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THE EXTENSION OF RANK ORDERING CRITERIA WEIGHTING METHODS IN FUZZY ENVIRONMENT

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Weight elicitation is an important part of multi-criteria decision analysis. In real-life decision-making problems precise information is seldom available, and providing weights is often cognitively demanding as well as very time- and effort-consuming. The judgment of decision-makers (DMs) depends on their knowledge, skills, experience, personality, and available information. One of the weights determination approaches is ranking the criteria and converting the resulting ranking into numerical values. The best known and most widely used are rank sum, rank reciprocal and centroid weights techniques. The goal of this paper is to extend rank ordering criteria weighting methods for imprecise data, especially fuzzy data. Since human judgments, including preferences, are often vague and cannot be expressed by exact numerical values, the application of fuzzy concepts in elicitation weights is deemed relevant. The methods built on the ideas of rank order techniques take into account imprecise information about rank. The fuzzy rank sum, fuzzy rank reciprocal, and fuzzy centroid weights techniques are proposed. The weights obtained for each criterion are triangular fuzzy numbers. The proposed fuzzy rank ordering criteria weighting methods can be easily implemented into decision support systems. Numerical examples are provided to illustrate the practicality and validity of the proposed methods.

Keywords: *multi-criteria decision analysis, criteria weights, criteria ranking, fuzzy criteria ranking, fuzzy criteria weights*

1. Introduction

Multi-criteria decision-making (MCDM) problems can be divided into two kinds. In classical MCDM problems, the information provided by the decision-makers (DMs) takes exact numerical values, and the ratings and the weights of criteria are measured in crisp numbers [31, 60]. In the fuzzy multi-criteria decision-making (FMCDM) problems, the ratings and the weights of criteria evaluated on incomplete information, imprecision, subjective judgment and vagueness are usually expressed by interval or fuzzy

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numbers [9, 17, 36], intuitionistic fuzzy numbers [4, 72], interval-valued intuitionistic fuzzy [63], linguistic terms [15, 76], evidence theory [70], ordered fuzzy numbers [51], among others.

The multi-criteria decision-making process is often constituted from four separate main stages: alternatives' formulation and criteria selection, criteria weighting, evaluation alternatives, and final aggregation and ranking [31, 58]. The second step, weight elicitation, is a vital part in multi-criteria decision analysis and the discussion of weights is a very important issue in both MCDM and FMCDM which can greatly influence the results of multi-criteria aiding techniques, and hence has been the focus of many papers [19, 48, 49, 55, 61, 73], among others.

As Choo et al. [19] point out, *the true meaning and the validity of criteria weights are crucial in order to avoid improper use of the MCDM models. Unfortunately, criteria weights are often misunderstood and misused, and there is no consensus on their meaning*. The authors also provide a list of possible interpretations of weights. In general, we can distinguish two main types of weights: importance coefficients, and trade-offs. The main difference between weights as the importance of criteria and weights as trade-offs is that of compensation between criteria, which refers to the fact that a good performance in some criteria can offset a bad performance in another one. What is also important, weights as importance coefficients require the use of non-compensatory multi-criteria methods.

For our paper, we assume that criteria weights show the relative importance of criteria in the problem considered. It should be taken into account that in the real-life decision-making problem precise information is seldom available and providing weights is often cognitively demanding as well as very time- and effort-consuming. The judgment of decision-makers depends on their knowledge, skills, experience and personality, and available information [22, 48]. Morton and Fasolo [43] report the implications of biases for multicriteria decision analysis modelling. They point out that *for most decision-makers, weighting criteria is the most cognitively demanding part of the MCDA process* and that *even under relatively favourable conditions where analytic support is available, weight judgements exhibit predictable biases*. Montibeller and von Winterfeldt [42] identify the cognitive and motivational biases in multicriteria analysis, including weight elicitation, and guiding the existing debiasing techniques to overcome these biases. Weber and Borcherding [72] investigate behavioural influences on weight judgments and examine biases in multicriteria weight assessment. This was the motivation for searching for weighting criteria methods which, on the one hand, are less cognitively demanding for decision-makers, and, on the other hand, may reduce biases in weights elicitation.

One of the approaches to weights determination is to rank the criteria and to convert the resulting ranking into numerical values [6, 56, 58]. Using ranks to elicit weights by formulas is sometimes more reliable than directly assigning weights to criteria. This is because decision-makers are usually more confident about the ranks of certain criteria and they can agree to them more easily than to their weights. The best known and most widely used techniques in this respect are rank sum, rank reciprocal [58], and centroid weights [6, 56].

The goal of this paper is to extend the rank ordering criteria of weighting methods for the imprecise data, especially fuzzy data, cases. The novel methods are built on the ideas of rank order methods that take into account imprecise information about rank. The weights obtained for all criteria are triangular fuzzy numbers. The fuzzy rank sum, fuzzy rank reciprocal and fuzzy centroid weights techniques are proposed. Certain properties of the ranking function are also studied. A comparative analysis is provided to show the effectiveness of the proposed methods. Finally, numerical examples are provided to illustrate the practicality and validity of the fuzzy rank ordering of weighing methods. The involvement of this work is three-fold. Firstly, it contributes to decision analysis with a proposition extending rank ordering criteria of weighting methods for imprecise data. Secondly, it argues the usability of such an approach from the behavioural perspective point of view. The proposed approach allows one to avoid or minimise biases in crisp weights evaluation. Thirdly, it shows that the proposed fuzzy rank ordering criteria weighting methods can be easily implemented in decision support systems.

The remainder of this paper is organised as follows. In Section 2, we introduce preliminary information: basic definitions and notations of crisp weights, fuzzy numbers, fuzzy normalization, and fuzzy weights. In Section 3, we present a brief overview of weighting techniques criteria. In Section 4, the classical rank ordering weighting techniques are outlined and a comparative analysis of rank ordering criteria weighing techniques is provided. In Section 5, fuzzy extensions of those methods are proposed. In Section 6, an illustrative example is presented. Finally, conclusions are formulated.

2. Preliminaries

2.1. Crisp weights elicitation

Consider a multiple criteria decision-making problem with a finite set $X = \{x_1, ..., x_n\}$ of n criteria. Let us assume that the decision-maker assigns to each criterion x_k individual weights r_k which represents the importance of criteria numerically. The individual weights are normalised by dividing by the sum of all individual weights

$$w_k = \frac{r_k}{\sum_{j=1}^{n} r_j}, \quad k = 1, 2, ..., n \tag{1}$$

The individual weights are converted to fractions between 0 and 1, with the sum of all normalised weights equal to 1.

Definition 1. Given the set $X = \{x_1, ..., x_n\}$ of criteria, the set $W = \{w_1, ..., w_n\}$ consists of the weights of these criteria, where w_k reflects the importance degree of x_k, if all w_k are non-negative and

$$\sum_{k=1}^{n} w_k = 1, \quad k = 1, 2, ..., n \tag{2}$$

2.2. Fuzzy numbers, fuzzy operations, and fuzzy weights elicitation

This subsection introduces the definitions and basic concepts related to fuzzy sets, fuzzy numbers, operational laws, and fuzzy normalisation (for details see [37]).

Definition 2. A fuzzy set $\tilde{a}$ in a universe of discourse X is characterised by a membership function $\mu_{\tilde{a}}(x)$ which associates with each element x in X a real number from the interval [0, 1]. The value $\mu_{\tilde{a}}(x)$ is called the grade of membership of x in $\tilde{a}$.

Definition 3. A fuzzy number is a fuzzy set of on real line R that is both convex and normal.

Definition 4. A triangular fuzzy number $\tilde{n}$ can be denoted by (n_l, n_m, n_u) and its membership function is defined as

$$\mu_{\tilde{n}}(x) = \begin{cases} 0 \text{ for } x \le n_l \\ \dfrac{x - n_l}{n_m - n_l} \text{ for } n_l < x \le n_m \\ 1 \text{ for } x = n_m \\ \dfrac{x - n_u}{n_m - n_u} \text{ for } n_m < x \le n_u \\ 0 \text{ for } x > n_u \end{cases} \tag{3}$$

where n_l represents the smallest value possible, n_m – the most probable value, and n_u – the largest possible value. A triangular number (n_l, n_m, n_u) is positive if $n_l > 0$.

Definition 5. Given any two positive triangular fuzzy numbers $\tilde{n} = (n_l, n_m, n_u)$ and $\tilde{k} = (k_l, k_m, k_u)(n_l, k_l > 0)$, and a positive real number r, the main operations of fuzzy numbers $\tilde{n}$ and $\tilde{k}$ can be expressed as follows

- addition

$$\tilde{n} \oplus \tilde{k} = \left(n_l + k_l, n_m + k_m, n_u + k_u \right) \tag{4}$$

- subtraction

$$\tilde{n} \ominus \tilde{k} = \left(n_l - k_u, n_m - k_m, n_u - k_l \right) \tag{5}$$

- multiplication by scalar

$$\tilde{n} \otimes r = \left(n_l r, n_m r, n_u r \right) \tag{6}$$

- multiplication

$$\tilde{n} \otimes \tilde{k} \cong \left(n_l k_l, n_m k_m, n_u k_u \right) \tag{7}$$

- inverse

$$\tilde{n}^{-1} \cong \left(\frac{1}{n_u}, \frac{1}{n_m}, \frac{1}{n_l} \right) \tag{8}$$

- max of TFN

$$\max\left(\tilde{n}, \tilde{k} \right) = \left(\max\left(n_l, k_l \right),\ \max\left(n_m, k_m \right),\ \max\left(n_u, k_u \right) \right) \tag{9}$$

- min of TFN

$$\min\left(\tilde{n}, \tilde{k} \right) = \left(\min\left(n_l, k_l \right),\ \min\left(n_m, k_m \right),\ \min\left(n_u, k_u \right) \right) \tag{10}$$

- the vertex distance

$$d\left(\tilde{n}, \tilde{k} \right) = \sqrt{\frac{1}{3}\left(\left(n_l - k_l \right)^2 + \left(n_m - k_m \right)^2 + \left(n_u - k_u \right)^2 \right)} \tag{11}$$

Ranking fuzzy numbers is often a necessary step in many mathematical models, and a large number of ranking methods have been proposed [17, 13]. Ranking methods are classified into four major classes [17]: preference relations, fuzzy mean and spread, fuzzy scoring, and linguistic expressions. In this paper, we use the concept of defuzzification for rank ordering of fuzzy numbers which consists of the selection of a specific real element based on the output fuzzy set and the conversion of fuzzy numbers into real numbers.

Definition 6. The crisp real number corresponding to the triangular fuzzy number $\tilde{n} = \left(n_l, n_m, n_u \right)$ is obtained as follows [35]

- first of maxima

$$d_{\text{FOM}}\left(\tilde{n}\right)=n_m \tag{12}$$

• center of gravity

$$d_{\text{COG}}\left(\tilde{n}\right)=\frac{n_l+n_m+n_u}{3} \tag{13}$$

We define the ranking of fuzzy triangular fuzzy numbers by defuzzification formulas as follows.

Definition 7. Given any two positive triangular fuzzy numbers $\tilde{n}=\left(n_l,n_m,n_u\right)$ and $\tilde{k}=\left(k_l,k_m,k_u\right)$,

$$\tilde{n}\succcurlyeq_X \tilde{k}\Leftrightarrow d_X\left(\tilde{n}\right)\geq d_X\left(\tilde{k}\right) \tag{14}$$

$$\tilde{n}\approx_X \tilde{k}\Leftrightarrow d_X\left(\tilde{n}\right)=d_X\left(\tilde{k}\right) \tag{15}$$

where $X\in$ {FOM, COG}.

In general, in the FOM (first of maxima) method, the defuzzification value is determined for the first value for which the membership function reaches the maximum value of one. In the case of the triangular fuzzy number, $\tilde{n}=\left(n_l,n_m,n_u\right)$, this occurs for n_m. Despite the simplicity, it is worth noting that the use of this method is associated with a large loss of potential information contained in the triangular fuzzy number (i.e., symmetry or its lack, the width of the fuzzy range, the position on the $0X$ axis). The centre of gravity (COG) defuzzification is one of the most popular and useful techniques which take into account all the parameters of a fuzzy number.

Fuzzy normalisation can be considered as a fuzzy extension of the crisp normalisation using fuzzy addition and fuzzy division [16]. In the literature, normalisation methods for both interval and fuzzy weights have been proposed [32, 46, 53, 67]. It is worth noting that in crisp normalisation only the individual weights need to be considered, while in fuzzy normalisation we have to consider all components of positive fuzzy numbers (the smallest, the most probable, the largest possible values), as well as the spreads of the fuzzy numbers.

It should also be noticed that there is no strict consensus which of the normalisation procedures is the best [16, 67, 54]. For our paper, to define fuzzy weights for positive triangular numbers, we adopt Wang and Elhag's [67] approach. Derived from the definition of normalised interval weights given by Wang and Elhag [67], the normalised fuzzy weights in the case of triangular fuzzy numbers can be defined as follows.

Definition 8. Let $X = \{x_1, ..., x_n\}$ be the set of criteria. We say that the set $W = \{\tilde{w}_1, ..., \tilde{w}_n\}$ is the set of normalised fuzzy criteria weights, where $\tilde{w}_k = (w_{kl}, w_{km}, w_{ku})$, $k = 1, 2, ..., n$, defined on [0, 1] express the importance degrees of x_k, if all $\tilde{w}_k$ are positive fuzzy numbers and the following holds

$$\sum_{j=1}^{n} w_{jm} = 1 \tag{16}$$

$$w_{kl} + \sum_{j=1, j\neq k}^{n} w_{ju} \geq 1 \tag{17}$$

$$w_{ku} + \sum_{j=1, j\neq k}^{n} w_{jl} \leq 1 \tag{18}$$

for every $k = 1, 2, ..., n$.

Let us assume that the decision-maker assigns to each criterion x_k individual positive fuzzy weights $\tilde{r}_k = (r_{kl}, r_{km}, r_{ku})$ representing the importance of the criteria ($k = 1, 2, ..., n$). To normalise the positive fuzzy numbers, $\tilde{r}_k = (r_{kl}, r_{km}, r_{ku})$, where $r_{kl} > 0$, we apply Wang and Elhag's formula [67] in the following way

$$\tilde{w}_k = \left(\frac{r_{kl}}{r_{kl} + \sum_{j=1, j\neq k}^{n} r_{ju}}, \frac{r_{km}}{\sum_{j=1}^{n} r_{jm}}, \frac{r_{ku}}{r_{ku} + \sum_{j=1, j\neq k}^{n} r_{jl}} \right), \quad k = 1, 2, ..., n \tag{19}$$

Wang and Elhag [67] show that the normalisation formula (19) satisfies the conditions (17) and (18). The condition (16) is easy for computation.

3. Methods of criteria weighting. A short overview

In this section, we present a short overview of the literature on the elicitation of criteria weights. In the literature, many criteria weighting methods have been proposed; for a comparison, see [61, 10, 11, 62]. Criteria weighting methods are usually classified into three categories: subjective weighting, objective weighting, and combination weighting methods. Criteria weights determined by the subjective weighting methods depend only on the preference of decision-makers. The subjective crisp methods include the tradeoff method and the pricing-out method [38], the ratio method [26], the Delphi method [31], conjoint methods [29], the ranking ordering method [56, 58], the analytic hierarchy process (AHP) [52], the swing method [39], the point allocation (PA) method [25], the direct rating (DR) method [12], the eigenvector method [59], LINear ProgrAmming of preference Comparisons [30], FITradeoff [2], and others [31].

By contrast, objective crisp weights are obtained by calculations based on the analysis of the initial data disregarding the subjective judgment information of the DM. They include the entropy method [31, 69, 75], the standard deviation (SD) method [24], the CRITIC (criteria importance through intercriteria correlation) method [24], the maximising deviation method [74] and ideal point method [41].

Fuzzy methods take into account the imprecise, vague, and incomplete information about criteria weights. The analytic hierarchy process is a useful weight estimation technique. In fuzzy AHP, the interval and fuzzy comparison matrixes are also used to express the DM's uncertain preference information. Saaty and Vargas [53] introduce a simulation approach to find interval weights from interval comparison matrices. Van Laarhoven and Pedrycz [64] consider treating elements in a comparison matrix as fuzzy numbers, and they employ a logarithmic least-squares method to generate fuzzy weights. Csutor and Buckley [20] propose a Lambda–Max method to find fuzzy weights. Lan et al. [40] present a method for deriving weights from an interval comparison matrix. Wang and Elhag [68] suggest a goal programming method to obtain interval weights from a consistent or inconsistent interval comparison matrix. Furthermore, Wang et al. [71] offer a linear programming method and DEA approach for generating the most favourable weights (LP–GFW) from pairwise comparison matrices. Wang and Zhang [72] give an intuitionistic fuzzy decision method based on prospect theory and the evidential reasoning approach in which the criteria values are intuitionistic fuzzy numbers and the information of attribute weights is unknown.

Some researchers suggest integrated methods for determining criteria weights. Recently, several combinations or optimal weighting methods have been proposed and developed. Integrated methods determine the weights of criteria using both the DM's subjective information and objective decision matrix information. Wang and Lee [68] propose a fuzzy TOPSIS method integrating subjective and objective weights. Subjective weights are assigned by decision-makers (DM) and normalised into a comparable scale, while objective weights are based on Shannon's entropy theory. Ma et al. [41] put forward a two-objective mathematical programming model. The integrated approach by Fan et al. [28] is based on the integration of the DM's fuzzy preference information on decision alternatives with objective decision matrix information. Wang and Parkan [66] integrate the DM's fuzzy preference relation on decision alternatives, the DM's multiplicative preference relation on the weights of criteria, and objective decision matrix information into a general model framework.

4. Rank ordering criteria weighting methods

4.1. Crisp rank ordering criteria weighting methods

Consider a multiple criteria decision-making problem with a finite set of n criteria, and let $X = \{x_1, x_2, ..., x_n\}$ be the set of criteria. We assume here that the weights reflect the relative importance of each decision criterion, and that they are normalised by making their sum equal to 1. The input is obtained as a list of n prioritised (ranked) criteria, where ranks are inversely related to weights. More exactly, the criteria are weighted by ranks in ascending order (the most important criterion is given rank 1, the second criterion rank 2, …, criterion k has rank k, and the least important criterion is given rank n). Our objective is to convert the list of ranks 1, 2, ..., n into numerical weights $w_1, w_2, ..., w_n$ for the n criteria.

Step 1. Ranking the criteria according to their importance
We have a list of n ranked criteria:

$$w_1 \geq ... \geq w_k \geq ... \geq w_n, \quad k = 1, 2, ..., n \tag{20}$$

Step 2. Weighting the criteria from their ranks
The numerical weights corresponding to the ranks are derived by a mathematical function of its rank and the total number of criteria. Stillwell et al. [58] propose three functions: rank reciprocal (inverse), rank sum (linear), and rank exponent weights, while Solymosi and Dompi [56] and Barron [6] the rank order centroid weights.

Rank sum weight method. In the rank sum (RS) procedure with n criteria, rank k is assigned the individual weight $n - k + 1$ which is normalised by dividing by the sum of the ranks [58]

$$w_k(\text{RS}) = \frac{n-k+1}{\sum_{j=1}^{n} n-j+1} = \frac{2(n-k+1)}{n(n+1)} \tag{21}$$

where k is the rank of the kth criterion, $k = 1, 2, ..., n$.

A generalisation of the rank sum method is the rank exponent weigh method (RE) [58]:

$$w_k(\text{RE}) = \frac{(n-k+1)^p}{\sum_{j=1}^{n}(n-j+1)^p} \tag{22}$$

where k is the rank of the kth criterion, p a parameter describing the weights, $k = 1, 2, ..., n$.

The parameter p controls the distribution of the weights: the higher the value of p, the steeper the weight distribution. For $p = 0$, all the criteria have the same weights, for $p = 1$, we obtain the rank sum technique.

Inverse or reciprocal weights [58]. The reciprocal weights (RS) method with n criteria uses the reciprocals of the ranks: rank k is assigned the individual weight $1/k$, which is normalised by dividing by the sum of the reciprocals

$$w_k(\mathrm{RR}) = \frac{\frac{1}{k}}{\sum_{j=1}^{n} \frac{1}{j}} \tag{23}$$

where k is the rank of the kth criterion, $k = 1, 2, ..., n$.

Rank order centroid (ROC) [56]. The rank order centroid (ROC) method uses the centroid (centre of mass) of the simplex defined by the ranking of the criteria; with more criteria, the error for ranked criteria will be much smaller. The formula is

$$w_k(\mathrm{ROC}) = \frac{1}{n}\sum_{j=k}^{n} \frac{1}{j} \tag{24}$$

where k is the rank of the kth criterion, $k = 1, 2, ..., n$.

4.2. A comparison of crisp rank ordering criteria weighting methods

There have been several studies comparing the decision quality of different weighting methods. Some of them also compare weight functions and found centroid weights to be superior in terms of accuracy and ease of use. For instance, Olson and Dorai [45] compare centroid weights to AHP on a student job selection problem. They state that *the results of the study clearly indicate that neither AHP nor the centroid approach would provide a tool that could be expected to totally reflect decision-maker preference. However, both approaches could be relied upon to generally reflect decision-maker preferences*. They also conclude that centroid weights provide almost the same accuracy while requiring much less input and mental effort from decision-makers. Edwards and Barron [27] extend SMART into SMARTER (SMART Exploiting Ranks), using centroid weights. Their paper also proposes tests for the usability of these approximations. Barron and Barrett [7] offer an analysis of the effectiveness of centroid weights in SMARTER. In another paper, Barron and Barrett [8] compare the centroid weights with rank sum and reciprocal (inverse) weights using a simulation study and report that all formulae are efficacious in determining the best multi-attribute alternative, but the ROC weights are more accurate than the other rank based formulae. They also recommend the ROC technique as a useful tool because ROC-based analysis is straightforward and efficacious. Srivastava et al. [57] use simulation experiments to compare five weight elicitation methods, including rank sum and centroid weights; they also find centroid weights to be superior to other methods. Also, Jia et al. [33] perform a detailed comparison of several weighting schemes and use simulation to compare centroid and rank sum weights with equal weighting and ratio weights. They report that equal weights do not always perform well, but rank ordered centroid weights based on an ordering of criteria only lead to the same choices as actual weights do. Noh and Lee [44] compare centroid weights with AHP and fuzzy methods and find that the simplicity and ease of use of centroid weights make it a practical method for determining criteria weights. Barron and Barrett [8] suggest that rank reciprocals are more accurate than rank sum, but in another paper Jia [34] prove that if knowledge of weights is highly limited, then rank sum weights are better. In a series of papers, Danielson and Ekenberg [21, 22] analyse the relevance of rank ordering weighting methods and suggest more robust methods as candidates for modelling and analysing multi-criteria decision problems. Alfares and Duffua [1] devise an empirically developed, evaluated, and validated methodology, based on a set of experiments involving students, to convert ordinal ranking of several criteria into numerical weights.

The decision regarding weight determination will strongly influence the final results of decision-making. The choice of the weighting method depends mainly on the knowledge of the underlying distributions of the "true" weight. The ROC approach to rank order has a clear statistical basis and interpretation, whereas other methods take a more heuristic approach. The rank sum weight (RS), reciprocal weight (RR), and centroid (ROC) methods differ as regards the steepness of the true weights. ROC weights are "steeper", while RS weights are much "flatter". In the ROC method, relatively larger weights are assigned to more important criteria. RS weights decrease linearly from the most important to the least important, while RR weights descend aggressively from the most important to the least important. For a comparison of weight distribution for several numbers of criteria, see [50].

5. Fuzzy rank ordering criteria weighing methods

Since human judgments, including preferences, are often vague and cannot be expressed by exact numerical values, the application of fuzzy concepts in elicitation weights is deemed relevant. In this part of the paper, we propose extension crisp rank-ordering criteria weighting methods. The methods built on the ideas of rank order techniques take into account imprecise information about rank. The fuzzy rank sum, fuzzy rank reciprocal and fuzzy centroid weights techniques are proposed. The weights obtained for each criterion are triangular fuzzy numbers. The input is obtained as a list of n prioritised (fuzzy ranked) criteria, with ranks inversely related to weights. Fuzzy ranks are represented by positive triangular numbers as follows

$$\tilde{k} = \left(k - 0.5,\ k,\ k + 0.5\right) = \left(\frac{2k-1}{2},\ k,\ \frac{2k+1}{2}\right),\quad k = 1, 2, ..., n \tag{25}$$

Note that from formulas (14) and (15), we have $d_X\left(\tilde{k}\right) = k$, where $X \in$ {FOM, COG}.

The criteria are weighted by fuzzy ranks in ascending order (the most important criterion is given rank $\tilde{1}$, the second criterion rank $\tilde{2}$, ..., and the least important rank $\tilde{n}$). We have: $\tilde{w}_k \succcurlyeq_X \tilde{w}_r \Leftrightarrow \tilde{k} \preccurlyeq_X \tilde{r}$. From formulas (14) and (15), we have $\tilde{k} \preccurlyeq_X \tilde{r} \Leftrightarrow d_X\left(\tilde{k}\right) \le d_X\left(\tilde{r}\right)$. Hence, $\tilde{w}_k \succcurlyeq_X \tilde{w}_r \Leftrightarrow k \le r$, where $X \in$ {FOM, COG}. Thus, the fuzzy weight ordering preserves the ascending order of fuzzy ranks and, after defuzzification, the ascending order of crisp ranks.

Step 1. Ranking the criteria according to their importance
We have a list of *n* fuzzy ranked criteria.

$$\tilde{w}_1 \succcurlyeq ...\tilde{w}_k \succcurlyeq ... \succcurlyeq \tilde{w}_n,\quad k = 1, 2, ..., n \tag{26}$$

In the latter analysis, we use defuzzification formulas (9) and (10) but other methods for ranking fuzzy numbers can be used [17, 13].

Step 2. Weighting the criteria from their fuzzy ranks
Once the fuzzy ranks are assigned, the fuzzy weights corresponding to the fuzzy ranks can be derived in different ways

Fuzzy rank sum weight method. In the FRS procedure with *n* criteria, rank $\tilde{k}$ receives the individual weight $n - \tilde{k} + 1 = \left(n - k + 0.5,\ n - k + 1,\ n - k + 1.5\right)$ and next is fuzzy normalized using formula (19)

$$\tilde{w}_k\left(\text{FRS}\right) = \left(\frac{n-k+0.5}{n-k+0.5+\sum\limits_{j=1, j\neq k}^{n}\left(n-j+1.5\right)},\quad \frac{n-k+1}{\sum\limits_{j=1}^{n}\left(n-j+1\right)},\right.$$

$$\left.\frac{n-k+1.5}{n-k+0.5+\sum\limits_{j=1, j\neq k}^{n} n-j+0.5}\right),\ k = 1, 2, ..., n \tag{27}$$

Fuzzy reciprocal weights method (FRS). The FRS method with n criteria uses the reciprocal of the fuzzy ranks, rank $\tilde{k}$ receives the individual weight

$$\tilde{k}^{-1} = \left(\frac{1}{k+0.5}, \frac{1}{k}, \frac{1}{k-0.5} \right)$$

which is fuzzy normalised using the formula (19)

$$\tilde{w}_k\left(\text{FRR}\right) = \left(\frac{\frac{1}{k+0.5}}{\frac{1}{k+0.5} + \sum_{j=1, j\neq k}^{n} \frac{1}{j-0.5}}, \frac{\frac{1}{k}}{\sum_{j=1}^{n} \frac{1}{j}}, \frac{\frac{1}{k-0.5}}{\frac{1}{k-0.5} + \sum_{j=1, j\neq k}^{n} \frac{1}{j+0.5}} \right),$$

$$k = 1, 2, ..., n \tag{28}$$

Fuzzy rank order centroid (FROC). In the FROC, the procedure with n criteria, rank $\tilde{k}$ receives the individual weight

$$\widetilde{iw_k} = \left(\frac{1}{n} \sum_{j=k}^{n} \frac{2}{2j+1}, \ \frac{1}{n} \sum_{j=k}^{n} \frac{1}{j}, \ \frac{1}{n} \sum_{j=k}^{n} \frac{2}{2j-1} \right)$$

Note that $\sum_{k=1}^{n} \left(\frac{1}{n} \sum_{j=k}^{n} \frac{1}{j} \right) = 1$, thus the formula for the fuzzy rank order centroid fuzzy weights is the following (19)

$$\tilde{w}_K\left(\text{FROC}\right) = \left(\frac{\frac{1}{n} \sum_{j=k}^{n} \frac{2}{2j+1}}{\frac{1}{n} \sum_{j=k}^{n} \frac{2}{2j+1} + \sum_{j=1, j\neq k}^{n} \left(\frac{1}{n} \sum_{j=k}^{n} \frac{2}{2j-1} \right)}, \ \frac{1}{n} \sum_{j=k}^{n} \frac{1}{j}, \right.$$

$$\left. \frac{\frac{1}{n} \sum_{j=k}^{n} \frac{2}{2j-1}}{\frac{1}{n} \sum_{j=k}^{n} \frac{2}{2j-1} + \sum_{j=1, j\neq k}^{n} \left(\frac{1}{n} \sum_{j=k}^{n} \frac{2}{2j+1} \right)} \right), \quad k = 1, 2, ..., n \tag{29}$$

Lemma. Let $\tilde{w}_k\left(FA\right) = (w_{kl}, w_{km}, w_{ku})$ for $k = 1, 2, ..., n$. We have the following:

i) $d_{\text{FOM}}\left(\tilde{w}_k(FA)\right) = w_k(A)$,

ii) $\sum_{j=1}^{n} w_{jm} = 1$,

iii) $w_{kl} + \sum_{j=1, j\neq k}^{n} w_{ju} \geq 1$,

iv) $w_{ku} + \sum_{j=1, j\neq k}^{n} w_{jl} \leq 1$.

where $A \in$ {RS, RR, ROC}, FA = {FRS, FRR, FROC}, $k = 1, 2, ..., n$.

Proof. i) follows for: FS from (21), (27) and (12); RR from (23), (28) and (12); ROC from (24), (29) and (12); (ii) follow from (27) after computation; (iii) and (iv) from (19) (see [67]).

6. Numerical example

To illustrate the input of rank order weighting methods in multi-criteria decision-making a numerical example is supplied. Three rank ordering crisp criteria weighing techniques (sum weight, reciprocal weights, and centroid weights), as well as three fuzzy rank ordering criteria weighing methods (fuzzy sum weight, fuzzy reciprocal weights and fuzzy centroid weights) were used.

Example. Let us assume that our multi-criteria decision problem consists of the set of four alternatives A_1, A_2, A_3, A_4, and the set of four criteria X_1, X_2, X_3, X_4, where, for simplicity, all criteria are of the benefit type. Let us also assume, for simplicity, that only fuzzy weights are considered in this example. The decision matrix is given in Table 1.

Table 1. The decision matrix

	X_1	X_2	X_3	X_4
A_1	15	5	4	6
A_2	18	4	7	4
A_3	12	8	10	2
A_4	17	4	3	3

At the first step, a set of four alternatives is evaluated using SAW (simple additive weighting) and FSAW (fuzzy simple additive weighting) methods. SAW (also known as weighted linear combination or scoring method) is a simple and most often used multi-criteria technique [18] based on the weighted average. An evaluation score is calculated for each alternative by multiplying the normalised value assigned to the alternative of that criterion by the weights of relative importance and then by summing the products for all criteria. The normalised values for the benefit criteria (more is better) are calculated using the following formula

$$c_{ij} = \frac{x_{ij} - \min_i x_{ij}}{\max_i x_{ij} - \min_i x_{ij}} \tag{30}$$

and for cost criteria (less is better) using the following formula

$$c_{ij} = \frac{\max_i x_{ij} - x_{ij}}{\max_i x_{ij} - \min_i x_{ij}} \tag{31}$$

where x_{ij} is the score of alternative i concerning jth criterion.

In *SAW*, the final score of each alternative is calculated as follows (crisp weights)

$$SAW_i = \sum_{j=1}^{n} c_{ij} w_j = \sum_{j=1}^{n} r_{ij} \tag{32}$$

where $SA\mathrm{W}_i$ is the score for alternative i, c_{ij} is the normalised value of ith alternative concerning jth criterion, w_{ij} is the weight of jth criterion, r_{ij} is the weighted normalised value of ith alternative concerning jth criterion. The final scores SAW_i are ranked: the higher the value of SAW_i, the higher the rank.

SAW can be easily extended to fuzzy values for alternatives as follows [51, 77] (fuzzy weights)

$$\mathrm{FSAW}_i = \sum_{j=1}^{n} c_{ij} \tilde{w}_j = \sum_{j=1}^{n} \hat{r}_{ij} \tag{33}$$

where FSAW_i is the fuzzy score for alternative i, c_{ij} is the normalised value of ith alternative concerning jth criterion and $\tilde{w}_j$ is the fuzzy weight of jth criterion, $\hat{r}_{ij}$ is the fuzzy weighted normalised value of ith alternative concerning jth criterion[1].

[1]Let us also note that some other fuzzy extensions of the weighted average operations are proposed in [47].

Table 2. Values for criteria weights
given by different formulas for four criteria

Crisp weights	w_1	w_2	w_3	w_4
Rank sum weight (RS)	0.40	0.30	0.20	0.10
Reciprocal weight (RR)	0.48	0.24	0.16	0.12
Rank order centroid (ROC)	0.52	0.27	0.15	0.06

The values for crisp weights given by different formulas for the four criteria are presented in Table 2, and for the fuzzy weights in Table 3.

Table 3. Approximations for fuzzy criteria weights given by different formulas for four criteria

Fuzzy weights	$\tilde{w}_1$	$\tilde{w}_2$	$\tilde{w}_3$	$\tilde{w}_4$
Fuzzy rank sum weight (FRS)	(0.32, 0.40, 0.50)	(0.23, 0.30, 0.39)	(0.14, 0.20, 0.28)	(0.05, 0.1, 0.17)
Fuzzy reciprocal weight (FRR)	(0.33, 0.48,0.69)	(0.13, 0.24, 0.36)	(0.09, 0.16, 0.24)	(0.07, 0,12, 0.17)
Fuzzy rank order centroid (FROC)	(0.40, 0.52,0.67)	(0.17, 0.27, 0.37)	(0.09, 0.15, 0.20)	(0.04, 0.06, 0.09)

In FSAW, since the final rating of each alternative is also a fuzzy number, the last step requires that fuzzy numbers be ranked. For our example, defuzzification formulas (12) and (13) are used. Next, the FSAW function is used to rank the alternatives. Table 4 provides the values computed by formula (32) and the ranking of the alternatives based on various criteria weighting methods.

Table 4. Scores for SAW and rank ordering of alternatives
for different criteria weighting methods

Alternative	Weight method					
	RS		RR		ROC	
	Value	Rank	Value	Rank	Value	Rank
A_1	0.404	3	0.443	2	0.409	4
A_2	0.564	1	0.631	1	0.636	1
A_3	0.500	2	0.400	4	0.420	3
A_4	0.358	4	0.430	3	0.448	2

Note that we obtain different alternative rank orderings, depending on weights determination. From Table 4, it can be shown that the preferred order of alternatives can be $A_2 \succ A_3 \succ A_1 \succ A_4$ (RS weight), $A_2 \succ A_1 \succ A_4 \succ A_3$ (RR weight) or $A_2 \succ A_4 \succ A_3 \succ A_1$ (ROC weights).

Table 5 provides the values computed by formula (33), while Table 6 provides the ranking of the alternatives based on various criteria weighting methods and defuzzification formulas.

Table 5. Fuzzy scores of alternatives for FSAW
for different fuzzy criteria weighting methods

Alternative	Weight method		
	FRS	FRR	FROC
A_1	(0.288,0.404, 0.558)	(0.280, 0.443, 0.639)	(0.295, 0.409, 0.546)
A_2	(0.425, 0.564,0.745)	(0.416, 0.631, 0.912)	(0.471, 0.636,0.829)
A_3	(0.370, 0.500, 0.650)	(0.220, 0.400, 0.600)	(0.260, 0.420, 0.570)
A_4	(0.279, 0.358,0.459)	(0.293, 0.430, 0.618)	(0.343, 0.448, 0.581)

Table 6. The ranking of alternatives for FSAW
for different fuzzy criteria weighting and defuzzification formulas

Alternative	Weight method					
	FRS		FRR		FROC	
	d_{FOM}	d_{COG}	d_{FOM}	d_{COG}	d_{FOM}	d_{COG}
A_1	0.404 (3)	0.416 (3)	0.443 (2)	0.454 (2)	0.409 (4)	0.4168 (3)
A_2	0.564 (1)	0.578 (1)	0.631 (1)	0.653 (1)	0.636 (1)	0.6455 (1)
A_3	0.500 (2)	0.513 (2)	0.400 (4)	0.407 (4)	0.420 (3)	0.4167 (4)
A_4	0.358 (4)	0.366 (4)	0.430 (3)	0.447 (3)	0.448 (2)	0.4575 (2)

Note that the proposed fuzzy rank ordering criteria weighting techniques are simply extensions of the equivalent methods for crisp weights using the FOM defuzzification formula (12). We can also see that the triangular numbers representing weights are not symmetrical, so the defuzzification formula (13) for COG gives different results than the analogous formula (12) for FOM does, but the ordering obtained is the same as in the crisp case for FRS and FRR method. Other ranking techniques can provide different results. In the analysis, we have to take into account not only the steepness of weight but also the properties of fuzzy numbers representing the fuzzy weights in the selection of the most appropriate fuzzy weights elicitation technique.

Thus finally, a set of four alternatives is evaluated using TOPSIS (technique for order performance by similarity to ideal solution) and FTOPSIS (fuzzy technique for order performance by similarity to ideal solution) methods. TOPSIS, proposed by Hwang and Yoon [31], is one of the most known methods for solving MCDM problems. This method is based on the concept that the chosen alternative should have the shortest distance to positive ideal solution (PIS) (the solution which minimises the cost criteria and maximises the benefit criteria) and the farthest distance to negative ideal solution (NIS).

The positive ideal solution and the negative ideal solution are defined as follows:

$$\text{PIS} = \left(v_1^+, ..., v_n^+\right) = \left(\max_i r_{i1}, ..., r_{in}\right) \tag{34}$$

$$\text{NIS} = \left(v_1^-, ..., v_n^-\right) = \left(\min_i r_{i1}, ..., \min_i r_{in}\right) \tag{35}$$

Next, the values of distances of the alternative from PIS and NIS are calculated using the Euclidean method

$$d_i^+(A_i,\ \text{PIS}) = \sqrt{\sum_{j=1}^{n}\left(r_{ij} - v_j^+\right)^2} \tag{36}$$

$$d_i^-(A_i,\ \text{NIS}) = \sqrt{\sum_{j=1}^{n}\left(r_{ij} - v_j^-\right)^2} \tag{37}$$

The relative closeness to the ideal solution is defined as follows:

$$\text{TOPSIS}_i = \frac{d_i^-(A_i,\ \text{NIS})}{d_i^-(A_i,\ \text{NIS}) + d_j^+(A_i,\ \text{PIS})} \tag{38}$$

where $0 \le \text{TOPSIS}_i \le 1$, $i = 1, 2, ..., m$.

The fuzzy positive ideal solution and the fuzzy negative ideal solution are defined as follows:

$$\text{FPIS} = \left(\hat{v}_1^+, ..., \hat{v}_n^+\right) = \left(\max_i \hat{r}_{i1}, ..., \max_i \hat{r}_{in}\right) \tag{39}$$

$$\text{FNIS} = \left(\hat{v}_1^-, ..., \hat{v}_n^-\right) = \left(\min_i \hat{r}_{i1}, ..., \min_i \hat{r}_{in}\right) \tag{40}$$

The values of distances of the alternative from FPIS and FNIS are calculated using the vertex distance between two triangular numbers in the following way

$$d_i^+(A_i,\ \text{FPIS}) = \sum_{j=1}^{n} d\left(\hat{r}_{ij}, \hat{v}_j^+\right), \quad i = 1, 2, ..., m \tag{41}$$

$$d_i^-(A_i,\ \text{FNIS}) = \sum_{j=1}^{n} d\left(\hat{r}_{ij}, \hat{v}_j^-\right), \quad i = 1, 2, ..., m \tag{42}$$

where $d(\hat{A}, \hat{B})$ is the vertex distance between two triangular numbers $\hat{A}$, $\hat{B}$ (see formula (11)).

The relative closeness to the fuzzy ideal solution is defined as follows

$$\text{FTOPSIS}_i = \frac{d_i^-(A_i,\ \text{FNIS})}{d_i^-(A_i,\ \text{FNIS}) + d_i^+(A_i,\ \text{FPIS})} \tag{43}$$

The TOPSIS function is used to rank the alternatives. Table 7 provides the values computed by formula (43) and the ranking of the alternatives based on various criteria weighting methods.

Table 7. Scores for TOPSIS and rank ordering of alternatives
for different criteria weighting methods

Alternative	Weight method					
	RS		RR		ROC	
	Value	Rank	Value	Rank	Value	Rank
A_1	0.407	4	0.455	3	0.438	3
A_2	0.570	1	0.657	1	0.654	1
A_3	0.467	3	0.368	4	0.371	4
A_4	0.472	2	0.562	2	0.572	2

Note that we obtain different alternative rank orderings, depending on weights determination. From Table 7, it can be shown that the preferred order of alternatives can be $A_2 \succ A_4 \succ A_3 \succ A_1$ (RS weight), or $A_2 \succ A_4 \succ A_1 \succ A_3$ (RR and ROC weights). It is also worth noting that we obtain different ranking than in SAW procedure. However, in all rankings, the alternative A_2 is the best.

Finally, Table 8 provides the values computed by formula (43) and the ranking of the alternatives based on various criteria weighting methods.

Table 8. The ranking of alternatives for FTOPSIS
for different fuzzy criteria weighting

Alternative	Weight method					
	FRS		FRR		FROC	
	Value	Rank	Value	Rank	Value	Rank
A_1	0.409	3	0.441	2	0.412	4
A_2	0.562	1	0.632	1	0.635	1
A_3	0.499	2	0.402	4	0.417	3
A_4	0.353	4	0.430	3	0.448	2

Note that we obtain alternative rank orderings, depending on weights determination. From Table 8, it can be shown that the preferred order of alternatives can be $A_2 \succ A_3 \succ A_1 \succ A_4$ for FRS method, $A_2 \succ A_1 \succ A_4 \succ A_3$ for FRR method, and $A_2 \succ A_4 \succ A_3 \succ A_1$ for FROC method. Also, in all rankings the alternative A_2 is the best.

7. Conclusions

Criteria weights are usually set subjectively, which means that they are more or less uncertain. Due to their vagueness and subjectivity, crisp data may be inadequate for the estimation of weights in real-life situations. When performance rating and weights cannot be given precisely, fuzzy set theory can introduce an uncertainty of human judgments into the model. Thus, the estimation of the relative weights of alternatives based on fuzzy sets plays an important role in the decision-making process.

In this paper, it has been shown how this kind of uncertain information about rank can be expressed employing fuzzy sets theory tools. To express uncertain normalised weights, normalised fuzzy weights, represented by fuzzy numbers, are introduced. Fuzzy rank ordering criteria weighing methods are useful weight estimation techniques and decision-making tools. These methods are selected for their simplicity and effectiveness. A numerical example is provided to demonstrate the capabilities of the proposed model. One of the practical possibilities of using the presented approach is evaluation negotiation issues while building negotiation scoring system. In the papers [77, 78] the fuzzy FSAW and fuzzy TOPSIS are used for scoring the negotiation offers in ill-structured negotiation problems. The rank ordering criteria weighting methods (crisp and fuzzy) are less cognitively demanding and may reduce biases in weights elicitation producing weights acceptable by DM. They can be also easily implemented in decision support systems.

However, the procedure for assigning fuzzy weights needs further elaboration. Since weights have fuzzy and uncertain characteristics, the choice of an appropriate fuzzy rank ordering weighting method should be supported by the decisionmakers' experience and goals. It should also be taken into consideration that there are several fuzzy multi-criteria techniques where fuzzy rank ordering weights can be applied. In the paper, we only use FSAW and FTOPSIS. It is worth noting that the defuzzification process in FSAW procedure may lose "lots of messages" not to mention that the decision-maker can be confused which of the defuzzification techniques will be more appreciative to him, so other techniques using ordered fuzzy numbers, such as fuzzy TOPSIS, can be useful.

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