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RUIN PROBABILITY OF A DISCRETE-TIME RISK PROCESS WITH PROPORTIONAL REINSURANCE AND INVESTMENT FOR THE EXPONENTIAL AND PARETO DISTRIBUTIONS

The paper focuses on a quantitative analysis of the probability of ruin in a finite time for a discrete risk process with proportional reinsurance and investment of the financial surplus. It is assumed that the total loss on a unit interval has either a light-tailed distribution – exponential distribution or a heavy-tailed distribution – Pareto distribution. The ruin probabilities for the finite-horizons 5 and 10 were determined from recurrence equations. Moreover, the upper bound of the ruin probability is given for the exponential distribution based on the Lundberg adjustment coefficient. This adjustment coefficient does not exist for the Pareto distribution, hence an asymptotic approximation is given for the ruin probability when the initial capital tends to infinity. The numerical results obtained are illustrated by tables and figures.

Keywords: *discrete time risk process, ruin probability, proportional reinsurance, Lundberg's inequality, regularly varying tail*

1. Introduction

In the risk theory, work concerning the financial surplus of insurance companies in continuous time has been proceeding for nearly a century. Very advanced models of the classical continuous risk process have been developed. Although discrete time models are more natural in describing reality, research on discrete processes of financial surplus

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is considerably more modest. A review of the results concerning discrete processes of financial surplus can be found in [6]. This paper is one of a series of papers which try to bring the classical discrete process of a financial surplus closer to the reality of insurance companies, namely, analysis of the investment of a financial surplus, which enhances the security of an insurance company. These problems are considered in [1–3, 9, 10]. Reinsurance has a considerable influence on increasing the security of an insurance company. Results concerning a discrete risk process with investment and reinsurance can be found in [4, 7].

In this paper, we consider the probability of ruin in a finite time for a discrete risk process with proportional reinsurance and investment of the financial surplus. Moreover, we obtain numerical results for particular cases: when the total loss has an exponential or Pareto distribution, and some asymptotic results.

A paper by Cai and Dickson [3] studies the ruin probability for a discrete time risk process with a Markov model describing interest rates. The paper also gives recursive equations for ruin probabilities, generalised Lundberg inequalities and approximations based on the recursive equations. Diaspara and Romera [4] introduced proportional reinsurance in discrete risk processes with investment.

For any form of reinsurance, not only proportional one, Jasiulewicz [7] obtained recursive equations and a Lundberg inequality for the ruin probability in a discrete-time risk process with the Markovian model of the interest rate. Moreover, for the case of proportional reinsurance and reinsurance of stop-loss, the optimal level of retention was considered based on maximising the Lundberg adjustment coefficient.

This paper is a continuation of the research initiated by Jasiulewicz [7]. In addition to theoretical results, we conduct a detailed quantitative analysis for particular distributions of the total loss in a unit period and proportional reinsurance. We consider the ruin probability for a light-tailed distribution (exponential) and a heavy-tailed distribution (Pareto), taking into account investment of the financial surplus according to a random interest rate. Based on these considerations, we give practical conclusions concerning the connections between the initial level of capital and the reinsurance level. We derive the required level of reinsurance of a loss for a fixed capital and the required initial capital for a fixed level of reinsurance in order to achieve a sufficiently low ruin probability.

The quality of the upper bound on the probability of ruin in a finite time obtained using the Lundberg coefficient is illustrated by the example of the exponential distribution. We observe that if an insurer and reinsurer use the same security loading, the adjustment coefficient is a convex function of the reinsurance level, which considerably improves estimation of an upper bound on the ruin probability. However, if the loading of a reinsurer is greater than the loading of an insurer, the adjustment coefficient is not a convex function, which lowers the quality of upper bound estimation. This observation was not taken into account in the numerical examples in Diaspara and Romera [4].

As is known, the Lundberg adjustment coefficient does not exist for heavy-tailed distributions. For distributions of this type, we give Theorem 3 about the approximation of the ruin probability when the initial capital is sufficiently large. The example of the Pareto distribution shows that such an approximation is appropriate and quickly tends to the limit value.

We assume that the expected loss in a unit period has a known monetary value. Without loss of generality, we may assume that the expected values for both considered distributions are equal to 1. For the assumed values of the parameters of the Pareto distribution, the variance does not exist.

In conclusion, below we list new elements, ideas and results which are introduced in this paper:

1. In a continuous risk process, the level of retention is optimal if it minimises the ruin probability which can be determined by maximising the adjustment coefficient according to the level of retention [5]. Thus we can pose the following natural question: Does this property hold for discrete risk processes?

2. The upper bound on the ruin probability obtained by the Lundberg coefficient in the case of proportional reinsurance is given by Diaspara and Romera [4]. A numerical example for the case $\eta = \theta$ shows that this estimate is reasonable. Is this estimate also good for the more natural case $\eta > \theta$?

3. In the case of heavy tailed claims, we give an approximation for the ruin probability. The question is: does the sequence of approximations converge quickly for a sufficiently large initial capital?

For light tailed claims in a discrete risk process with proportional reinsurance, the level of retention minimising a ruin probability cannot be achieved by maximising Lundberg's coefficient as in a continuous risk process. This is pointed out by the calculations contained in Tables 1, 2 and their graphical illustrations given in Figs. 1, 2. The level of retention should be determined using Theorem 1 in such a way as to set the risk of ruin at a level acceptable to an insurer, for example, at the level of 0.05. The retention level for heavy tailed distributions should be determined in the same way (Tables 3, 4 and Figs. 4, 5).

Regarding the second question, our results indicate that the answer is also negative. If an insurer uses a security loading θ which is lower than the loading η of a reinsurer, the Lundberg type upper bound on the ruin probability given by Theorem 2 is a very bad estimate. This is illustrated in Fig. 3. This observation restricts the practical application of Theorem 2.

For heavy tailed claims, based on Theorem 3 in this paper, the approximation of a ruin probability converges quickly for a sufficiently large initial capital. Figure 6 illustrates this.

2. Notation and theorems

The notation, assumptions and theorems 1 and 2 given below come from the paper by Jasiulewicz [7]. In that paper, the following notation and assumptions were used.

1. Let Z_n denote the total loss in the unit period $(n-1, n]$. The loss is calculated at the end of each period. Let us assume that $\{Z_n, n=1,2,...\}$ is a sequence of independent and identically distributed random variables with a common distribution function $W(z)$. The complementary distribution function is denoted by $\overline{W}(z) = 1 - W(z)$.

2. The premium is calculated based on the expected value principle with loading factor $\theta > 0$. The constant premium $c = (1+\theta)\mathrm{E}Z_n$ is paid at the end of each unit period $(n-1, n]$.

3. The insurer's surplus at the moment n is denoted by U_n and is calculated after paying off claims. The surplus U_n is invested at the beginning of the period $(n, n+1]$ at a random interest rate I_n.

4. Let us assume that the interest rates $\{I_n, n=0,1,...\}$ follow a time-homogeneous Markov chain. We further assume that for all $n = 0, 1, \ldots$, the rate I_n takes possible values $i_1, i_2, ..., i_l$. For all n and all states, the transition probability is denoted by

$$\Pr(I_{n+1} = i_t \mid I_n = i_s) = p_{st}$$

and the initial distribution is denoted by

$$\Pr(I_0 = i_s) = \pi_s$$

5. Suppose that the insurer uses reinsurance and that the amount paid by the insurer when the loss Z_n occurs is $h(Z_n, b)$, where the parameter $b > 0$ denotes the retention level. The meaning of the parameter b will be explained based on two examples of the most frequently adopted forms of reinsurance applied in insurance practice.

A. Proportional reinsurance, the function $h(x, b)$ has the form

$$h(x, b) = bx$$

where $b \in (0, 1]$.

B. Stop loss reinsurance, the function $h(x, b)$ has the form

$$h(x,b)=\begin{cases} x, & x\le b \\ b, & x>b \end{cases}$$

where $b>0$.

The following assumption regarding h is obvious: $0\le h(x,b)\le x$. The part of the loss retained by the insurer is denoted by $Z_n^{ce}=h(Z_n,b)$ and its distribution function by $V(z)$. Therefore, $Z_n^{re}=Z_n-Z_n^{ce}$ is the reinsured part of the loss Z_n.

6. Let us assume that the reinsurer calculates the premium rate c_{re} according to the expected value rule with a loading factor η, i.e.

$$c_{re}=(1+\eta)\mathrm{E}\left(Z_n-h(Z_n,b)\right)$$

We assume that $\eta\ge\theta>0$, so an insurer Z_n does not earn without risk if he retains only zero value claims.

7. The premium rate retained by an insurer in a unit period is denoted by $c(b)$ and is given by

$$c(b)=c-c_{re}=(1+\eta)\mathrm{E}h(Z_n,b)-(\eta-\theta)\mu$$

8. Let U_n^b denote the financial surplus of an insurer at the end of the unit period $(n-1,n]$ after the payment of premiums and paying off claims. The process U_n^b considered in the paper is given by

$$U_n^b=U_{n-1}^b(1+I_n)+c(b)-h(Z_n,b)$$

9. The ultimate probability of ruin for this risk process in a finite time is denoted by $\Psi_n^b(u,i_s)$ and is defined by

$$\begin{aligned}\Psi_n^b(u,i_s)&=\Pr\left(\bigcup_{i=1}^{n}\left(U_i^b<0\right)\mid U_0^b=u,I_0=i_s\right)\\ &=\Pr\left(U_i^b<0 \quad \text{for some} \quad i\le n\mid U_0^b=u,\quad I_0=i_s\right)\end{aligned}$$

The ultimate probability of ruin in an infinite time is given by

$$\Psi^b(u, i_s) = \Pr\left(\bigcup_{i=1}^{\infty}\left(U_i^b < 0\right) \mid U_0^b = u, I_0 = i_s\right)$$
$$= \Pr\left(U_i^b < 0 \quad \text{for some} \quad i \geq 1 \mid U_0^b = u, I_0 = i_s\right).$$

Obviously,

$$\Psi^b(u, i_s) = \lim_{n\to\infty} \Psi_n^b(u, i_s)$$

Further research is conducted on proportional reinsurance. The premium rate retained by an insurer is

$$c(b) = \left((1+\eta)b - (\eta - \theta)\right)\mu \tag{1}$$

To avoid the possibility that ruin could occur with the probability 1, it is assumed that

$$\mathrm{E}h(Z_1, b) < c(b) \tag{2}$$

To write a self-contained paper, we give two theorems (1 and 2) from Jasiulewicz [7] which will be used in the analysis of the ruin probability. In a special case of reinsurance, namely proportional reinsurance, theorems analogous to Theorems 1 and 2 were given by Diaspara and Romera [4].

Theorem 1. The probability of an insurer's ruin in a finite time is given recursively in the following way:

$$\Psi_1^b(u, i_s) = \sum_{j=1}^{l} p_{sj}\bar{V}\left(u\left(1+i_j\right)+c(b)\right) \tag{3}$$

$$\begin{aligned}\Psi_{n+1}^b(u, i_s) = \sum_{j=1}^{l} p_{sj} \Big\{ & \bar{V}\left(u\left(1+i_j\right)+c(b)\right) \\ & + \int_0^{u(1+i_j)+c(b)} \Psi_n^b\left(u\left(1+i_j\right)+c(b)-z, i_j\right) dV(z) \Big\}\end{aligned} \tag{4}$$

The probability of ruin in an infinite time:

$$\Psi^b(u,i_s)=\sum_{j=1}^{l}p_{sj}\left\{\bar{V}\left(u\left(1+i_j\right)+c\left(b\right)\right)\right.$$
$$\left.+\int_0^{u\left(1+i_j\right)+c\left(b\right)}\Psi^b\left(u\left(1+i_j\right)+c\left(b\right)-z,i_j\right)dV\left(x\right)\right\}$$

where

$$c\left(b\right)=\left(1+\eta\right)\mathrm{E}h\left(Z_n,\ b\right)-\left(\eta-\theta\right)\mu \tag{5}$$

Proof. Let $Z_1^{ce}=z,\ I_1=i_j$. If $z>u\left(1+i_j\right)+c\left(b\right)$, then ruin will occur in the first period $\left(0,1\right]$. Therefore,

$$\Psi_1^b\left(u,i_s\right)=\sum_{j=1}^{l}p_{sj}\Pr\left(Z_1^b>u\left(1+i_j\right)+c\left(b\right)\mid I_1=i_1,I_0=i_s\right)$$
$$=\sum_{j=1}^{l}p_{sj}\bar{V}\left(u\left(1+i_j\right)+c\left(b\right)\right)$$

Ruin in the first $n+1$ periods can occur in one of two mutually exclusive ways: ruin occurs in the first period or ruin does not occur in the first period but occurs in one of the following periods.

Since the process U_n^b is stationary with independent increments, then

$$\Psi_{n+1}^b\left(u,i_s\right)=\sum_{j=1}^{l}p_{sj}\int_0^{\infty}\Pr\left(\bigcup_{k=1}^{n+1}\left(u_k^b<0\mid Z_1^b=z,I_1=i_s\right)\right)dV\left(z\right)$$
$$=\sum_{j=1}^{l}p_{sj}(\bar{V}\left(u\left(1+i_j\right)+c\left(b\right)\right)+\int_0^{u\left(1+i_j\right)+c\left(b\right)}\Psi_n^b\left(u\left(1+i_j\right)+c\left(b\right)-z,i_j\right)dV\left(z\right))$$

The probability of ruin in an infinite time is obtained by taking a two-sided limit in the above formula for $n\to\infty$. □

Recurrence formulae for the ruin probability can be presented in a matrix form which simplifies calculations when using computer packages[3].

Let

$$\Psi_n^b(u) = \left[\Psi_n^b(u, i_1), \Psi_n^b(u, i_2), \ldots, \Psi_n^b(u, i_l)\right]$$

and

$$\vec{V}_n = \left[v_1^{(n)}, v_2^{(n)}, \ldots, v_l^{(n)}\right]$$

where

$$v_j^{(1)} = \bar{V}\left(u\left(1+i_j\right)+c(b)\right)$$

and for $n \geq 2$

$$v_j^{(n+1)} = v_j^{(1)} + \int_0^{u\left(1+i_j\right)+c(b)} \Psi_n^b\left(u\left(1+i_j\right)+c(b)-z, i_j\right) dV(z)$$

Then we can write Eqs. (3) and (4) in a matrix form

$$\Psi_n^b(u) = \vec{V}_n P^T$$

Theorem 2. If $\mathrm{E}h(Z_1, b) < c(b)$ and there exists a positive constant $R(b)$ fulfilling the equation

$$\mathrm{E}\mathrm{e}^{R(b)\,h(Z_1,b)} = \mathrm{e}^{R(b)c(b)} \tag{6}$$

then an upper bound on the probabilities of ruin in finite and infinite time is given by

$$\Psi_n^b(u, i_s) \leq \Psi^b(u, i_s) \leq \xi(b)\,\mathrm{E}\left(\mathrm{e}^{-R(b)\,u(1+I_1)} \mid I_0 = i_s\right) \tag{7}$$

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[3]The calculations were done by means of the Maxima package: http://maxima.sourceforge.net/

where

$$\xi(b) = \sup_{x \ge c(b)} \frac{\mathrm{e}^{R(b)x}\overline{V}(x)}{\int_x^{\infty} \mathrm{e}^{R(b)z} dV(z)}, \quad 0 < \xi(b) \le 1 \tag{8}$$

Proof. For every $x \ge 0$ we have

$$\begin{aligned}\overline{V}(x+c(b)) &= \frac{\mathrm{e}^{R(b)x}\overline{V}(x+c(b))}{\int_x^{\infty} \mathrm{e}^{R(b)z}\, dV(z+c(b))} \mathrm{e}^{-R(b)x} \int_x^{\infty} \mathrm{e}^{R(b)z}\, dV(z+c(b)) \\ &= \frac{\mathrm{e}^{R(b)(x+c(b))}\overline{V}(x+c(b))}{\int_{x+c(b)}^{\infty} \mathrm{e}^{R(b)y}\, dV(y)} \mathrm{e}^{-R(b)x} \int_{x+c(b)}^{\infty} \mathrm{e}^{R(b)(y-c(b))}\, dV(y)\end{aligned} \tag{9}$$

Let

$$g(t) = \frac{\mathrm{e}^{R(b)(t)}\overline{V}(t)}{\int_t^{\infty} \mathrm{e}^{R(b)y}\, dV(y)}$$

Then

$$\begin{aligned}\overline{V}(x+c(b)) &\le \sup_{x \ge 0}\{g(x+c(b))\} \mathrm{e}^{-R(b)x} \int_{x+c(b)}^{\infty} \mathrm{e}^{R(b)(y-c(b))}\, dV(y) \\ &= \beta \mathrm{e}^{-R(b)x} \int_{x+c(b)}^{\infty} \mathrm{e}^{R(b)(y-c(b))}\, dV(y)\end{aligned} \tag{10}$$

where

$$\beta = \sup_{y \ge c(b)} g(y)$$

From Equation (6), we obtain

$$\overline{V}(x+c(b)) \le \beta \mathrm{e}^{-R(b)x} \int_{-\infty}^{\infty} \mathrm{e}^{R(b)(y-c(b))}\, dV(y) = \beta \mathrm{e}^{-R(b)x} \tag{11}$$

Whereas inequality (8) follows from the fact that for $z \geq t$ the inequality $\exp\left(R(b)z\right)$ $\geq \exp\left(R(b)t\right)$ holds. Therefore,

$$\frac{\int_t^\infty \mathrm{e}^{R(b)z}\, dV(z)}{\mathrm{e}^{R(b)t}\overline{V}(t)} \geq \frac{\mathrm{e}^{R(b)t}\int_t^\infty dV(z)}{\mathrm{e}^{R(b)t}\overline{V}(t)} = 1$$

Inequality (8) is obtained by transforming this inequality. In the next step, we prove (7) inductively. From Theorem 1 and inequality (11), we have

$$\Psi_1^b(u, i_s) \leq \sum_{j=1}^{l} p_{sj}\beta \mathrm{e}^{-R(b)u(1+i_j)} = \beta \mathrm{E}\left(\mathrm{e}^{-R(b)u(1+I_1)} \mid I_0 = i_s\right)$$

From the induction assumption

$$\Psi_n^b(u, i_s) \leq \beta \mathrm{E}\left(\mathrm{e}^{-R(b)u(1+I_1)} \mid I_0 = i_s\right)$$

and Theorem 1 we have

$$\Psi_{n+1}^b(u, i_s) \leq \sum_{i=j}^{l} p_{sj}\left(\beta \mathrm{e}^{-R(b)u(1+i_j)} \int_{u(1+i_j)+c(b)}^{\infty} \mathrm{e}^{R(b)(y-c(b))}\, dV(y)\right.$$
$$\left. + \int_0^{u(1+i_j)+c(b)} \beta \mathrm{E}\left(\mathrm{e}^{-R(b)\left(u(1+i_j)-z+c(b)\right)(1+I_1)} \mid I_0 = i_s\right)\right)$$

Since

$$\mathrm{E}\left(\mathrm{e}^{-R(b)\left(u(1+i_j)-z+c(b)\right)(1+I_1)} \mid I_0 = i_s\right) \leq \mathrm{e}^{-R(b)\left(u(1+i_j)-z+c(b)\right)} \tag{12}$$

then

$$\Psi_{n+1}^b(u, i_s) \leq \sum_{i=j}^{l} p_{sj}\beta \mathrm{e}^{-R(b)u(1+i_j)} \int_{-\infty}^{\infty} \mathrm{e}^{R(b)(y-c(b))}\, dV(y) = \beta \mathrm{E}\left(\mathrm{e}^{-R(b)u(1+I_1)} \mid I_0 = i_s\right)$$

Taking limits for $n \to \infty$, we obtain inequality (7). □

Theorem 1 gives recurrence formulae for the ruin probability and Theorem 2 gives an upper bound on the ruin probability using the Lundberg adjustment coefficient, which exists only for light-tailed distributions. Therefore, one cannot use Theorem 2 to estimate the ruin probability for heavy-tailed distributions. In this case, we will use the asymptotic ruin probability in the case where initial capital tends to infinity, whereas the total loss has a distribution with a regularly varying tail.

Definition 1. A distribution F on $(-\infty, \infty)$ has a regularly varying tail if there exists some constant $\alpha \geq 0$ such that for every $y > 0$ the following holds:

$$\lim_{x\to\infty} \frac{\bar{F}(xy)}{\bar{F}(x)} = y^{-\alpha}$$

The class of such distributions is denoted by $R_{-\alpha}$.

For such a class of distributions, we give an approximation of the ruin probability with proportional reinsurance, which has not previously been considered in the actuarial literature.

Theorem 3. Let the total loss Z_n have cumulative distr ibution function (cdf) $W \in R_{-\alpha}$ for some $\alpha > 0$. If $1 + I_n > 0$ for any fixed $i_0 = i_s$, and there exists a finite positive moment of rank α of the discount factor $(1 + I_1)^{-1}$, then in the case of proportional reinsurance for every $I_0 = i_s$ and every n we have

$$\Psi_n^b(u, i_s) \sim c_n(i_s)\bar{V}(u) \tag{13}$$

if $u \to \infty$, where $c_n(i_s)$ are given recursively

$$I_0 = i_s c_n(i_s) = \mathrm{E}\left((1 + c_{n-1}(I_1)) \left(\frac{1}{1+I_1} \right)^{\alpha} \mid I_0 = i_s \right) \tag{14}$$

with the initial condition $c_0(i_s) = 0$ for $n = 1, 2, \ldots$

Proof. In the paper by Cai and Dickson [3], the above theorem was proved in the case where an insurer does not apply reinsurance, but invests the financial surplus. It is sufficient to remark that with proportional reinsurance $Z_n^{ce} = bZ_n$, if Z_n has a distribution with a regularly changing tail of index α, then Z_n^{ce} also has a distribution with a regularly varying tail of index α. This follows from

$$\lim_{x\to\infty}\frac{\bar{V}(xy)}{\bar{V}(x)} = \lim_{x\to\infty}\frac{\bar{W}\left(\frac{yx}{b}\right)}{\bar{W}\left(\frac{x}{b}\right)} = \lim_{z\to\infty}\frac{\bar{W}(yz)}{\bar{W}(z)} = y^{-\alpha}$$

where $z = x/b \to \infty$, if $x \to \infty$, because $b > 0$. Therefore, Theorem 3 is fulfilled for Z_{ce} by Theorem 5.1 from Cai and Dickson [3]. If we substitute V by G, our proof repeats the arguments given in Theorem 5.1 from that paper. □

In the following sections, we will consider particular cases where the total loss in a unit period has an exponential distribution with mean 1, i.e. $W(x) = 1 - \mathrm{e}^{-x}$ or has the Pareto distribution with the same mean: $W(x) = 1 - (\beta/x)^{\alpha}$, $x > \beta$, $\alpha > 1$, $\beta = (\alpha - 1)/\alpha$. In Section 3, we give analytical formulae only for the cases $l = 1$, $i_1 = 0$ (i.e. the financial surplus is not invested) and small values of the parameter n. To determine these formulae we use the Maxima package, which can carry out symbolic calculations.

Numerical results will be presented for the case $l = 2$ and for selected values of the parameters α, β, η, θ and b.

3. Ruin probability

Calculations of values of the function $\Psi^b(u, i_s)$ given by Theorem 1 were conducted for b = 0.2, 0.3,..., 1.0, u = 0, 1, 2, 3, 4, 5 and n = 1, 2,..., 10. We considered the cases

- $l = 1$ for $i_1 = 0$,
- $l = 2$ for $i_1 = 0.3$, $i_2 = 0.5$ with the transition matrix

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \end{bmatrix}$$

The values $\eta = 0.25$ and $\theta = 0.2$ were used. For $\mathrm{E}h(Z_n, b) = b$, from (5) we obtain the formula

$$c(b)=(1+\eta)b-(\eta-\theta)=1.25b-0.05$$

Condition (2) is fulfilled for $b>1-\theta/\eta=0.2$.

3.1. Exponential distribution

Let us assume that Z_n has the exponential distribution with the mean 1. Hence, $Z_n^{ce}=bZ_n$ has the distribution function

$$V(x)=1-\mathrm{e}^{-x/b} \tag{15}$$

for $x\geq 0$ and $\mathrm{E}Z_n^{ce}=b,\ \operatorname{Var}Z_n^{ce}=b^2.$

The explicit formulae for the function $\Psi_{1n}^b(u,i_s)$ for $n\geq 2$ are too complicated to present. We take $l=1$ and $i_1=0.$

$$\Psi_1^b(u)=\mathrm{e}^{(-u-\theta+(-b)(\eta+1)+\eta)/b}$$

$$\Psi_2^b(u)=\frac{\left(e^{2\eta/b}u+\mathrm{e}^{2\eta/b}\theta+\left((b-1)\eta+b\right)\mathrm{e}^{2\eta/b}\right)\mathrm{e}^{-u/b-2\theta/b-2\eta-2}}{b}+\mathrm{e}^{(-u-\theta-b(\eta+1)+\eta)/b}$$

The formulae for $\Psi_n^b(u)$ when $n\leq 5$ obtained from Maxima were used to verify the correctness of numerical algorithms which are used for larger *n* and *l*. Therefore, the ruin probability given by the recurrence formulae (3) and (4) in Theorem 1 should be determined numerically. The results of calculations for some chosen n, u, i_s and b are given in Tables 1 and 2 for claims with exponential distribution and in Table 3 and 4 for claims with the Pareto distribution.

From Table 1 we obtain the following conclusions.

- As the initial capital grows, for any time horizon *n* the proportion of the insurer's retained loss which maintains the ruin probability at a constant level is also increasing.
- As the initial investment rate grows, for any time horizon n the level of retention b which maintains the ruin probability at a constant level is also increasing..
- As the time horizon n grows, then the ruin probability grows for every fixed $u\geq 0.2$ and interest rate $I_0=i_s$. The greater *u*, the smaller the ruin probability.

Table 1. Values of ruin probabilities for the exponential distribution

n	i_s	u	b								
			0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
5	3%	1	0.0087	0.0385	0.0776	0.1164	0.1512	0.1814	0.2073	0.2299	0.2494
		2	0.0001	0.0029	0.0119	0.0271	0.0460	0.0665	0.0871	0.1074	0.1265
		3	0.0000	0.0002	0.0017	0.0060	0.0134	0.0236	0.0357	0.0491	0.0630
		4	0.0000	0.0000	0.0002	0.0013	0.0038	0.0081	0.0143	0.0220	0.0308
		5	0.0000	0.0000	0.0000	0.0003	0.0010	0.0027	0.0056	0.0096	0.0148
	5%	1	0.0046	0.0256	0.0580	0.0934	0.1267	0.1568	0.1832	0.2067	0.2271
		2	0.0001	0.0015	0.0077	0.0196	0.0357	0.0542	0.0734	0.0927	0.1113
		3	0.0000	0.0001	0.0010	0.0040	0.0098	0.0183	0.0288	0.0409	0.0539
		4	0.0000	0.0000	0.0001	0.0008	0.0026	0.0061	0.0111	0.0178	0.0257
		5	0.0000	0.0000	0.0000	0.0002	0.0007	0.0020	0.0042	0.0076	0.0121
10	3%	1	0.0112	0.0493	0.0978	0.1448	0.1856	0.2203	0.2494	0.2749	0.2964
		2	0.0003	0.0049	0.0190	0.0411	0.0669	0.0936	0.1193	0.1442	0.1669
		3	0.0000	0.0005	0.0035	0.0113	0.0236	0.0391	0.0564	0.0748	0.0932
		4	0.0000	0.0000	0.0006	0.0030	0.0081	0.0160	0.0262	0.0383	0.0515
		5	0.0000	0.0000	0.0001	0.0008	0.0027	0.0064	0.0119	0.0193	0.0281
	5%	1	0.0049	0.0282	0.0654	0.1064	0.1452	0.1800	0.2103	0.2372	0.2605
		2	0.0001	0.0020	0.0101	0.0256	0.0462	0.0695	0.0932	0.1168	0.1392
		3	0.0000	0.0001	0.0015	0.0061	0.0146	0.0267	0.0411	0.0574	0.0742
		4	0.0000	0.0000	0.0002	0.0014	0.0046	0.0101	0.0180	0.0280	0.0393
		5	0.0000	0.0000	0.0000	0.0003	0.0014	0.0038	0.0078	0.0135	0.0206

Table 2. Maximal level of retention b, for which the ruin probability does not exceed 0.05 when claims have an exponential distribution

Initial capital u		1	2	3	4	5
n	i_s [%]					
5	3	0.3289	0.6188	0.9062	1.0000	1.0000
	5	0.3752	0.6775	0.9700	1.0000	1.0000
10	3	0.3005	0.5339	0.7626	0.9876	1.0000
	5	0.3585	0.6160	0.8549	1.0000	1.0000

Figure 1 depicts the graphs of $\Psi_n^b(u, i_s)$ for the exponential distribution with $n = 5$ and $n = 10$, each of $b = 0.2, 0.4, \ldots, 1.0$ and $i_2 = 0.05$. In Figure 2, the graphs of $\Psi_n^b(u, i_s)$ for $n = 5$ and $n = 10$ are depicted, with $u = 1, 2, 3, 4, 5$ and $i_2 = 0.05$. The graphs for $i_1 = 0.03$ look almost identical, so we omit them. The differences are easy to observe in Table 1.

Table 2 implies that with initial capital $u \geq 4$ and interest rate $I_0 = i_s = 0.03$ for any b, the ruin probability does not exceed 0.05 for time horizons $n = 5$ and $n = 10$. In the table, the number 1 means that even without reinsurance an insurer will have a ruin probability of below 5%.

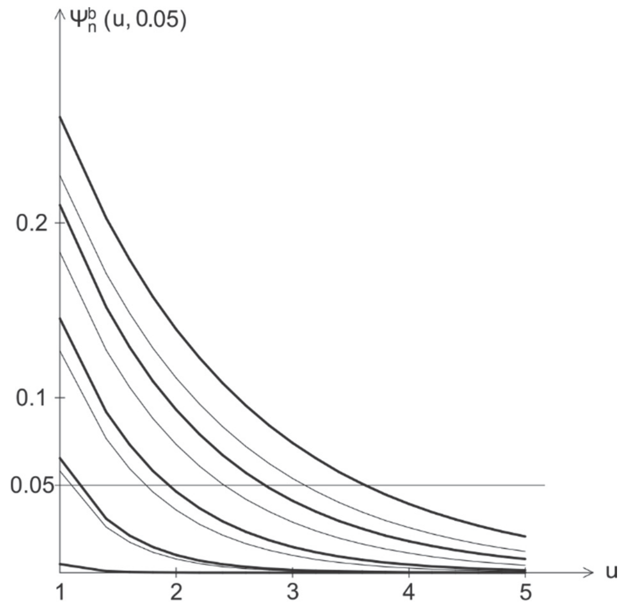

Fig. 1. Ruin probability for claims with an exponential distribution as a function of *u*. $\Psi_5^b(u, 0.05)$ – thin lines, $\Psi_{10}^b(u, 0.05)$ – thick lines, for *b* = 0.2, 0.4, 0.6, 0.8, 1.0 (lowest to highest), respectively

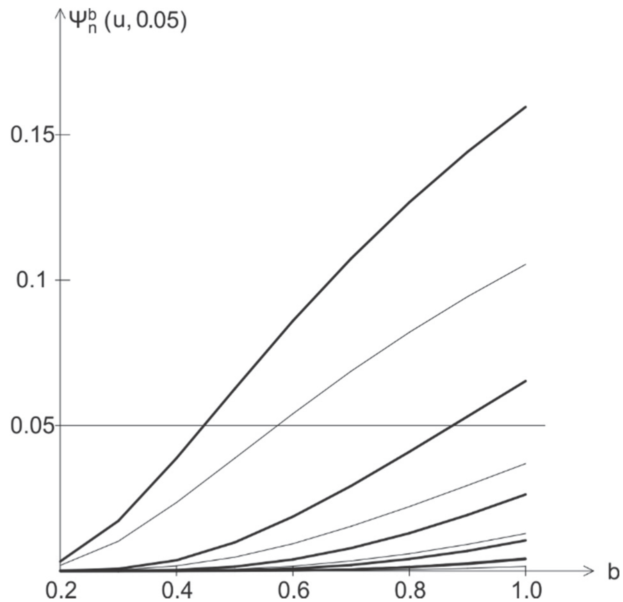

Fig. 2. Ruin probability for claims with an exponential distribution as a function of *b*. $\Psi_5^b(u, 0.05)$ – thin lines, $\Psi_{10}^b(u, 0.05)$ – thick lines, for $u = 1, 2, 3, 4, 5$ (highest to lowest), respectively

Now, let us answer question 2 from Section 1. We calculate the parameter $\xi(b)$ from Eq. (8) for $V(x)$ defined by (15):

$$\xi(b) = \sup_{x \geq c(b)} \frac{e^{R(b)x}\overline{V}(x)}{\int_x^{\infty} e^{R(b)z} dV(z)} = \sup_{x \geq c(b)} \frac{e^{R(b)x} e^{-x/b}}{\int_x^{\infty} e^{R(b)z} \frac{1}{b} e^{-z/b}\, dz} \tag{16}$$

Next, we calculate the integral under the assumption that $bR(b) < 1$:

$$\int_x^{\infty} e^{R(b)z} \frac{1}{b} e^{-z/b}\, dz = \frac{1}{R(b) - \frac{1}{b}} e^{(R(b)-1/b)z} \Bigg|_{z=x}^{z=\infty} = \frac{1}{1 - bR(b)} e^{(R(b)-1/b)x}$$

After substituting into (16) we have

$$\xi(b) = \sup_{x \geq c(b)} \frac{e^{(R(b)-1/b)x}}{\frac{1}{1-bR(b)} e^{(R(b)-1/b)x}}$$

Hence,

$$\xi(b) = 1 - bR(b) \tag{17}$$

When $b > 1 - \theta/\eta$, the adjustment coefficient $R(b)$ is the positive solution of Eq. (6). Since the moment generating function $V(x)$ has the form

$$M(z) = \frac{1}{1 - bz}$$

where $z < 1/b$, then Eq. (6) has the form

$$\frac{1}{1 - bR(b)} = e^{R(b)c(b)}$$

from which we determine $R(b)$. The function $R(b)$ is a concave function for $\theta = \eta$, whereas $R(b)$ is convex for $\theta < \eta$.

Based on Theorem 2, the following gives an upper bound on the ruin probability:

$$\Psi_n^b(u, i_s) \le \left(1 - bR(b)\right) \sum_{t=1}^{l} p_{st} \mathrm{e}^{-R(b)u(1+i_t)}, \qquad n = 1, 2, \ldots \tag{18}$$

Let us denote the right-hand-side of inequality (18) by $g^b(u, i_s)$.

If an insurer uses a smaller security loading θ than the loading η of a reinsurer, Theorem 2 indicates that an upper bound of Lundberg's type is a very bad estimate, useless in insurance practice. This is justified by Fig. 3.

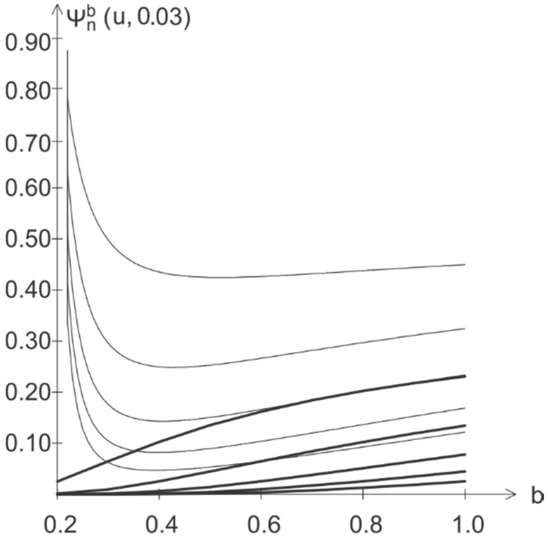

Fig. 3. Upper bounds $g^b(u, 0.03)$ – thin lines, on the probabilities $\Psi_{10}^b(u, 0.03)$ – thick lines, for u = 1, 2, 3, 4, 5 (highest to lowest), η = 0.25, θ = 0.20, respectively

3.2. Pareto distribution

We assume that the total loss Z_n has the Pareto distribution with the distribution function

$$W(x) = 1 - \left(\frac{\beta}{x}\right)^{\alpha} \tag{19}$$

for $x \ge \beta > 0.$ The random variable Z_n has expected value

$$\mathrm{E}X = \frac{\alpha\beta}{\alpha - 1}$$

for $\alpha > 1$ and variance

$$\mathrm{Var}\, X = \frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)}$$

for $\alpha > 2$.

We assume that $\mathrm{E}Z_n = 1$. Hence, the parameter β must be of the form

$$\beta = \frac{\alpha - 1}{\alpha}$$

The loss $Z_n^{ce} = bZ_n$ retained by an insurer has cdf

$$V(x) = 1 - \left(\frac{b\beta}{x}\right)^{\alpha} \tag{20}$$

for $x \geq b\beta$.

In the numerical calculations, we assume that $\alpha = 1.25$, as in the paper by Palmowski [8]. It was shown that the greatest losses, which came at the turn of the eighties and nineties of the XX century, have the Pareto distribution with the parameter approximately equal to 1.24138. With such a value of α, the variance is infinite.

From (5), we have

$$c(b) = (1+\eta)b - (\eta - \theta)$$

The function $\Psi_1^b(u, i_s)$ can be derived by (3) in explicit form only for $n = 1$, $l = 1$, and $i_1 = 0$.

$$\Psi_1^b(u) = \left(\frac{b\beta}{u + \theta + b(\eta+1) - \eta}\right)^{\alpha} \tag{21}$$

The cases $n > 1$ require numerical integration. Let us consider the case $n = 2$. In this case, it is necessary to calculate the integral

$$\alpha(b\beta)^{\alpha}\int_{b\beta}^{x+c(b)}\left(\frac{b\beta}{u+\theta+b(\eta+1)-\eta-z}\right)^{\alpha} z^{-(\alpha+1)}dz$$

Substituting $A = u + \theta + b(\eta + 1) - \eta$, we come to the problem of calculating the integral

$$\int \frac{1}{(A-z)^{\alpha} z^{\alpha+1}}dz = -\frac{\left(1-\frac{z}{A}\right)_2 F_1\left(-\alpha, \alpha; 1; 1-\alpha, \frac{x}{A}\right)}{\alpha(A-z)^{\alpha} z^{\alpha}},$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function.

Table 3. Values of ruin probabilities for the Pareto distribution

n	i_s	u	b								
			0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
5	3%	1	0.0471	0.0663	0.0818	0.0945	0.1050	0.1156	0.1214	0.1280	0.1337
		2	0.0255	0.0384	0.0499	0.0602	0.0693	0.0787	0.0846	0.0912	0.0972
		3	0.0169	0.0263	0.0352	0.0434	0.0510	0.0590	0.0644	0.0704	0.0759
		4	0.0124	0.0197	0.0267	0.0335	0.0399	0.0468	0.0515	0.0569	0.0618
		5	0.0097	0.0156	0.0214	0.0270	0.0325	0.0384	0.0427	0.0474	0.0519
	5%	1	0.0421	0.0603	0.0754	0.0880	0.0986	0.1092	0.1154	0.1222	0.1281
		2	0.0234	0.0356	0.0466	0.0566	0.0655	0.0748	0.0807	0.0873	0.0933
		3	0.0158	0.0247	0.0332	0.0411	0.0485	0.0563	0.0617	0.0676	0.0730
		4	0.0118	0.0187	0.0254	0.0320	0.0382	0.0448	0.0495	0.0548	0.0597
		5	0.0092	0.0148	0.0204	0.0259	0.0312	0.0370	0.0411	0.0458	0.0502
10	3%	1	0.0685	0.0947	0.1150	0.1312	0.1442	0.1599	0.1640	0.1718	0.1785
		2	0.0405	0.0599	0.0765	0.0907	0.1029	0.1175	0.1226	0.1309	0.1382
		3	0.0282	0.0432	0.0568	0.0690	0.0798	0.0929	0.0980	0.1060	0.1131
		4	0.0213	0.0334	0.0448	0.0552	0.0648	0.0765	0.0814	0.0888	0.0956
		5	0.0170	0.0270	0.0367	0.0458	0.0542	0.0647	0.0694	0.0762	0.0826
	5%	1	0.0582	0.0829	0.1028	0.1191	0.1325	0.1480	0.1532	0.1615	0.1687
		2	0.0354	0.0533	0.0691	0.0829	0.0949	0.1091	0.1148	0.1232	0.1307
		3	0.0252	0.0391	0.0519	0.0635	0.0740	0.0866	0.0921	0.1000	0.1072
		4	0.0194	0.0306	0.0413	0.0513	0.0605	0.0717	0.0768	0.0841	0.0908
		5	0.0156	0.0250	0.0341	0.0427	0.0509	0.0609	0.0656	0.0723	0.0786

Table 4. Maximal level of retention b, for which the ruin probability does not exceed 0.05 when claims have the Pareto distribution

Initial capital u		1	2	3	4	5
n	i_s [%]					
5	3	0.2190	0.4052	0.5907	0.7696	0.9621
	5	0.2468	0.4379	0.6209	0.8133	0.9996
10	3	lack	0.2567	0.3582	0.4588	0.5588
	5	lack	0.2884	0.3933	0.4958	0.5974

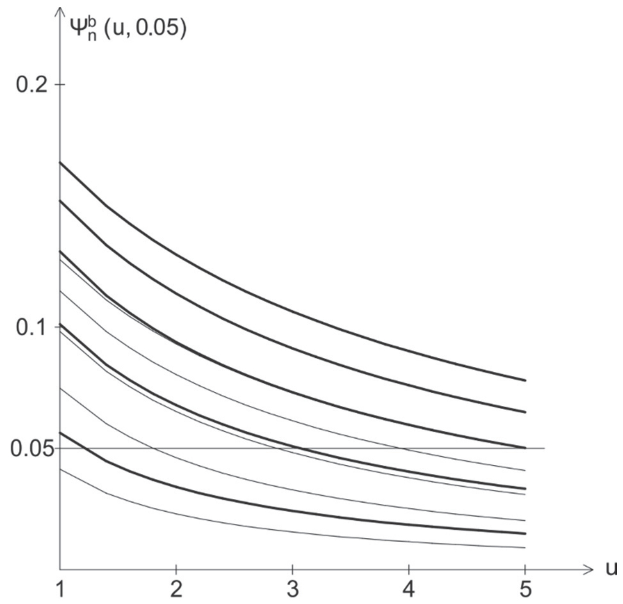

Fig. 4. Ruin probability for claims from the Pareto distribution as a function of *u*. $\Psi_5^b(u, \ 0.05)$ – thin lines, $\Psi_{10}^b(u, 0.05)$ – thick lines, for *b* = 0.2, 0.4, 0.6, 0.8, 1.0 (lowest to highest), respectively

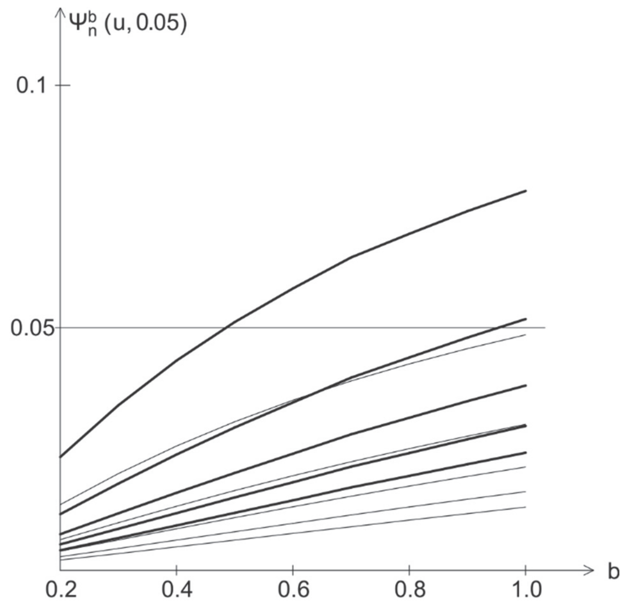

Fig. 5. Ruin probability for claims from the Pareto distribution as a function of *b*. $\Psi_5^b(u, 0.05)$ – thin lines $\Psi_{10}^b(u, 0.05)$ – thick lines, for *u* = 1, 2, 3, 4, 5 (highest to lowest), respectively

Table 3 gives similar conclusions as for the exponential distribution. The word lack in Table 4 indicates that for any level of retention $b \in (0.2, 1]$ with initial capital $u = 1$, the ruin probability exceeds 0.05 both for a five-year-time horizon and for a ten-year--time horizon. In Figure 4, the graphs of $\Psi_n^b(u, i_s)$ for $n = 5$ and $n = 10$ when claims have the Pareto distribution are depicted for $i_2 = 0.05$. Figure 5 presents the graphs of $\Psi_n^b(u, i_s)$ for $n = 5$ and $n = 10$, $u = 1, 2, 3, 4, 5$ and $i_2 = 0.05$. The graphs for $i_1 = 0.03$ look almost identical, so we omit them. The differences are easy to observe in Table 3.

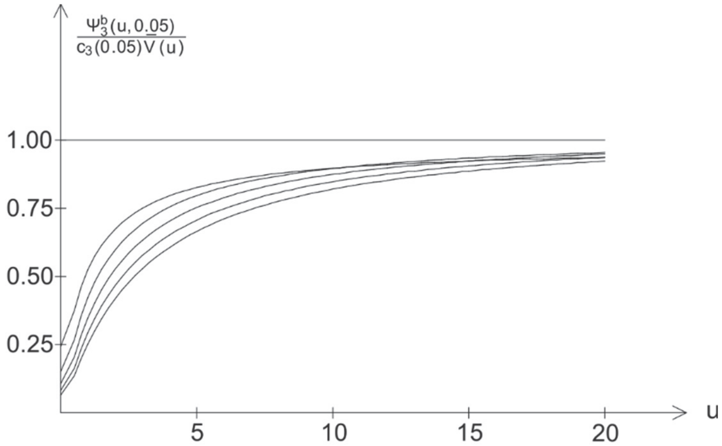

Fig. 6. Asymptotic approximation of the ruin probability for claims from the Pareto distribution – graphs $\Psi_n^b(u, i_s)/c_n(i_s)\bar{V}(u)$ for $b = 0.2, 0.4, 0.6, 0.8, 1.0$ (highest to lowest), respectively

Using Theorem 3, we will present results concerning an approximation of the ruin probability for claims from the Pareto distribution. Figure 6 illustrates the ratio

$$\frac{\Psi_n^b(u, i_s)}{c_n(i_s)\bar{V}(u)}$$

for $n = 3, b = 0.2, 0.4, \ldots, 1.0$ and $0 \le u \le 20$.

4. Conclusions

In a continuous risk process, the optimal level of retention can be determined by maximising the adjustment coefficient with respect to the level of retention. This statement is not true for a discrete risk process.

For any fixed initial capital $u \geq 1$, the probability of ruin is an increasing function of the retention level b. Therefore, the probability of ruin is minimised when the retention level is minimised. This means that an insurer retains only very small losses, which leads to a very low income and is very unfavourable for him. It seems that the right approach is based on fixing an acceptable level for the ruin probability and determining the retention level appropriate for this probability.

If the loading of a reinsurer is greater than the loading of an insurer ($\eta > \theta$), then the adjustment coefficient is not a convex function which lowers the quality of the upper bound. Based on our numerical examples, we conclude that such an upper bound is very imprecise, and, in practice, is worthless.

For heavy tailed claims, we present a theorem about the approximation of the ruin probability when the initial capital is sufficiently large. The example of the Pareto distribution shows that such an approximation is appropriate and converges quickly to the limit value.

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