aligned} &\max \otimes z =_G \otimes C_B \otimes X_B + \otimes C_N \otimes X_N \\ &\text{subject to} \\ &\otimes X_B \geq_G \otimes 0, \otimes X_N \geq_G \otimes 0 \end{aligned} \tag{13}$$

Table 1. The simplex tableau of the GLP in equation (13)

Basis	$\otimes Z$	$\otimes X_B$	$\otimes X_N$	R.H.S
$\otimes Z$	1	0	$\otimes C_B B^{-1} N - \otimes C_N$	$\otimes y_{00} =_G \otimes C_B B^{-1} \otimes b$
$\otimes X_B$	0	I	$B^{-1} N$	$\otimes y_{00} =_G B^{-1} \otimes b$

The cost row gives $\otimes C_B B^{-1} N - \otimes C_N$ which consists of the $\otimes y_{0j} =_G \otimes C_B B^{-1} a_j - \otimes C_j =_G \otimes Z_j - \otimes C_j$'s for the non-basic variables and consists of the $\otimes y_{0j} =_G \otimes 0$'s, $j = B_i, i = 1, 2, ..., m$ for the basic variables.

Definition 3.4. Basic feasible solution

Consider the grey system in equation (7), where A is a $m \times n$ matrix, $\otimes b$ and $\otimes x$ are an m vector and n vector, respectively, satisfying $\text{rank}(A, \otimes b) = \text{rank}(A) = m$. After possibly rearranging the columns of A, let $A = [B, N]$, where B is a $m \times m$ invertible matrix and N is a $m \times (n-m)$ matrix. The solution $\otimes X^T = [\otimes x_B^T, \otimes x_N^T]$ to the equations $A \otimes X =_G \otimes b$, where $\otimes X_B = [\otimes x_{B_1}, \otimes x_{B_2}, ..., \otimes x_{B_m}]^T$ and $\otimes X_N =_G \otimes 0$, is called a basic solution of the system. If $\otimes x_B \geq_G \otimes 0$, then $\otimes X$ is called a basic feasible solution of the system.

Below are some results related to grey linear programming (taken from [37]).

Theorem 3.1. If there is a basic feasible solution with grey objective value $\otimes Z$ such that $\otimes y_{0j} =_G \otimes Z_j - \otimes C_j <_G \otimes 0$ or $\otimes Z_j <_G \otimes C_j$ for some non-basic variable $\otimes x_j$, and

$y_j = B^{-1}a_j \not\leq 0 \ \ 1 \leq j \leq n,$ then it is possible to obtain a new basic feasible solution with a new grey objective value $\otimes Z'$ that satisfies $\otimes Z \leq_G \otimes Z'$.

Theorem 3.2. If there is a basic feasible solution satisfying

$$\otimes y_{0k} =_G \otimes C_B B^{-1} a_k - \otimes C_k =_G \otimes Z_k - \otimes C_k <_G \otimes 0$$

for some non-basic variable $\otimes x_k$, and $y_{ik} \leq_G 0, i = 1, 2, ..., m,$ then the problem in equation (7) has an unbounded optimal solution.

Theorem 3.3. If a basic solution $\otimes X_B =_G B^{-1} \otimes b, \ \otimes X_N =_G \otimes 0$ is feasible to the problem in equation (7) and $\otimes c_B B^{-1} a_j \geq_G \otimes c_j,$ for all $j = 1, 2, ..., n,$ then the basic solution is an optimal solution to the problem in equation (7).

Although different algorithms are developed to solve GLP, let us focus on the GLP simplex algorithm.

Algorithm 3.1. The simplex algorithm of GLP

Suppose that a basic feasible solution is also accompanied with a basis B and corresponding simplex table.

1. The basic feasible solution is given by $\otimes x_B =_G B^{-1} \otimes b =_G \otimes y_0$ and $\otimes x_N =_G \otimes 0.$

Then the grey objective function value will be: $\otimes Z =_G \otimes c_B B^{-1} \otimes b =_G \otimes y_{00}.$

2. Calculate $\otimes y_{\circ j} =_G \otimes Z_j - \otimes C_j, \quad j = 1, 2, ..., n, j \neq B_i, i = 1, 2, ..., m.$

Let $\otimes y_{0k} =_G \min_{1 \leq j \leq n} \{\otimes y_{0j}\}.$

3. If $\otimes y_{0k} \geq_G \otimes 0,$ then stop; the solution is optimal.

4. If $\otimes y_{0k} <_G \otimes 0$ and $y_{ik} \leq 0, i = 1, 2, ..., m,$ the problem has an unbounded solution.

5. If $\otimes y_{0k} <_G \otimes 0$ and there is $i = 1, 2, ..., m$ so that $y_{ik} > 0,$ then determine an index r corresponding to a variable x_{Br} that leaves the basis as follows

$$\frac{\otimes y_{r0}}{y_{rk}} =_G \min_{1 \leq i \leq m} \left\{ \frac{\otimes y_{i0}}{y_{ik}} \middle| y_{ik} > 0 \right\}$$

6. Pivot on y_{rk} and update the simplex tableau. Go to step 2.

In the following, we will look at an example to demonstrate how the above algorithm plays out in real life. Some applications in the real-life problem that can be modelled as an uncertain system with inaccurate data by using interval, fuzzy or grey parameters are mentioned in [6, 8, 36].

Example 3.1. Consider the following GLP.

$$\begin{aligned}
&\max \otimes z =_G \otimes[2, 4] \otimes x_1 + \otimes[1, 6] \otimes x_2 \\
&\text{subject to} \\
&\otimes[3, 3] \otimes x_1 + \otimes[4, 4] \otimes x_2 \leq_G \otimes[7, 9] \\
&\otimes[1,1] \otimes x_1 + \otimes[2,2] \otimes x_2 \leq_G \otimes[5, 6] \\
&\otimes x_1, \otimes x_2 \geq_G \otimes 0
\end{aligned}$$

Table 2. Starting tableau of the simplex method

Basis	$\otimes z$	$\otimes x_1$	$\otimes x_2$	$\otimes s_1$	$\otimes s_2$	R.H.S
$\otimes z_\circ$	[1, 1]	–[2, 4]	–[1, 6]	[0, 0]	[0, 0]	[0, 0]
$\otimes s_1$	[0, 0]	[3, 3]	[4, 4]	[1, 1]	[0, 0]	[7, 9]
$\otimes s_2$	[0, 0]	[1, 1]	[2, 2]	[0, 0]	[1, 1]	[5, 6]

Table 3. Optimal tableau of the simplex method

Basis	$\otimes z$	$\otimes x_1$	$\otimes x_2$	$\otimes s_1$	$\otimes s_2$	R.H.S
$\otimes z_\circ$	$[1,1]$	$\left[-\frac{13}{3}, 7\right]$	$-[1,6]$	$[0,0]$	$[0,0]$	$\left[\frac{7}{4}, \frac{54}{4}\right]$
$\otimes s_1$	$[0,0]$	$\left[\frac{3}{4}, \frac{3}{4}\right]$	$[1,1]$	$\left[\frac{1}{4}, \frac{1}{4}\right]$	$[0,0]$	$\left[\frac{7}{4}, \frac{9}{4}\right]$
$\otimes s_2$	$[0,0]$	$\left[-\frac{1}{2}, -\frac{1}{2}\right]$	$[0,0]$	$\left[-\frac{1}{2}, -\frac{1}{2}\right]$	$[1,1]$	$\left[\frac{1}{2}, \frac{5}{2}\right]$

The next section presents how the sensitivity analysis of GLP problems is carried out.

4. Sensitivity analysis

The purpose of the sensitivity analysis in GLP is to investigate how changing some parameters of the problem of concern do not need to resolve the problem. Assume that the basic optimal solution to a GLP problem is available. In particular, the following variations in the problem will be considered:

- Change in objective function coefficients $(\otimes c_j)$ for non-basic variables.
- Change in objective function coefficients $(\otimes c_j)$ for basic variables.
- Change in right-hand-side values $(\otimes b_i)$.

The effect of the above changes can appear in the following two ways.

1. The current grey solution remains optimal.
2. The current grey solution changes.

Note 1. Since the coefficients of the slack variables in the objective functions are zero, it is unnecessary to study changes of coefficients of the slack variables in the objective function.

Algorithm 1. Determine the ranges of objective function coefficients for non-basic decision variables that only affect the optimal condition of the grey simplex panel. Since changes in the values of the non-basic variables only affect their new values in the optimal panel, to keep the answer obtained in the final panel, let us do the following:

By considering $\otimes c_{x_j}$ as an unknown value, calculate the new values of the non--basic variables $\otimes \overline{y}_{0_j} =_G \otimes c_B \overline{p}_{x_j} - \otimes c_{x_j}$, $j = 1, 2, ..., n$.

By $\hat{\otimes} \overline{y}_{0_j} \geq 0$, $j = 1, 2, ..., n$ (the centre of new values of non-basic variables), investigate the positive results.

Share the results of step 2 to determine the desired ranges to keep the optimal results of the grey simplex panel constant.

Algorithm 2. Determine ranges of changes of the objective function coefficients for the basic decision variables. If the coefficients of the primary decision variable changes in the objective function, all the elements of the zero line of the non-basic and total $\otimes Z$ in the final panel of the original model will change. So, to keep the answer obtained in the final panel, let us do the following:

By considering $\otimes c_B$ as an unknown value, calculate non-basic variables from $\otimes \overline{y}_{0_j} =_G \otimes c_B \overline{p}_{x_j} - \otimes c_{x_j}$, $j = 1, 2, ..., n$.

Using $\hat{\otimes} \overline{y}_{0_j} \geq 0$, $j = 1, 2, ..., n$ (the centre of the new values of the non-basic variables), check the positivity of the results.

Determine the unity of the obtained results from step 2 to identify the desired ranges to keep the optimal results of the grey simplex panel constant.

Algorithm 3. Determine the range of the right-hand-side values of the grey linear programming ($\otimes b_i$).

Changes in the right-hand-side values of the model in the constraints affect the conditions of feasible solution (the right-hand-side values being positive) of the simplex tableau or the optimal solution. In other words, any change in $\otimes b_i$ affects the right-hand-side values of the optimal tableau and causes a change in $\otimes \overline{b}$ and total $\otimes Z$.

Calculate the new right-hand-side values of the model by using $\otimes\bar{b} =_G B^{-1} \otimes b_i$, $i = 1, 2, ..., m$.

2. Calculate the right-hand-side values of the model by $\hat{\otimes}\bar{b}_i \geq 0, i = 1, 2, ..., m$ (the center of the right-hand-side values of the model) and check the positivity of the results.

3. The obtained domains will be the desired domains to keep the current basic variables in the final tableau constant, i.e., the optimality.

In the following, we will examine a sensitivity analysis by looking at an example.

5. Numerical example

In this section, we provide an example to examine how changes in model parameters can affect the optimal answer of a grey linear programming model to better understand the previously proposed algorithms.

Example 5.1. Consider the following GLP problem.

$$\begin{aligned}
&\max \otimes z =_G \otimes[1, 3] \otimes x_1 + \otimes[2, 5] \otimes x_2 \\
&\text{subject to} \\
&\otimes[2, 2] \otimes x_1 + \otimes[3, 3] \otimes x_2 \leq_G \otimes[5, 7] \\
&\otimes[3, 3] \otimes x_1 + \otimes[1, 1] \otimes x_2 \leq_G \otimes[3, 6] \\
&\otimes x_1, \otimes x_2 \geq_G \otimes 0
\end{aligned}$$

Table 4. Starting tableau of the simplex method

Basis	$\otimes z$	$\otimes x_1$	$\otimes x_2$	$\otimes s_1$	$\otimes s_2$	R.H.S
$\otimes z_\circ$	$[1, 1]$	$-[1, 3]$	$-[2, 5]$	$[0, 0]$	$[0, 0]$	$[0, 0]$
$\otimes s_1$	$[0, 0]$	$[2, 2]$	$[3, 3]$	$[1, 1]$	$[0, 0]$	$[5, 7]$
$\otimes s_2$	$[0, 0]$	$[3, 3]$	$[1, 1]$	$[0, 0]$	$[1, 1]$	$[3, 6]$

Table 5. The optimal tableau of the simplex method

Basis	$\otimes z$	$\otimes x_1$	$\otimes x_2$	$\otimes s_1$	$\otimes s_2$	R.H.S
$\otimes z_\circ$	$[1, 1]$	$\left[-\frac{5}{3}, 3\right]$	$[0, 0]$	$\left[\frac{2}{3}, \frac{5}{3}\right]$	$[0, 0]$	$\left[\frac{10}{3}, \frac{35}{3}\right]$
$\otimes x_2$	$[0, 0]$	$\left[\frac{2}{3}, \frac{2}{3}\right]$	$[1, 1]$	$\left[\frac{1}{3}, \frac{1}{3}\right]$	$[0, 0]$	$\left[\frac{5}{3}, \frac{7}{3}\right]$
$\otimes s_2$	$[0, 0]$	$\left[\frac{7}{3}, \frac{7}{3}\right]$	$[0, 0]$	$\left[\frac{-1}{3}, \frac{-1}{3}\right]$	$[1, 1]$	$\left[\frac{2}{3}, \frac{13}{3}\right]$

5.1. The effect of changes in the objective function coefficients for non-basic variables

Since the coefficients of the slack variables in the objective function are zero, so changing the coefficients of slack variables is not required in the objective function. Therefore, we examine the changes in the coefficients of non-basic decision variables.

According to the relationship $\otimes\overline{y}_{0_j} =_G \otimes c_B \overline{p}_{x_j} - \otimes c_{x_j}$, $j = 1, 2, ..., n,$ it is clear that changes in the coefficients of the objective function only affect the optimal condition of the simplex tableau. Here, $\otimes x_1$ is non-basic variable; and, we have

$$\otimes\overline{y}_{0_1} =_G \otimes c_B \overline{p}_{x_1} - \otimes c_{x_1} \geq_G \otimes 0$$

$$\left(\otimes[2,5], \otimes[0,0]\right)\begin{bmatrix}\frac{2}{3}\\ \frac{7}{3}\end{bmatrix} - \otimes c_{x_1} \geq_G \otimes 0 \Rightarrow \frac{2}{3}[2,5] + \frac{7}{3}[0,0] - \otimes c_{x_1} \geq_G \otimes 0$$

$$\Rightarrow \left[\frac{4}{3} - \overline{c}_{x_1}, \frac{10}{3} - \underline{c}_{x_1}\right] \geq_G \otimes 0 \Rightarrow 0 \leq \otimes\hat{c}_{x_1} \leq \frac{7}{3}$$

That is, as long as $0 \leq \otimes\hat{c}_{x_1} \leq 7/3,$ in other words, as long as the centre of the grey numbers which are placed as a coefficient, is between zero and 7/3, the current optimal answer will remain unchanged.

5.2. The effect of changes in the objective function coefficients for basic variables

If the coefficient of the basic variable in the objective function changes, all elements of the zero line of the non-basic variables and the total $\otimes Z$ in the optimal tableau of the starting model will change.

Set the range $\otimes c_{x_2}$ so that the current optimal solution remains unchanged. Then,

$$\otimes\overline{y}_{0_j} =_G \otimes c_B \ \overline{p}_{x_j} - \otimes c_{x_j}, \ \otimes c_B = \left(\otimes c_{x_2}, \otimes s_2\right)$$

$$\otimes\overline{y}_{0_1} =_G \otimes c_B \overline{p}_{x_1} - \otimes c_{x_1} \geq_G \otimes 0$$

$$\left(\otimes c_{x_2}, \otimes[0,0]\right)\begin{bmatrix}\frac{2}{3}\\ \frac{7}{3}\end{bmatrix} - \otimes[1,3] \geq_G \otimes 0 \Rightarrow \frac{2}{3} \otimes c_{x_2} + \frac{7}{3}[0,0] - \otimes[1,3] \geq_G \otimes 0$$

$$\Rightarrow \left[\frac{2}{3}\overline{c}_{x_2} - 3, \ \frac{2}{3}\underline{c}_{x_2} - 1\right] \geq_G \otimes 0$$

Now, according to Theorem 2.2, we have:

$$\frac{\frac{2}{3}\overline{c}_{x_2}-3+\frac{2}{3}\underline{c}_{x_2}-1}{2}\geq 0\Rightarrow\frac{\frac{2}{3}\left(\overline{c}_{x_2}+\underline{c}_{x_2}\right)-4}{2}\geq 0\Rightarrow\frac{2}{3}\left(\frac{\overline{c}_{x_2}+\underline{c}_{x_2}}{2}\right)-2\geq 0$$

$$\Rightarrow\left(\frac{\overline{c}_{x_2}+\underline{c}_{x_2}}{2}\right)\geq 3\Rightarrow\otimes\hat{c}_{x_2}\geq 3 \tag{14}$$

$$\otimes\overline{y}_{0_1}=_G\otimes c_B\overline{p}_{s_1}-\otimes c_{s_1}\geq_G\otimes 0$$

$$\left(\otimes c_{x_2},\otimes[0,0]\right)\begin{bmatrix}\frac{1}{3}\\ -\frac{1}{3}\end{bmatrix}-\otimes[0,0]\geq_G\otimes 0\Rightarrow\frac{1}{3}\otimes c_{x_2}\geq_G\otimes 0\Rightarrow\left[\frac{2}{3}\overline{c}_{x_2},\frac{2}{3}\underline{c}_{x_2}\right]\geq_G\otimes 0$$

$$\frac{\frac{2}{3}\overline{c}_{x_2}+\frac{2}{3}\underline{c}_{x_2}}{2}\geq 0\Rightarrow\frac{\frac{2}{3}\left(\overline{c}_{x_2}+\underline{c}_{x_2}\right)}{2}\geq 0$$

$$\Rightarrow\frac{2}{3}\left(\frac{\overline{c}_{x_2}+\underline{c}_{x_2}}{2}\right)\geq 0\Rightarrow\left(\frac{\overline{c}_{x_2}+\underline{c}_{x_2}}{2}\right)\geq 0\Rightarrow\otimes\hat{c}_{x_2}\geq 0 \tag{15}$$

A combination of the results from equations (14) and (15) shows that for all grey numbers $\otimes\hat{c}_{x_2}\geq 3$, the current optimal solution will remain unchanged.

5.3. Change of a right-hand side value of a constraint $(\otimes b_i)$

Calculate the new right-hand side values as follows: $\otimes\overline{b}=_G B^{-1}\otimes b$, B^{-1} the matrix of technical coefficients of the slack variables in the optimal tableau, and

$$\otimes\overline{Z}=_G\otimes C_B\otimes\overline{b}$$

To answer the question of to what extent $\otimes b_1=\left[\underline{b}_1,\overline{b}_1\right]$ will remain unchanged in the current basic optimal solution, compute

$$\otimes\bar{b} =_G B^{-1} \otimes b$$

$$\otimes\bar{b} =_G \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} \otimes\left[\underline{b}_1, \bar{b}_1\right] \\ \otimes[3, 6] \end{bmatrix} =_G \begin{bmatrix} \frac{1}{3}\otimes\left[\underline{b}_1, \bar{b}_1\right] \\ -\frac{1}{3}\otimes\left[\underline{b}_1, \bar{b}_1\right] + \otimes[3, 6] \end{bmatrix}$$

The final tableau will be optimal if all the right-hand-side values are non-negative. Therefore, we have

$$\frac{1}{3}\otimes\left[\underline{b}_1, \bar{b}_1\right] \geq_G \otimes 0 \Rightarrow \otimes\left[\underline{b}_1, \bar{b}_1\right] \geq_G \otimes 0 \Rightarrow \hat{\otimes} b_1 \geq 0$$

$$-\frac{1}{3}\otimes\left[\underline{b}_1, \bar{b}_1\right] + \otimes[3, 6] \geq_G \otimes^{\circ} \Rightarrow \otimes\left[-\frac{1}{3}\bar{b}_1, -\frac{1}{3}\underline{b}_1\right] + \otimes[3, 6] \geq_G \otimes 0$$

$$\Rightarrow \otimes\left[-\frac{1}{3}\bar{b}_1 + 3, -\frac{1}{3}\underline{b}_1 + 6\right] \geq_G \otimes 0 \Rightarrow \hat{\otimes} b_1 \leq \frac{27}{2}$$

Therefore, as long as $0 \leq \hat{\otimes} b_1 \leq \frac{27}{2}$ is satisfied, the current basic variables in the optimal tableau will not change. That is, the condition of optimality is maintained.

To answer the question of to what extent $\otimes b_2 = \left[\underline{b}_2, \bar{b}_2\right]$ of the current basic optimal solution will remain unchanged, we compute

$$\otimes\bar{b} =_G B^{-1} \otimes b$$

$$\otimes\bar{b} =_G \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} \otimes[5,7] \\ \otimes\left[\underline{b}_2, \bar{b}_2\right] \end{bmatrix} =_G \begin{bmatrix} \frac{1}{3}\otimes[5,7] \\ -\frac{1}{3}\otimes[5,7] + \otimes\left[\underline{b}_2, \bar{b}_2\right] \end{bmatrix}$$

The final tableau will be optimal if all the right-hand side values are non-negative. That is,

$$-\frac{1}{3}\otimes[5, 7] + \otimes\left[\underline{b}_2, \bar{b}_2\right] \geq \otimes 0 \Rightarrow \otimes\left[-\frac{7}{3}, -\frac{5}{3}\right] + \otimes\left[\underline{b}_2, \bar{b}_2\right] \geq \otimes 0 \Rightarrow \hat{\otimes} b_2 \geq 2$$

Hence, as long as $\hat{\otimes} b_2 \geq 2$ is satisfied, the current basic variables in the optimal tableau will not change. That is, the condition of optimality will be maintained.

6. Conclusion

Uncertainty is an integral and inherent feature of problems encountered in social, economic, agricultural, educational, etc., areas. However, the decision systems established to describe such systems makes them problematic. Therefore, to optimise real-world problems that are highly inaccurate due to the limited availability of data and the limited amount of information and cannot be resolved by using theories of fuzzy systems and random systems, the theory of grey systems will come to the rescue. This theory is developed to deal with systems that contain uncertain and incomplete information. To solve problems of GLP, different methods have been advanced. Some of these methods are developed by whitenising parameters to create additional information, while others have to deal with complexity and place high demands on computational time due to the uncertainty involved. In this paper, by using concepts and theorems of the theory of grey systems, we analyse the sensitivity of GLP model parameters, which enables us to find the intervals within which the optimal solution remains unchanged. And we can practically reflect the uncertainty in the parameters of a GLP model in the obtained ranges. For further research, one can consider the sensitivity analysis of GLP for simultaneous changes of more than one parameter or a more general situation.

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